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FEEDBACK STABILIZATION OF 2D SYSTEMS

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The paper is concerned with the problem of constructing feedback compensators which ensure stability properties of a 2D system. The feedback invariant subset of zeros of the closed loop characteristic polynomial is specified and conditions are stated for the existence of stabilizing compensators.

Introduction

The first papers dealing with the class of 2D systems appeared in the literature nearly ten years ago[1-4] and one of the aspects that this theory afforded from the beginning was the stability analysis[5,6]. Nevertheless only very recently the stabilization problem using feedback compensators has been tackled[7-9]. Having in mind the 1D systems theory, static feedback compensators have been considered first but, differently from the 1D situation, they have shown poor efficiency[7].

The dynamics of 2D systems depends on two independent variables, so the idea of using compensators which are in some way static with respect to one variable and dynamic with respect to the other was also pursued. The efficiency of the compensators can then be considerably improved, but the system resulting by feedback connection does not keep any longer the quarter plane causality[7,8].

A deeper understanding of the techniques related to polynomial matrices in two indeterminates and their connections with system properties allowed to tackle the synthesis problem of dynamic feedback compensators that are realizable by 2D systems[9].

Our aim in this contribution is to discuss the stabilization problem by using state and output feedback compensators. We shall present conditions for the existence of such stabilizing compensators and the relative constructing techniques. As we shall see, particular interest deserves the set of zeros of the characteristic polynomial of the resulting feedback system that are invariant under feedback dynamic compensation.

Preliminary definitions and results

In this section we shall briefly review some definitions and results about polynomial and rational matrices in two indeterminates, that will be used in the following sections.

Also we will introduce the state equations of 2D systems and some properties related to the stabilization problem.

Consider a strictly proper transfer matrix $W(z_1,z_2) \in \mathbb{R}(z_1,z_2)^{p\times m}$ and let

$$N_R(z_1,z_2)D_R^{-1}(z_1,z_2) \qquad (1)$$

be a right matrix fraction description (MFD) of W. The polynomial matrices N_R and D_R are said to be right factor coprime (and consequently $N_R D_R^{-1}$ is a right coprime MFD) if for any polynomial matrix $X(z_1,z_2)$ such that

$$N_R(z_1,z_2) = \tilde{N}_R(z_1,z_2)X(z_1,z_2)$$

$$D_R(z_1,z_2) = \tilde{D}_R(z_1,z_2)X(z_1,z_2)$$

with $\tilde{N}_R$ and $\tilde{D}_R$ polynomial matrices, we have $\det X(z_1,z_2) = \text{const.}$

The following facts are equivalent[10]:

(i) N_R and D_R are right factor coprime

(ii) the Bézout equation

$$X_R D_R + Y_R N_R = I \qquad (2)$$

is solvable both with X_R and Y_R having elements in $\mathbb{R}(z_1)[z_2]$ and in $\mathbb{R}(z_2)[z_1]$.

(iii) the matrix

$$\begin{bmatrix} N_R(z_1,z_2) \\ D_R(z_1,z_2) \end{bmatrix} \qquad (3)$$

is full rank for any generic point of $\mathcal{Y}$, $\mathcal{Y}$ being the algebraic curve associated with the equation

$$\det D_R(z_1,z_2) = 0$$

over the complex field $\mathbb{C}$.

A rational matrix $W(z_1,z_2)$ is strictly proper if and only if, for any right coprime MFD $N_R D_R^{-1} = W$, we have

(i) $N_R(0,0) = 0$

(ii) $\det D_R(0,0) \neq 0$

We shall say that two polynomial matrices N_R and D_R are zero right coprime if

$$\text{rank} \begin{bmatrix} N_R(z_1,z_2) \\ D_R(z_1,z_2) \end{bmatrix} \qquad (3)$$

is full for any (z_1,z_2) in $\mathbb{C}x\mathbb{C}$. It can be shown that N_R and D_R are zero right coprime if and only if there exist X_R and Y_R with elements in $\mathbb{R}[z_1,z_2]$ such that the Bézout identity (2) holds.

Clearly all definitions and statements have analogs for left MFDs.

In general, factor coprimeness assumption does not imply zero coprimeness. In fact, even when N_R and D_R are factor coprime, the rank of (3) may be less than m on a finite subset of $\mathbb{C}x\mathbb{C}$. The elements of this set, called "rank singularities", are the intersections of the algebraic curves associated with the minors of maximal order of (3).

Since the set of rank singularities does not depend[11] on the (right or left) factor coprime MFD of W, in the sequel it will be denoted as $\mathcal{V}(W)$.

The state updating equations of a 2D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ are given by[6]

$$\begin{cases} x(h+1,k+1) = A_1 x(h,k+1) + A_2 x(h+1,k) \\ \qquad + B_1 u(h,k+1) + B_2 u(h+1,k) \\ y(h,k) = Cx(h,k) + Du(h,k) \end{cases} \qquad (4)$$

where $x(h,k) \in \mathbb{R}^n$ is the local state, $u(h,k) \in \mathbb{R}^m$, $y(h,k) \in \mathbb{R}^p$ are the input and output vectors at $(h,k) \in \mathbb{Z}x\mathbb{Z}$ and A_1, A_2, B_1, B_2, C, D are real matrices of suitable sizes

Denote by

$$\mathscr{X}_0 = \sum_{i=-\infty}^{+\infty} x(i,-i) z_1^i z_2^{-i}$$

the global state on the separation set

$$\mathscr{C}_0 = \{(i,j): i+j = 0\}$$

and by

$$X(z_1,z_2) = \sum_{i+j\geq 0} x(i,j) z_1^i z_2^j$$

$$U(z_1,z_2) = \sum_{i+j\geq 0} u(i,j) z_1^i z_2^j$$

$$Y(z_1,z_2) = \sum_{i+j\geq 0} y(i,j) z_1^i z_2^j$$

the state, input and output functions respectively.

Then from (4) one gets

$$X(z_1,z_2) = (I-A_1z_1-A_2z_2)^{-1}\left[\mathscr{X}_0 + (B_1z_1+B_2z_2)U(z_1,z_2)\right]$$

$$Y(z_1,z_2) = C\, X(z_1,z_2) + D\, U(z_1,z_2)$$

Assuming zero initial conditions $\mathscr{X}_0 = 0$, the input-output map of Σ is given by

$$Y(z_1,z_2) = W_\Sigma(z_1,z_2)U(z_1,z_2)$$

where

$$W_\Sigma(z_1,z_2) \stackrel{\Delta}{=} C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2) + D$$

is the transfer matrix of Σ.

Given a rational matrix $W(z_1,z_2)$, Σ is a realization of $W(z_1,z_2)$ if $W_\Sigma(z_1,z_2) = W(z_1,z_2)$.

A system Σ is <u>strictly causal</u> when D=0 and is (internally) stable with a degree of stability $\rho(\rho\geq 1)$ if for any set of bounded initial conditions $\mathscr{X}_0$, the free state evolution satisfies

$$\lim_{h+k\to\infty} x(h,k)\rho^{h+k} = 0$$

A simple characterization of the degree of stability of Σ is given by the following Theorem[9].

<u>Theorem 1</u>. A system Σ is stable with a degree of stability $\rho(\rho\geq 1)$ if and only if its characteristic polynomial

$$\Delta_\Sigma = \det(I-A_1z_1-A_2z_2) \qquad (6)$$

does not vanish on the closed polydisc

$$\mathscr{P}_\rho = \{(z_1,z_2): |z_1| \leq \rho,\ |z_2| \leq \rho\}. \qquad (7)$$

Let now introduce the following structural properties that will play an essential role in the construction of feedback compensators.

<u>Definition 1.</u> Σ is <u>ρ-stabilizable</u> if and only if the polynomial matrix

$$\begin{bmatrix} I-A_1z_1-A_2z_2 & B_1z_1+B_2z_2 \end{bmatrix} \qquad (8)$$

has full rank in the closed polydisc $\mathscr{P}_\rho$.

<u>Definition 2.</u> Σ is ρ-detectable if and only if the polynomial matrix

$$\begin{bmatrix} I-A_1z_1-A_2z_2 \\ C \end{bmatrix} \qquad (9)$$

has full rank in the closed polydisc $\mathscr{P}_\rho$.

In Definitions 1 and 2 ρ-stabilizability and ρ-detectability have been introduced using abstract arguments. Nevertheless they can be related to concrete properties of the dynamics of Σ. In fact, Σ is ρ-stabilizable if for any given initial local state $x(0,0)$, there exists an input u that produces a state evolution $x(h,k)$ whose norm is bounded by $c\,\rho^{-h-k}$, $c>0$, h,k non negative.

Similarly, Σ is ρ-detectable if the norm of every free state evolution $x(h,k)$, producing an identically zero output function, is bounded by $c\,\rho^{-h-k}$, $c>0$, h,k nonnegative.

The case $\rho=\infty$ in Definitions 1 and 2 deserves to be evidentiated. In fact if (8) has full rank for all $(z_1,z_2)\in\mathbb{C}\times\mathbb{C}$, then for any initial state $x(0,0)$ there exists an input u such that the corresponding state evolution goes to zero in a finite number of steps. In this case Σ will be said controllable. Similarly, if (9) has full rank for all $(z_1,z_2)\in\mathbb{C}\times\mathbb{C}$, every free state evolution that produces identically zero output, goes to zero in a finite number of steps. Then Σ will be said causally reconstructible[12].

As in 1D case these properties are strictly related to state estimation and control problems. Nevertheless the 1D and 2D theories differ to this respect by some important facts.

It is well known that any proper rational, matrix in one indeterminate $W(z)$ can be realized by a 1D system that is ρ- stabilizable and ρ-detectable for any $\rho\geq 1$. Moreover any minimal realization of $W(z)$ is jointly reachable and observable, and a fortiori ρ-stabilizable and ρ-detectable, for any $\rho\geq 1$.

The situation is quite different for 2D systems, where jointly ρ-stabilizable and ρ-detectable 2D realizations of a proper rational matrix $W(z_1,z_2)$ may not exist. This is a direct consequence of the following result[11]:

Theorem 2. Let $N_R D_R^{-1}$ be a right coprime MFD of a 2D proper rational matrix W and $\Sigma=(A_1,A_2,B_1,B_2,C)$ be a realization of W. Denote by $\mathcal{M}$ and $\mathcal{N}$ respectively the subsets of $\mathbb{C}\times\mathbb{C}$ where (8) and (9) do not have full rank. Then we have

a) $\det D_R$ divides $\det(I-A_1z_1-A_2z_2)$

b) $\mathcal{M}\cup\mathcal{N}\supseteq\mathcal{V}(W)$

Moreover, the following facts are equivalent:

(i) $\det D_R=\det(I-A_1z_1-A_2z_2)$

(ii) $\mathcal{M}\cup\mathcal{N}=\mathcal{V}(W)$

(iii) $(C,\ I-A_1z_1-A_2z_2)$ (10)

and

$(I-A_1z_1-A_2z_2,\ B_1z_1+B_2z_2)$ (11)

are factor coprime.

In view of the regulator design, it is essential to point out that given any 2D proper rational matrix W and any right coprime MFD $N_R D_R^{-1}=W$, it is possible to construct[11] realizations $\Sigma=(A_1,A_2,B_1,B_2,C,D)$ that satisfy

$$\det D_R=\det(I-A_1z_1-A_2z_2)$$

Because of Theorem 2, these realizations have the property that matrices (10) and (11) are factor coprime and the subsets of $\mathbb{C}\times\mathbb{C}$ where one of them does not have full rank are subsets of $\mathcal{V}(W)$.

Stabilization by state feedback and state estimation

The first problem we shall tackle in this section is the following: assuming that the state of a strictly proper system $\Sigma=(A_1,A_2,B_1,B_2,C)$ is available, design a 2D system $\bar{\Sigma}=(\bar{A}_1,\bar{A}_2,\bar{B}_1\bar{B}_2,\bar{C},\bar{D})$ so that the system of fig. 1, obtained by state feedback connecting Σ and $\bar{\Sigma}$, is internally stable.

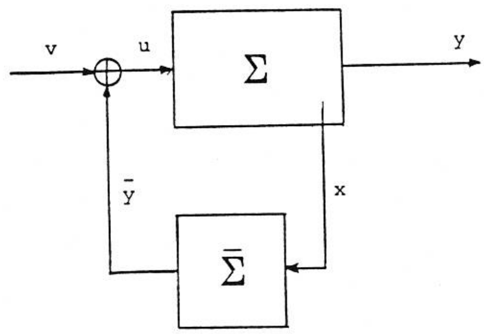

fig. 1

The solution is based on the following Theorem.

Theorem 3. Let $E(z_1,z_2)$ denote the greatest common left divisor (gcld) of $I-A_1z_1-A_2z_2$ and $B_1z_1+B_2z_2$. Let

$$[I-A_1z_1-A_2z_2\quad B_1z_1+B_2z_2]=E(z_1,z_2)\,[V(z_1,z_2)\quad T(z_1,z_2)]$$

and denote by $\mathcal{M}$ the finite subset of $\mathbb{C}\times\mathbb{C}$ where $[V\ T]$ is not full rank. Then the characteristic polynomial $\Delta_f(z_1,z_2)$ of the system of fig. 1 is multiple of $\det E(z_1,z_2)$ and vanishes on $\mathcal{M}$ for any choice of $\bar{\Sigma}$.

Moreover given any polynomial $q(z_1,z_2)$ vanishing on $\mathcal{M}$, there exist $\bar{\Sigma}$ and an integer $r>0$ such that

$$\Delta_f(z_1,z_2)=q^r(z_1,z_2)\det E(z_1,z_2) \quad (12)$$

proof. Let $R\,S^{-1}$ be a right MFD of the transfer matrix of $\bar{\Sigma}$ such that

$$\det S = \det(I-\bar{A}_1 z_1-\bar{A}_2 z_2)$$

Since

$$\Delta_f = \det E \det(VS+TR)$$

noting that det(VS+TR) vanishes on $\mathcal{M}$ proves the first part of the Theorem.

Conversely, consider a 2D polynomial q vanishing on $\mathcal{M}$. Denote by L_i, $i=1,2,\dots,\nu$ the submatrices of maximal order of $[V\ T]$ and by ℓ_i the corresponding minors. By Hilbert Nullstellensatz, there exists an integer t such that q^t belongs to the ideal generated by $\ell_1,\ell_2,\dots\ell_\nu$

$$q^t(z_1,z_2) \overset{\Delta}{=} \sum_{1}^{\nu} {}_i\, \ell_i(z_1,z_2)a_i(z_1,z_2)$$

We prove now that there exist matrices S and R such that

$$VS+TR = I_n q^t \tag{13}$$

In fact, we can write the submatrices L_i as

$$L_i = V K_i + T Y_i, \qquad i=1,2,\dots,\nu$$

so that:

$$\ell_i I_n = L_i \mathrm{Adj}\, L_i =$$
$$= V K_i \mathrm{Adj}\, L_i + T Y_i \mathrm{Adj}\, L_i$$

Then we have

$$q^t I_n = \sum_{1}^{\nu} {}_i\, \ell_i a_i I_n = V \sum_{1}^{\nu} {}_i\, K_i a_i \mathrm{Adj}\, L_i +$$
$$+ T \sum_{1}^{\nu} {}_i\, Y_i a_i \mathrm{Adj}\, L_i$$

so that (13) is solved assuming

$$S \overset{\Delta}{=} \sum_{1}^{\nu} {}_i\, K_i a_i \mathrm{adj}\, L_i$$

$$T \overset{\Delta}{=} \sum_{1}^{\nu} {}_i\, Y_i a_i \mathrm{adj}\, L_i$$

The integer r in (12) can be taken as $r = tn$.

Corollary. Σ is ρ-stabilizable if and only if det E does not vanish on $\mathcal{P}_\rho$ and $\mathcal{M}$ does not intersect $\mathcal{P}_\rho$.

When the state of Σ is not available, it can be estimated from the knowledge of system inputs and outputs, by constructing an asymptotic observer. The technique is based on the following theorem[12].

Theorem 4. Let $\hat{q}(z_1,z_2)$ be a polynomial devoid of zeros in $\mathcal{P}_\rho$ ($\rho \geq 1$) and $P(z_1,z_2)$ and $Q(z_1,z_2)$ be polynomial matrices such that

$$Q(I-A_1 z_1-A_2 z_2) + P\,C = I_n \hat{q} \tag{14}$$

Then any causally reconstructible realization of the matrix $[Q(B_1 z_1+B_2 z_2)\ \ P]\,(I_n \hat{q})^{-1}$ furnishes an asymptotic observer with a rate of convergence ρ.

The construction goes through the following essential lines. Denoting by $\hat{E}(z_1,z_2)$ the gcrd of $(I-A_1 z_1-A_2 z_2)$ and C, we have

$$\begin{bmatrix} I-A_1 z_1-A_2 z_2 \\ C \end{bmatrix} = \begin{bmatrix} \hat{V}(z_1,z_2) \\ \hat{T}(z_1,z_2) \end{bmatrix} \hat{E}(z_1,z_2)$$

for some polynomial matrices $\hat{V}(z_1,z_2)$ and $\hat{T}(z_1,z_2)$. Let now $\mathcal{N}$ be the (finite) subset of $\mathbb{C}\times\mathbb{C}$ where $[\hat{V}'\ \ \hat{T}']$ is not full rank. Using similar arguments as in the proof of Theorem 3, we obtain that given any polynomial q multiple of det $\hat{E}$ and vanishing on $\mathcal{N}$, there exists an integer r such that (14) can be solved with $q^r = \hat{q}$.

Hence, if $\mathcal{N}$ does not intersect $\mathcal{P}_\rho$ and det $\hat{E}$ does not vanish on $\mathcal{P}_\rho$, we can construct an asymptotic observer with a rate of convergence ρ.

Output feedback stabilization

Let $\hat{\Sigma} = (\hat{A}_1,\hat{A}_2,[\hat{B}_1\ \hat{L}_1],[\hat{B}_2\ \hat{L}_2],\hat{C})$ be a state observer of Σ with transfer matrix $[Q(B_1 z_1+B_2 z_2)\ \ P](I_n \hat{q})^{-1}$ and let $\bar{\Sigma} = (\bar{A}_1,\bar{A}_2,\bar{B}_1,\bar{B}_2,\bar{C})$ be a state feedback controller with transfer matrix $-NM^{-1}$.

The state updating equations of the feedback system with a combined observer-controller compensator are given by

$$\begin{bmatrix} x(h+1,k+1) \\ \hat{x}(h+1,k+1) \\ x(h+1,k+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1\bar{C} \\ L_1 C & \hat{A}_1 & \hat{B}_1\bar{C} \\ 0 & \bar{B}_1\hat{C} & \bar{A}_1 \end{bmatrix} \begin{bmatrix} x(h,k+1) \\ \hat{x}(h,k+1) \\ \bar{x}(h,k+1) \end{bmatrix} +$$

$$+ \begin{bmatrix} A_2 & 0 & B_2\bar{C} \\ L_2 C & \hat{A}_2 & \hat{B}_2\bar{C} \\ 0 & \bar{B}_2\hat{C} & \bar{A}_2 \end{bmatrix} \begin{bmatrix} x(h+1,k) \\ \hat{x}(h+1,k) \\ \bar{x}(h+1,k) \end{bmatrix} +$$

$$+ \begin{bmatrix} B_1 \\ \hat{B}_1 \\ 0 \end{bmatrix} v(h,k+1) + \begin{bmatrix} B_2 \\ \hat{B}_2 \\ 0 \end{bmatrix} v(h+1,k)$$

$$y(h,k) = [C\ 0\ 0] \begin{bmatrix} x(h,k) \\ \hat{x}(h,k) \\ \bar{x}(h,k) \end{bmatrix} \tag{15}$$

The characteristic polynomial of system (15) is then

$$\tilde{\Delta}(z_1,z_2) =$$

$$\det\begin{bmatrix} I-A_1z_1-A_2z_2 & 0 & -(B_1z_1+B_2z_2)\bar{C} \\ -(L_1z_1+L_2z_2)C & I-\hat{A}_1z_1-\hat{A}_2z_2 & -(\hat{B}_1z_1+\hat{B}_2z_2)\bar{C} \\ 0 & -(\bar{B}_1z_1+\bar{B}_2z_2)\hat{C} & I-\bar{A}_1z_1-\bar{A}_2z_2 \end{bmatrix} =$$

$$= \det(I-A_1z_1-A_2z_2)\det(I-\hat{A}_1z_1-\hat{A}_2z_2)\det\big[I-\bar{A}_1z_1-\bar{A}_2z_2 -$$

$$-(\bar{B}_1z_1+\bar{B}_2z_2)\hat{C}(I-\hat{A}_1z_1-\hat{A}_2z_2)^{-1}(\hat{L}_1z_1+\hat{L}_2z_2)C(I-A_1z_1-A_2z_2)^{-1}$$

$$(B_1z_1+B_2z_2)\bar{C} - (\bar{B}_1z_1+\bar{B}_2z_2)\hat{C}(I-\hat{A}_1z_1-\hat{A}_2z_2)^{-1}(\hat{B}_1z_1+\hat{B}_2z_2)\bar{C}\big] \quad (16)$$

Recalling the structure of the transfer matrix of the observer and the equation $P(z_1,z_2)C + Q(z_1,z_2)(I-A_1z_1-A_2z_2) = I\,\hat{q}(z_1,z_2)$, the last factor in the right-hand term of (15) becomes:

$$\det\big[I-\bar{A}_1z_1-\bar{A}_2z_2-(\bar{B}_1z_1+\bar{B}_2z_2)\hat{q}(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)\bar{C}\big]$$

$$=\det(I-\bar{A}_1z_1-\bar{A}_2z_2)\det\big[I+\hat{q}(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)NM^{-1}\big]$$

$$=\det(I-\bar{A}_1z_1-\bar{A}_2z_2)\det(I-A_1z_1-A_2z_2)\det\big[(I-A_1z_1-A_2z_2)M +$$

$$+ \hat{q}(B_1z_1+B_2z_2)N\big]\det M^{-1} .$$

So, the characteristic polynomial is given by

$$\det(I-\hat{A}_1z_1-\hat{A}_2z_2)\det(I-\bar{A}_1z_1-\bar{A}_2z_2)\det M^{-1}$$

$$\det\big[(I-A_1z_1-A_2z_2)M+\hat{q}(B_1z_1+B_2z_2)N\big] .$$

Now, if the state feedback controller is a realization of $-NM^{-1}$ with $\det M = \det(I-\bar{A}_1z_1-\bar{A}_2z_2)$, the zeros of the characteristic polynomial coincide with the zeros of

$$\det(I-\hat{A}_1z_1-\hat{A}_2z_2)\det\big[(I-A_1z_1-A_2z_2)M + \hat{q}(B_1z_1+B_2z_2)N\big] \quad (17)$$

For any M and N, the second factor in (17) vanishes on $\mathcal{M}$, on the zeros of det E and, in case, on further zeros common to $\hat{q}$ and $\det(I-A_1z_1-A_2z_2)$.

Now, since we can realize observers such that the zeros of $\det(I-\hat{A}_1z_1-\hat{A}_2z_2)$ coincide with those of $\hat{q}$, the variety associated with the characteristic polynomial of the whole system $\tilde{\Delta}(z_1,z_2)$ can be arbitrarily assi - gned, provided it contains the varieties associated with det E, det $\hat{E}$ and the sets $\mathcal{M}$ and $\mathcal{N}$. In particular, if C and $(I-A_1z_1-A_2z_2)$ are right factor coprime and $(I-A_1z_1-A_2z_2)$ and $B_1z_1+B_2z_2$ are left factor coprime, the variety of $\tilde{\Delta}(z_1,z_2)$ is only constrained to contain $\mathcal{M}$ and $\mathcal{N}$.

These constraints cannot be further weakened. This descends from the properties of the characteristic polynomials of output feedback systems and from the fact that $\tilde{\Delta}(z_1,z_2)$ coincides with the characteristic polynomial of the system we get by feeding the observer with the pair $(\bar{y},y)$ rather than (u,y). The system we obtain in this way has the structure of a system with dynamic output feedback, whose characteristic polynomial [11] vanishes on $\mathcal{M}$, $\mathcal{N}$ and on the subsets $\det E = 0$ and $\det \hat{E} = 0$.

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