On Some Connections between BIBO and Internal Stability of Two-Dimensional Filters

M. BISIACCO, E. FORNASINI, AND G. MARCHESINI

Abstract —Necessary and sufficient conditions are given ensuring that a BIBO stable two-dimensional (2-D) filter admits an internally stable state-space realization.

These correspond to the nonexistence of common zeros of the relatively prime numerator and denominator on the unit bidisc distinguished boundary.

An equivalent condition is the existence of detectable and stabilizable state space realizations of the transfer function.

I. Introduction

In [1] Goodman has proved that there exist transfer functions $p(z_1, z_2)/q(z_1, z_2)$, with p and q coprime, which are BIBO stable and have nonessential singularities of the second kind on the distinguished boundary

$$T = \{(z_1, z_2): |z_1| = |z_2| = 1\}.$$

A consequence of this fact is that BIBO stability does not imply that the denominator $q(z_1, z_2)$ is devoid of zeros in the closed polydisc:

$$\mathscr{P}_1 = \{(z_1, z_2): |z_1| \leqslant 1; \ |z_2| \leqslant 1\}$$

while, as we know, internal stability does [2]. So, there exist 2-D BIBO stable transfer functions which don't admit internally stable realizations in state-space form.

A similar situation does not arise in 1-D case, since the minimal realization of a BIBO stable transfer function is always internally stable. For this reason, one may expect that the connections between 2-D internal and external stability are more complex than in 1-D case. As we shall see, this is not completely true, and the purpose of this note is to discuss these connections and in particular to analyze the conditions which guarantee that a BIBO stable system is internally stable. Actually, we shall prove that a BIBO stable 2-D system is internally stable if and only if it is stabilizable and detectable, a condition which is exactly the same we have for 1-D systems.

Nevertheless, given a 1-D transfer function, a stabilizable and detectable realization can always be computed, while for the 2-D case there are situations where this is not possible. More precisely, this happens when both p and q vanish at some common point in $\mathscr{P}_1$, which corresponds to the case of the transfer functions examined by Goodman.

II. Definitions and Preliminary Results

Consider a 2-D transfer function

$$W(z_1, z_2) = \frac{p(z_1, z_2)}{q(z_1, z_2)}$$

with p and q coprime and $q(0,0) = 1$. It is well known that $W(z_1, z_2)$ is BIBO stable if and only if the coefficients of its power series expansion

$$W(z_1, z_2) = \sum_{i,j=0}^{+\infty} w_{ij} z_1^i z_2^j$$

satisfy the following inequality [1]:

$$\sum_{i,j=0}^{+\infty} |w_{ij}| < +\infty. \tag{1}$$

If $q(z_1, z_2)$ is devoid of zeros in $\mathscr{P}_1$, then $W(z_1, z_2)$ is BIBO stable. As shown by Goodman's counterexample, the vice versa is not true.

The internal stability concept refers to the state evolution of realizations of $W(z_1, z_2)$ in state space form. We recall [3] that a 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ given by

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h, k+1) + A_2 x(h+1, k) \\ &\quad + B_1 u(h, k+1) + B_2 u(h+1, k) \\ y(h,k) &= Cx(h,k) + Du(h,k) \end{aligned} \tag{2}$$

where the *local state* x is an n-dimensional vector over the real field $\mathbb{R}$, and A_1, A_2, B_1, B_2, C, D are matrices of suitable dimensions with entries in $\mathbb{R}$, realizes $W(z_1, z_2)$ if

$$W(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D. \tag{3}$$

The free state evolution of system (2) is given by

$$\begin{aligned} X(z_1, z_2) &= \sum_{i+j \geqslant 0} x(i,j) z_1^i z_2^j = (I - A_1 z_1 - A_2 z_2)^{-1} \mathscr{X}_0 \\ &= \sum_{i=0}^{+\infty} (A_1 z_1 + A_2 z_2)^i \mathscr{X}_0 \end{aligned} \tag{4}$$

Manuscript received November 5, 1984; revised February 25, 1985.
The authors are with the Institute of Electrical and Electronic Engineering, 35131 Padova, Italy.

0098-4094/85/0900-0948$01.00 ©1985 IEEE

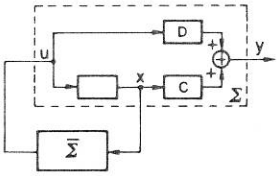

Fig. 1.

where

$$\mathscr{X}_0 = \sum_{i=-\infty}^{+\infty} x(i,-i) z_1^i z_2^{-i}$$

is called the *global state* associated with the sequence of initial local states $\{x(i,-i)\colon i \in Z\}$. The 2-D system Σ is internally stable [2] if, given any bounded sequence $\{x(i,-i),\ i\in\mathbb{Z}\}$, the state evolution (4) satisfies

$$\lim_{n\to+\infty} \sup_{h\in\mathbb{Z}} \{\|x(n-h,h)\|\} = 0. \qquad (5)$$

A necessary and sufficient condition for internal stability [2] is that the polynomial $\det(I - A_1 z_1 - A_2 z_2)$ is devoid of zeros in $\mathscr{P}_1$.

Detectability and stabilizability are the structural properties needed for connecting BIBO and internal stability of 2-D systems. We recall [4]–[9] that a system Σ is detectable if there exists an asymptotic observer of the state $x(h,k)$ whose estimate error vanishes as $h+k\to\infty$ and is stabilizable if there exists a 2-D system $\bar{\Sigma}$ such that the state feedback connection of Fig. 1 is internally stable.

The role played by detectability and stabilizability in the stability analysis is based on the equivalences stated in Theorems 1 and 2.

Theorem 1. The following facts are equivalent:

(i) Σ *is stabilizable*.

(ii) *The matrix*

$$[B_1 z_1 + B_2 z_2 \ \vdots\ I - A_1 z_1 - A_2 z_2] \qquad (6)$$

is full rank for every $(z_1, z_2) \in \mathscr{P}_1$.

(iii) *There exist rational matrices* $M(z_1,z_2)$ *and* $N(z_1,z_2)$, *whose denominators are devoid of zeros in* $\mathscr{P}_1$, *such that the Bézout identity*

$$(B_1 z_1 + B_2 z_2) N(z_1,z_2) + (I - A_1 z_1 - A_2 z_2) M(z_1,z_2) = I \qquad (7)$$

holds.

Proof i) $\Rightarrow$ ii): Let $\bar{\Sigma}$ be a 2-D system represented by the following state equations

$$\bar{x}(h+1,k+1) = \bar{A}_1 \bar{x}(h,k+1) + \bar{A}_2 \bar{x}(h+1,k) + \bar{B}_1 \bar{u}(h,k+1) + \bar{B}_2 \bar{u}(h+1,k)$$

$$\bar{y}(h,k) = \bar{C}\bar{x}(h,k) + \bar{D}\bar{u}(h,k)$$

where $\bar{u}(h,k)\in\mathbb{R}^n$, $\bar{y}(h,k)\in\mathbb{R}$, and assume that the feedback connection of Fig. 1 is internally stable.

Defining

$$F_i \triangleq \begin{bmatrix} A_i + B_i\bar{D} & B_i\bar{C} \\ \bar{B}_i & \bar{A}_i \end{bmatrix}, \qquad i = 1,2 \qquad (8)$$

the internal stability of the feedback system is equivalent to assume that the matrix

$$I - F_1 z_1 - F_2 z_2 = \begin{bmatrix} I - A_1 z_1 - A_2 z_2 - (B_1 z_1 + B_2 z_2)\bar{D} & -(B_1 z_1 + B_2 z_2)\bar{C} \\ -(\bar{B}_1 z_1 + \bar{B}_2 z_2) & I - \bar{A}_1 z_1 - \bar{A}_2 z_2 \end{bmatrix} \qquad (9)$$

is full rank for any (z_1,z_2) in $\mathscr{P}_1$. Then (6) is full rank in $\mathscr{P}_1$. For example, assume

$$v^T\left[I - A_1 z_1^0 - A_2 z_2^0 \quad B_1 z_1^0 + B_2 z_2^0\right] = 0$$

for some nonzero v in $\mathbb{C}^n$ and (z_1^0, z_2^0) in $\mathscr{P}_1$. This implies $[v^T \quad 0]. (I - F_1 z_1^0 - F_2 z_2^0) = 0$, which is impossible since (9) is full rank.

ii) $\Rightarrow$ iii): Let $S_1(z_1,z_2), S_2(z_1,z_2)\cdots S_{n+1}(z_1,z_2)$ be the submatrices of order n in (6) and $s_1(z_1,z_2),\ s_2(z_1,z_2)\cdots s_{n+1}(z_1,z_2)$ be their determinants. Denote by $\mathscr{I}\subseteq\mathbb{C}[z_1,z_2]$ the ideal of polynomials p which satisfy the following equation

$$(I - A_1 z_1 - A_2 z_2)\tilde{M}(z_1,z_2) + (B_1 z_1 + B_2 z_2)\tilde{N}(z_1,z_2) = Ip(z_1,z_2) \qquad (10)$$

for some polynomial matrices $\tilde{M}$ and $\tilde{N}$.

First of all we shall prove that the algebraic variety $\mathscr{V}(\mathscr{I})$ is the variety of the polynomials $s_1, s_2, \cdots, s_{n+1}$:

$$\mathscr{V}(\mathscr{I}) = \mathscr{V}(s_1, s_2, \cdots, s_{n+1}).$$

From the definition of $S_k(z_1,z_2)$, $k=1,2,\cdots,n+1$, it is clear that there exist constant matrices M_k, N_k such that

$$S_k(z_1,z_2) = (I - A_1 z_1 - A_2 z_2)M_k + (B_1 z_1 + B_2 z_2)N_k.$$

Then, postmultiplying both sides by $\operatorname{adj} S_k(z_1,z_2)$ yields

$$s_k(z_1,z_2)I = (I - A_1 z_1 - A_2 z_2)[M_k \operatorname{adj} S_k(z_1,z_2)] + (B_1 z_1 + B_2 z_2)[N_k \operatorname{adj} S_k(z_1,z_2)]$$

which shows that s_k is in $\mathscr{I}$ for $k = 1,2,\cdots,n+1$ and

$$\mathscr{V}(s_1, s_2, \cdots, s_{n+1}) \supseteq \mathscr{V}(\mathscr{I}).$$

To prove the inverse inclusion, choose any nonzero polynomial p in $\mathscr{I}$ and polynomial matrices $\tilde{M}$ and $\tilde{N}$ satisfying (10). Then, for any $(z_1^0, z_2^0)\notin\mathscr{V}(p)$, we have

$$(B_1 z_1^0 + B_2 z_2^0)\frac{\tilde{N}(z_1^0,z_2^0)}{p(z_1^0,z_2^0)} + (I - A_1 z_1^0 - A_2 z_2^0)\frac{\tilde{M}(z_1^0,z_2^0)}{p(z_1^0,z_2^0)} = I.$$

Therefore, the column span of

$$\left[I - A_1 z_1^0 - A_2 z_2^0 \quad B_1 z_1^0 + B_2 z_2^0\right]$$

is $\mathbb{C}^n$ and, consequently, (z_1^0, z_2^0) cannot be a common zero of $s_1, s_2, \cdots, s_{n+1}$. This means that $(z_1^0,z_2^0)\notin\mathscr{V}(s_1,s_2,\cdots,s_{n+1})$ and proves the inclusion

$$\mathscr{V}(\mathscr{I}) \supseteq \mathscr{V}(s_1, s_2, \cdots, s_{n+1}).$$

Next step is to prove that there exists a polynomial $q(z_1,z_2)$ in $\mathscr{I}$ such that $\mathscr{V}(q)\cap\mathscr{P}_1 = \varnothing$.

Let $c = \mathrm{GCD}(s_1, s_2, \cdots, s_{n+1})$ and factorize the polynomials s_i as

$$s_1 = h_1 c, \cdots, s_{n+1} = h_{n+1} c.$$

Then

$$\mathscr{V}(\mathscr{I}) = \mathscr{V}(c) \cup \mathscr{V}(h_1, \cdots, h_{n+1})$$

where $\mathscr{V}(h_1,\cdots,h_{n+1})$ is a conjugate symmetric finite set of points $\{(a_i,b_i),\ i=1,2,\cdots,k\}$ which does not intersect $\mathscr{P}_1$ because of ii).

Define the polynomials $d_i(z_1, z_2)$, $i = 1, 2, \cdots, k$ as

$$d_i(z_1, z_2) = \begin{cases} (z_1 - a_i)(z_1 - a_i^*), & \text{if } |a_i| > 1 \\ (z_2 - b_i)(z_2 - b_i^*), & \text{if } |a_i| \leqslant 1. \end{cases}$$

The polynomial

$$d(z_1, z_2) = c(z_1, z_2) \prod_{i=1}^{k} d_i(z_1, z_2)$$

belongs to $\mathbb{R}[z_1, z_2]$ and $\mathscr{V}(d) \supseteq \mathscr{V}(\mathscr{I})$. By Hilbert Nullstellensatz, $d^h \in \mathscr{I}$ for some integer h, so that we can assume $q = d^h$.

Finally, by the definition of $\mathscr{I}$, there exist polynomial matrices $\tilde{M}(z_1, z_2)$ and $\tilde{N}(z_1, z_2)$ such that

$$(B_1 z_1 + B_2 z_2)\tilde{N}(z_1, z_2) + (I - A_1 z_1 - A_2 z_2)\tilde{M}(z_1, z_2) = Iq(z_1, z_2)$$

and (7) follows by assuming

$$M(z_1, z_2) = \tilde{M}(z_1, z_2)/q(z_1, z_2)$$
$$N(z_1, z_2) = \tilde{N}(z_1, z_2)/q(z_1, z_2).$$

iii) $\Rightarrow$ i): Let $d(z_1, z_2)$ be the least common denominator of the elements in $M(z_1, z_2)$ and $N(z_1, z_2)$. Then

$$(B_1 z_1 + B_2 z_2)\overline{N}(z_1, z_2) + (I - A_1 z_1 - A_2 z_2)\overline{M}(z_1, z_2) = Id(z_1, z_2)$$

where $\overline{N} = Nd$ and $\overline{M} = Md$ are polynomial matrices and $d(z_1, z_2)$ is devoid of zeros in $\mathscr{P}_1$.

Since $M(0,0) = Id(0,0) \neq 0$, $\overline{M}(z_1, z_2)$ is nonsingular in $\mathscr{P}_1$. Then the row matrix

$$-\overline{N}(z_1, z_2)\overline{M}(z_1, z_2)^{-1} = [\det \overline{M}(z_1, z_2)]^{-1}[-\overline{N}(z_1, z_2) \operatorname{adj} \overline{M}(z_1, z_2)]$$

can be realized [3] by a 2-D system $\overline{\Sigma} = (\overline{A}_1, \overline{A}_2, \overline{B}_1, \overline{B}_2, \overline{C}, \overline{D})$ where

$$\det(I - \overline{A}_1 z_1 - \overline{A}_2 z_2) = \det \overline{M}(z_1, z_2).$$

Furthermore, F_1 and F_2 in (8) are the state updating matrices of the system obtained by connecting $\overline{\Sigma}$ and Σ as in Fig. 1, and

$$\begin{aligned}
&\det(I - F_1 z_1 - F_2 z_2) \\
&= \det(I - \overline{A}_1 z_1 - \overline{A}_2 z_2) \\
&\quad \cdot \det\big[I - A_1 z_1 - A_2 z_2 - (B_1 z_1 + B_2 z_2)\overline{D} \\
&\quad - (B_1 z_1 + B_2 z_2)\overline{C}(I - \overline{A}_1 z_1 - \overline{A}_2 z_2)^{-1}(\overline{B}_1 z_1 + \overline{B}_2 z_2)\big] \\
&= \det(I - \overline{A}_1 z_1 - \overline{A}_2 z_2) \\
&\quad \cdot \det\big[I - A_1 z_1 - A_2 z_2 + (B_1 z_1 + B_2 z_2)\overline{N}\overline{M}^{-1}\big] \\
&= \det(I - \overline{A}_1 z_1 - \overline{A}_2 z_2) \\
&\quad \cdot \det\big[(I - A_1 z_1 - A_2 z_2)\overline{M} + (B_1 z_1 + B_2 z_2)\overline{N}\big]\det(\overline{M}^{-1}) \\
&= \det(I - \overline{A}_1 z_1 - \overline{A}_2 z_2)\det(Id(z_1, z_2))\det(\overline{M}^{-1}) \\
&= d(z_1, z_2)^n.
\end{aligned} \tag{11}$$

This implies that the feedback connection is internally stable.

Dual arguments can be used to prove the following theorem (see also [7], [9]).

Theorem 2. The following facts are equivalent:

i) *Σ is detectable.*

ii) *The matrix*

$$\begin{bmatrix} C \\ \hdashline I - A_1 z_1 - A_2 z_2 \end{bmatrix} \tag{12}$$

is full rank for every $(z_1, z_2) \in \mathscr{P}_1$.

iii) *There exist rational matrices $P(z_1, z_2)$ and $Q(z_1, z_2)$, whose denominators are devoid of zeros in $\mathscr{P}_1$, such that the Bézout identity*

$$P(z_1, z_2)C + Q(z_1, z_2)(I - A_1 z_1 - A_2 z_2) = I \tag{13}$$

holds.

III. The Main Result

Let us first introduce the following lemma:

Lemma: Assume that the 2-D transfer matrix

$$W(z_1, z_2) = \sum_{i,j=0}^{+\infty} W_{ij} z_1^i z_2^j$$

is BIBO-stable, and let $u(h, k)$ be a bounded input satisfying:

i) $$u(h, k) = 0 \qquad \text{for } h + k < 0$$

ii) $$\lim_{n \to +\infty} \sup_{i \in \mathbb{Z}} \|u(n - i, i)\| = 0.$$

Then the output function is bounded and satisfies

$$\lim_{n \to +\infty} \sup_{i \in \mathbb{Z}} \|y(n - i, i)\| = 0.$$

Proof: Let

$$a_n \triangleq \sup_{i \in \mathbb{Z}} \|u(n - i, i)\|$$

and

$$b_n \triangleq \sum_{k=0}^{n} \|W_{n-k,k}\|.$$

For any (h, k), with $h + k \geqslant 0$, the input–output relation in time domain is

$$y(h, k) = \sum_{\substack{t, \tau \geqslant 0 \\ t + \tau \leqslant h + k}} W_{t,\tau} u(h - t, k - \tau). \tag{14}$$

We, therefore, have

$$\|y(h, k)\| \leqslant \sum_{\substack{t, \tau \geqslant 0 \\ t + \tau \leqslant h + k}} \|W_{t,\tau}\| \|u(h - t, k - \tau)\| \leqslant \sum_{\substack{t, \tau \geqslant 0 \\ t + \tau \leqslant h + k}} \|W_{t,\tau}\| a_{(h+k)-(t+\tau)}$$

and introducing the new variable $r = t + \tau$

$$\|y(h, k)\| \leqslant \sum_{r=0}^{h+k} \left(\sum_{t=0}^{r} \|W_{t,r-t}\| \right) a_{(h+k)-r} = \sum_{r=0}^{t+k} b_r a_{(h+k)-r}.$$

So, defining

$$c_n = \sum_{k=0}^{n} b_k a_{n-k}$$

we have

$$\sup_{i \in \mathbb{Z}} \|y(n - i, i)\| \leqslant \sum_{r=0}^{n} b_r a_{n-r} = c_n.$$

Since $\{a_n\}$ is infinitesimal and $\{b_n\}$ is absolutely summable by the BIBO stability hypothesis, the sequence $\{c_n\}$ is infinitesimal, and the proof is complete.

We are now in a position to prove the main result. Theorem 3 shows that 2-D detectability and stabilizability are good generalizations of the analogous 1-D concepts, in the sense that they relate internal and external stability in the same way as in 1-D case.

Theorem 3: The following facts are equivalent:

i) *System (2) is internally stable.*

ii) *System (2) is detectable and stabilizable, and its transfer function (3) is BIBO-stable.*

Proof (i) ⇒ (ii): Let system (2) to be internally stable. Then the denominator of the rational matrix

$$(I - A_1z_1 - A_2z_2)^{-1} = \frac{\operatorname{adj}(I - A_1z_1 - A_2z_2)}{\det(I - A_1z_1 - A_2z_2)}$$

is devoid of zeros in $\mathscr{P}_1$.

The Bézout identities (7) and (13) are satisfied assuming

$$P(z_1, z_2) = 0 \qquad N(z_1, z_2) = 0$$

$$Q(z_1, z_2) = M(z_1, z_2) = (I - A_1z_1 - A_2z_2)^{-1}.$$

Hence, by Theorems 1 and 2, system (2) is detectable and stabilizable. To complete the proof, note that the denominator of the transfer function (3) is devoid of zeros in $\mathscr{P}_1$.

(ii) ⇒ (i): Assume system (2) to be BIBO-stable, detectable and stabilizable. Then, by Theorems 1 and 2 there exist rational matrices P, Q, N, M in z_1 and z_2, with denominators devoid of zeros in $\mathscr{P}_1$, that satisfy the Bézout identities (7) and (13),

From (7) and (13) one gets

$$(I - A_1z_1 - A_2z_2)^{-1} = Q + PC(I - A_1z_1 - A_2z_2)^{-1} \quad (15)$$

$$(I - A_1z_1 - A_2z_2)^{-1} = M + (I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)N. \quad (16)$$

Substitution of (16) in (15) gives

$$(I - A_1z_1 - A_2z_2)^{-1} = Q + PCM + PC(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)N$$

and, recalling (3), we get

$$(I - A_1z_1 - A_2z_2)^{-1} = Q + PCM - PDN + PWN. \quad (17)$$

This is a BIBO stable transfer matrix. In fact P, Q, M, N can be viewed as BIBO stable transfer matrices, and BIBO stability is preserved under multiplication, by Cauchy's theorem [10] on the product of absolutely summable series.

Consider now the free state evolution of system (2) starting from any initial global state $\mathscr{X}_0$ associated with a bounded sequence $\{x(h, -h)\}$ of local states. Then the state evolution is

$$X(z_1, z_2) = (I - A_1z_1 - A_2z_2)^{-1}\mathscr{X}_0 = (Q + PCM - PDN + PWN)\mathscr{X}_0.$$

It follows that $x(h, k)$ can be viewed as the output of a multi-input, multi-output BIBO stable 2-D filter with transfer matrix $(Q + PCM - PDN + PWN)$, driven by the input

$$\hat{u}(h, k) = \begin{cases} x(h, -h), & \text{if } k = -h \\ 0, & \text{elsewhere.} \end{cases}$$

Then $\hat{u}(h, k)$ satisfies the hypothesis of Lemma 1, so that

$$\lim_{n \to +\infty} \sup_{i \in \mathbf{Z}} \{\|x(n - i, i)\|\} = 0.$$

This proves the internal stability of system (2).

Remark 1. The key of the proof above is (17), which directly depends on Bézout identities (7) and (13). This equation allows us to relate the free state evolution, expressible in terms of $(I - A_1z_1 - A_2z_2)^{-1}$, with the forced output evolution which is expressible in terms of the transfer function $W(z_1, z_2)$.

Remark 2. In [11] it was proved that a realization $\Sigma = (A_1, A_2, B_1, B_2, C)$ of a transfer function $W(z_1, z_2) = p/q$, p and q coprime, is internally stable if a) $W(z_1, z_2)$ is BIBO stable and does not exhibit nonessential singularities of the second kind in $\mathscr{P}_1$, and b) Σ is a coprime realization, i.e., $C, I - A_1z_1 - A_2z_2$ are left coprime and $B_1z_1 + B_2z_2$, $I - A_1z_1 - A_2z_2$ are right coprime.

In fact, b) implies $\det(I - A_1z_1 - A_2z_2) = q$ (see [12]) and a) implies that q is devoid of zeros in the closed polydisc $\mathscr{P}_1$.

Assumptions a) and b) are more restrictive than condition ii) in Theorem 3. Actually coprimeness implies that matrices (6) and (12) are full rank in $\mathbb{C}^2 \setminus \mathscr{P}_1$, except possibly a finite set of points, while detectability and stabilizability refer to the rank of (6) and (12) in $\mathscr{P}_1$ only.

Theorem 3 reduces the problem of constructing an internally stable realization of a BIBO stable 2-D transfer function to obtaining a detectable and stabilizable realization.

Necessary and sufficient conditions for the existence of a detectable and stabilizable realization are provided by the following Theorem.

Theorem 4: Let Σ be a detectable and stabilizable realization of the transfer function

$$W(z_1, z_2) = \frac{p(z_1, z_2)}{q(z_1, z_2)} = p_{00} + \frac{\sum_{i+j>0} p_{ij}z_1^iz_2^j}{\sum_{i+j\geq 0} q_{ij}z_1^iz_2^j} \quad (18)$$

with p and q coprime and $q_{00} = 1$. Then p and q have no common zeros in $\mathscr{P}_1$. Conversely, let $W(z_1, z_2)$ as in (18) and assume p and q having no common zeros in $\mathscr{P}_1$. Then $W(z_1, z_2)$ admits a detectable and stabilizable realization.

Proof: Let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be a detectable and stabilizable realizable of $W(z_1, z_2)$. Detectability and stabilizability properties imply (17). This can be written as

$$\operatorname{adj}(I - A_1z_1 - A_2z_2) = (Q + PCM - PDN)\det(I - A_1z_1 - A_2z_2) + PNp(z_1, z_2)\frac{\det(I - A_1z_1 - A_2z_2)}{q(z_1, z_2)}. \quad (19)$$

By the coprimeness assumption on p and q, $\det(I - A_1z_1 - A_2z_2)$ factorizes through $q(z_1, z_2)$, so that we may express (19) as

$$\operatorname{adj}(I - A_1z_1 - A_2z_2) = (Q + PCM - PDN)r(z_1, z_2)q(z_1, z_2) + PNr(z_1, z_2)p(z_1, z_2) \quad (20)$$

where $r(z_1, z_2) = \det(I - A_1z_1 - A_2z_2)/q(z_1, z_2)$.

Assume now that both p and q vanish in $(z_1^0, z_2^0) \in \mathscr{P}_1$. Since the matrices P, Q, N, M have no singularities in (z_1^0, z_2^0), both sides of (20) are well defined in (z_1^0, z_2^0), and $p(z_1^0, z_2^0) = q(z_1^0, z_2^0) = 0$ implies

$$\operatorname{adj}(I - A_1z_1^0 - A_2z_2^0) = 0.$$

We, therefore, have that every minor of order $(n-1)$ in $(I - A_1z_1^0 - A_2z_2^0)$ is zero, i.e.

$$\operatorname{rank}[I - A_1z_1^0 - A_2z_2^0] \leq n - 2 \quad (21)$$

Then neither the $(n+1)\times n$ matrix (12), nor the $n \times (n+1)$ matrix (6) is full rank in (z_1^0, z_2^0), which contradicts the detectability and stabilizability hypotheses (see Theorems 1 and 2).

Conversely, the assumption $\mathscr{V}(p, q) \cap \mathscr{P}_1 = \varnothing$ implies that $W(z_1, z_2)$ admits a stabilizable and detectable realization, as we shall prove by direct construction of such a realization.

The proof is divided in 3 steps.

Step 1. The 2-D system Σ characterized by the following matrices:

$$A_1=\left[\begin{array}{ccccccccccc}
 & & & & & & & & & & \vdots\\
 & & & & & & & & & & 0\\
\vdots & & & & & & & & & & 0\\
 & & & & & & & & & & 0\\
 & & & & & & & & & & -q_{40}\\
 & 0 & 0 & 0 & 0 & & & & & & 0\\
 & 0 & 0 & 0 & 0 & & & & & & 0\\
 & 0 & 0 & 0 & 1 & & & & & & -q_{30}\\
 & & & & & 0 & 0 & 0 & & & 0\\
 & & & & & 0 & 0 & 1 & & & -q_{20}\\
 & & & & & & & & 0 & 1 & -q_{10}
\end{array}\right]$$

$$A_2=\left[\begin{array}{ccccccccccc}
 & & & & & & & & & & \vdots\\
 & & & & & & & & & & -q_{04}\\
 & & & & & & & & & & -q_{13}\\
\vdots & & & & & & & & & & -q_{22}\\
 & & & & & & & & & & -q_{31}\\
 & 1 & 0 & 0 & 0 & & & & & & -q_{03}\\
 & 0 & 1 & 0 & 0 & & & & & & -q_{12}\\
 & 0 & 0 & 1 & 0 & & & & & & -q_{21}\\
 & & & & & 1 & 0 & 0 & & & -q_{02}\\
 & & & & & 0 & 1 & 0 & & & -q_{11}\\
 & & & & & & & & 1 & 0 & -q_{01}
\end{array}\right]$$

$$B_1=\begin{bmatrix}\vdots\\0\\0\\0\\p_{40}\\0\\0\\p_{30}\\0\\p_{20}\\p_{10}\end{bmatrix}\qquad B_2=\begin{bmatrix}\vdots\\p_{04}\\p_{13}\\p_{22}\\p_{31}\\p_{03}\\p_{12}\\p_{21}\\p_{02}\\p_{11}\\p_{01}\end{bmatrix}$$

$$C=[0\ \ 0\ \ \cdots\ \ 0\ \ 1]\qquad D=[p_{00}] \tag{22}$$

is a realization of $W(z_1,z_2)$. In fact $\det(I-A_1z_1-A_2z_2)=q(z_1,z_2)$, and the last row of $\operatorname{adj}(I-A_1z_1-A_2z_2)$ has the following structure:

$$\left[\cdots z_2^3\ \ z_2^2z_1\ \ z_2z_1^2\ \ z_1^3\ \ z_2^2\ \ z_2z_1\ \ z_1^2\ \ z_2\ \ z_1\ \ 1\right].$$

Step 2. System (22) is detectable. In fact

$$\operatorname{rank}\left[\frac{C}{I-A_1z_1-A_2z_2}\right]=\operatorname{rank}\left[\frac{C}{I-(A_1+L_1C)z_1-(A_2+L_2C)z_2}\right] \tag{23}$$

for any pair of vectors L_1, L_2. Assuming

$$L_1^T=[\cdots\ \ 0\ \ 0\ \ d_{30}\ \ 0\ \ d_{20}\ \ d_{10}]$$
$$L_2^T=[\cdots\ \ d_{03}\ \ d_{12}\ \ d_{21}\ \ d_{02}\ \ d_{11}\ \ d_{01}]$$

we have $\det[I-(A_1+L_1C)z_1-(A_2+L_2C)z_2]=1$, so that (23) is full rank for any $(z_1,z_2)\in\mathbb{C}\times\mathbb{C}$.

Step 3. The matrix $[B_1z_1+B_2z_2|I-A_1z_1-A_2z_2]$ is full rank if and only if $(z_1,z_2)\notin\mathscr{V}(p,q)$. In fact, consider $(\bar z_1,\bar z_2)$ in $\mathbb{C}\times\mathbb{C}$ and let v be a real vector indexed in the following way:

$$v^T=[\cdots\ \ v_{02}\ \ v_{11}\ \ v_{20}\ \ v_{01}\ \ v_{10}\ \ v_{00}]$$

such that

$$v^T[B_1\bar z_1+B_2\bar z_2|I-A_1\bar z_1-A_2\bar z_2]=0$$

or equivalently,

$$v^T(B_1\bar z_1\bar z_1+B_2\bar z_2)=0 \tag{24}$$
$$v^T(I-A_1\bar z_1-A_2\bar z_2)=0. \tag{25}$$

By using (22), it is easy to check that (25) implies

$$v_{10}=\bar z_1v_{00}\qquad v_{01}=\bar z_2v_{00}$$
$$v_{20}=\bar z_1v_{10}=\bar z_1^2v_{00}$$
$$v_{11}=\bar z_2v_{10}=\bar z_2\bar z_1v_{00}$$
$$v_{02}=\bar z_2v_{01}=\bar z^2v_{00}$$
$$\cdots$$

Hence v turns out to have the following structure:

$$v^T=v_{00}\left[\cdots\ \ \bar z_2^2\ \ \bar z_2\bar z_1\ \ \bar z_1^2\ \ \bar z_2\ \ \bar z_1\ \ 1\right]. \tag{26}$$

Furthermore, substituting (26) in (24) and (25) and recalling (18) and (22), we have

$$v_{00}q(\bar z_1,\bar z_2)=0$$
$$v_{00}(p(\bar z_1,\bar z_2)-p_{00}q(\bar z_1,\bar z_2))=0 \tag{27}$$

Now, if $(\bar z_1,\bar z_2)\notin\mathscr{V}(p,q)$, (27) implies $v_{00}=0$. Then v is zero, which implies the full rank condition.

On the other hand, if $(\bar z_1,\bar z_2)\in\mathscr{V}(p,q)$, assume $v_{00}=1$, so that v is a nonzero vector that satisfies (24) and (25). This implies that the full rank condition is not satisfied.

We, therefore, have that $\mathscr{V}(p,q)\cap\mathscr{P}_1=\varnothing$ implies that the matrix (6) is full rank in $\mathscr{P}_1$ and hence that Σ is stabilizable.

Goodman gave some examples of BIBO stable and BIBO unstable 2-D transfer functions whose denominator and numerator have common zeros in the distinguished boundary T. In all cases, the existence of common zeros on T implies that detectable and stabilizable realizations do not exist. Then, by Theorem 3, any realization of those transfer functions is internally unstable, independently on their input–output stability behavior.

Moreover any BIBO stable transfer function $W(z_1,z_2)$ having nonessential singularities of the second kind on T does admit neither an internally stable realization nor a realization which can be stabilized by output feedback. The reason is that stabilizability and detectability are necessary conditions for output feedback stabilizability.

It is interesting to note that, for the case of one nonessential singularity at $(\pm1,\pm1)$, the (zero initial state) external stability can be robust with respect to parameter quantization error [13].

V. Conclusions

The Goodman example of a transfer function whose denominator vanishes in $\mathscr{P}_1$ was taken here as the starting point for clarifying the connections between BIBO and internal stability for 2-D filters and hence, in the light of state space realization properties, for pointing out the peculiar character of the Goodman's transfer function.

Dealing with external stability presumes that we have assumed zero initial conditions. However, nonzero initial conditions may arise from parasite excitations which cannot be incorporated into the input–output description. The internal description through the state variables permits us to incorporate the effect of this type of phenomena and the internal stability accounts for the corresponding dynamics.

The state model properties relevant in this matter are detectability and stabilizability which are characteristic of every internally stable realization. Using Bézout's identities (7) and (13) the results presented in the paper for the single-input single-output case can be extended in a straightforward manner to the multi-input multi-output case.

References

[1] D. Goodman, "Some stability properties of two-dimensional linear shift-invariant digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 201–208, 1977.

[2] E. Fornasini and G. Marchesini, "Stability analysis of 2D systems," *IEEE Trans. Circuits Syst.*, vol. CAS-27, pp. 1210–1217, 1980.

[3] ——, "Double-indexed dynamical systems: State models and structural properties." *Math. Syst. Theory*, vol. 12, pp. 59–72, 1978.

[4] M. Bisiacco and E. Fornasini, "State feedback stabilizability of 2D filters: A dead beat controller approach," presented at the *IEEE Int. Conf. on Computers, Systems and Signal processing*, Bangalore, India, Dec. 1984.

[5] M. Bisiacco, "State and output feedback stabilizability of 2D systems," *IEEE Circuits Syst.*, to be published.

[6] M. Bisiacco and E. Fornasini, "On the state reconstruction of 2D systems," presented at the *Int. Conf. TELECON* '84, Halkidiki, Greece, Aug. 1984.

[7] M. Bisiacco and G. Marchesini, "2D-systems: the structure of asymptotic observers," presented at the *IEEE Int. Conf. on Computers, Systems and Signal Processing*, Bangalore, India, Dec. 1984.

[8] M. Bisiacco, "On the algebraic theory of 2D state observers," *Syst. Contr. Lett.*, vol. 5, pp. 347–353, 1985.

[9] ——, "On the structure of 2D-observers," *IEEE Automat. Contr.*, to be published.

[10] K. Knopp, *Infinite Sequences and Series*. New York, Dover, 1956.

[11] E. Fornasini and G. Marchesini, "On the internal stability of two-dimensional filters," *IEEE Trans. Automat. Contr.*, vol. AC-24, Feb. 1979, pp. 129–130, and Aug. 1979.

[12] S. Kung, B. C. Lévy, M. Morf, and T. Kailath, "New results in 2D systems theory, Pt. II: 2D state space models-realizations and notions of controllability, observability and minimality," *Proc. IEEE*, vol. 65, pp. 861–872, 1977.

[13] R. K. Alexander, J. W. Woods, "2D digital filter stability in the presence of second kind nonessential singularities", *IEEE Trans. Circuits Syst.*, vol. CAS-29, pp. 604–612, Sept. 1982.