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CONTROLLER DESIGN FOR 2D SYSTEMS

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In this paper the problem of constructing output feedback compensators for 2D systems is considered. The bounds on the algebraic variety associated to the closed loop characteristic polynomial are evidentiated and an explicit technique for synthesizing the compensator is given.

INTRODUCTION

Very recently some papers dealing with 2D systems have been concerned with the problem of constructing dynamic compensators having observer-controller structure [1,2].

The principal tool considered for solving this problem is constituted by 2D analogs of the polynomial matrices involved in the PBH tests for controllability and reconstructibility. In particular, the rank analysis of these matrices provides the bounds on the performances of asymptotic observers and state feedback controllers.

The aim of this paper is to analyze the 2D output feedback scheme using the polynomial matrix approach and to investigate what is the structure of the characteristic polynomial as the output feedback compensator is varied. Moreover we shall introduce a general procedure for synthesizing the compensator.

In this framework the 2D analogs of the polynomial matrices involved in the PBH tests play a fundamental role, in the sense that the unique constraint on the variety of the characteristic polynomial is that it has to include the set of points where at least one of these matrices does not have full rank.

Obviously, this set depends on the system that realizes the transfer matrix. However, it contains the so called "rank singularities", that are characteristic of the transfer matrix only, and are invariant both with respect to the realization and to the output feedback compensation policy.

SOME PROPERTIES OF 2D TRANSFER MATRICES

Many papers appeared in the recent literature, concerning with the theory of matrix fractions in two indeterminates [see for instance 3]. In the following we shall briefly review some well known fundamental properties of this class of matrices with the purpose of stating notations and terminology. We shall rather dwell upon the analysis of the so called "rank singularities" that provide the extension to the matrix case of the non essential singularities of the second kind for rational functions in two variables.

As we shall see, rank singularities constitute the fundamental constraint we are faced by, when we use output feedback techniques for modifying the dynamics of 2D systems.

Consider a strictly proper 2D transfer matrix $W(z_1,z_2) \in \mathbb{R}(z_1,z_2)^{p \times m}$ and let

$$N_R(z_1,z_2)D_R^{-1}(z_1,z_2) = W(z_1,z_2) \tag{1}$$

be a right matrix fraction description (MFD) of W.

The polynomial matrices N_R and D_R (*) are said <u>right factor coprime</u> (and, consequently, $N_R D_R^{-1}$ is a right factor coprime MFD) if for any polynomial matrix X such that

$$N_R = \tilde{N}_R X \ , \qquad D_R = \tilde{D}_R X$$

with $\tilde{N}_R$ and $\tilde{D}_R$ polynomial matrices, we have $\det X = \text{const.}$

The following theorem provides an extension of coprimeness condition to polynomial matrices in two indeterminates.

<u>Theorem 1 [3]</u> Let $N_R \in \mathbb{R}[z_1,z_2]^{p \times m}$ and $D_R \in \mathbb{R}[z_1,z_2]^{m \times m}$ and consider the matrix fraction $N_R D_R^{-1}$. Then the following facts are equivalent

(i) N_R and D_R are right factor coprime;

(ii) the Bézout equation

$$X_R D_R + Y_R N_R = I \tag{2}$$

is solvable both with X_R and Y_R having elements in $\mathbb{R}(z_1)[z_2]$ and in $\mathbb{R}(z_2)[z_1]$;

(iii) N_R and D_R are 1D right coprime both on $\mathbb{R}(z_1)[z_2]$ and $\mathbb{R}(z_2)[z_1]$;

(iv)
$$[N_R' \quad D_R'] \tag{3}$$

is full rank for any generic point of $\mathcal{V}$, $\mathcal{V}$ being the algebraic curve associated with the equation $\det D_R = 0$ over the complex field $\mathbb{C}$.

Assuming $W(z_1,z_2)$ is strictly proper is equivalent to require that every right coprime MFD $N_R D_R^{-1} = W$ satisfies

(i) $N_R(0,0) = 0$

(ii) $D_R(0,0)$ is invertible, so we can assume $D_R(0,0) = I$.

Clearly, all definitions and statements have analogs for left MFDs.

We shall say that N_R and D_R are <u>zero right coprime</u> if (3) has full rank for any $(z_1,z_2) \in \mathbb{C} \times \mathbb{C}$. It can be shown that N_R and D_R are zero right coprime if and only if there exist polynomial matrices X_R and Y_R such that the Bézout identity (2) holds.

(*) In the sequel the arguments z_1 and z_2 of polynomial and rational matrices in two variables will be omitted, when unnecessary.

In general, factor coprimeness assumption does not imply zero coprimeness. In fact, even when N_R and D_R are factor coprime, the rank of (3) may be less than m on a finite subset of $\mathbb{C} \times \mathbb{C}$. The elements of this set, called "rank singularities", are the intersections of the algebraic curves associated with the minors of maximal order of (3).

As a consequence of the following Theorem, the set of rank singularities does not depend on the factor coprime MFD of W and for this reason it will be denoted by $\mathscr{V}(W)$.

Theorem 2. Let $N_R D_R^{-1}$ and $D_L^{-1} N_L$ be right and left coprime MFDs of W. If $\mathscr{V}_R(W)$ and $\mathscr{V}_L(W)$ are the (finite) subsets of $\mathbb{C} \times \mathbb{C}$ where $[N_R' \quad D_D']$ and $[N_L \quad D_L]$ are not full rank, then $\mathscr{V}_R(W) = \mathscr{V}_L(W)$.

proof. By contradiction. Assume $(z_{10}, z_{20}) \in \mathscr{V}_L(W)$ and $(z_{10}, z_{20}) \notin \mathscr{V}_R(W)$. Consider the line

$$\begin{aligned} z_1 &= z_{10} + \alpha t \\ z_2 &= z_{20} + \beta t \end{aligned} \tag{4}$$

with α and β such that (4) does not intersect $\mathscr{V}_R(W)$ and let $N_R(t) D_R^{-1}(t)$ and $D_L^{-1}(t) N_L(t)$ denote the restrictions of $N_R D_R^{-1}$ and $D_L^{-1} N_L$ to the line (4).

$N_R(t) D_R^{-1}(t)$ is a 1D coprime MFD since $[N_R'(t) \quad D_R'(t)]$ is full rank for every complex t. On the contrary, $[N_L(0) \quad D_L(0)]$ is not full rank, so $D_L^{-1}(t) N_L(t)$ is not a 1D coprime MFD and

$$\begin{aligned} N_L(t) &= \tilde{N}_L(t) R(t) \\ D_L(t) &= \tilde{D}_L(t) R(t) \end{aligned}$$

where $\tilde{N}_L$ and $\tilde{D}_L$ are polynomial matrices, R(t) is not unimodular and $\tilde{D}_L^{-1}(t) \tilde{N}_L(t)$ is a left coprime MFD.

Then

$$\det \tilde{D}_L(t) = \det D_R(t) \tag{5}$$

On the other side, because of 2D coprimeness, we have

$$\det D_L(z_1, z_2) = \det D_R(z_1, z_2)$$

hence

$$\det D_L(t) = \det D_R(t) \tag{6}$$

Since

$$\det D_L(t) = \det \tilde{D}_L(t) \det R(t)$$

(5) and (6) lead to a contradiction.

As a consequence of Theorem 2 if $N_R D_R^{-1}$ and $\bar{N}_R \bar{D}_R^{-1}$ are two right coprime MFDs and $D_L^{-1} N_L$ is a left coprime MFD, denoting by $\mathscr{V}_R(W)$, $\bar{\mathscr{V}}_R(W)$ and $\mathscr{V}_L(W)$ their rank singularity sets, we have

$$\mathcal{V}_R(W) = \mathcal{V}_L(W) = \bar{\mathcal{V}}_R(W)$$

that proves the invariance of the rank singularities with respect to coprime MFDs.

CONNECTIONS BETWEEN $\mathcal{V}(W)$ AND THE STRUCTURE OF REALIZATIONS OF $N_R D_R^{-1}$

Consider a 2D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ given by

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) + B_1 u(h,k+1) + B_2 u(h+1,k) \\ y(h,k) &= C\, x(h,k) + D\, u(h,k) \end{aligned} \tag{7}$$

where the local state x is an n-dimensional vector over the real field $\mathbb{R}$, input and output functions take values in $\mathbb{R}^m$ and $\mathbb{R}^p$, A_1, A_2, B_1, B_2, C and D are matrices of suitable dimensions with entries in $\mathbb{R}$ [4].

Denote by

$$\mathcal{X}_0 = \sum_{i=-\infty}^{+\infty} x(i,-i) z_1^i z_2^{-i}$$

the global state on the separation set

$$\mathcal{C}_0 = \{(i,j): i+j = 0\}$$

and by

$$X(z_1,z_2) = \sum_{i+j\geq 0} x(i,j) z_1^i z_2^j, \quad U(z_1,z_2) = \sum_{i+j\geq 0} u(i,j) z_1^i z_2^j$$

$$Y(z_1,z_2) = \sum_{i+j\geq 0} y(i,j) z_1^i z_2^j$$

the state, input and output functions respectively.

Then from (7) one gets

$$(I - A_1 z_1 - A_2 z_2) X(z_1,z_2) - (B_1 z_1 + B_2 z_2) U(z_1,z_2) = \mathcal{X}_0 \tag{8}$$

and

$$Y(z_1,z_2) = C\, X(z_1,z_2) + D\, U(z_1,z_2). \tag{9}$$

Assuming zero initial conditions $\mathcal{X}_0 = 0$, the input-output map is given by

$$Y(z_1,z_2) = W_\Sigma(z_1,z_2) U(z_1,z_2)$$

where

$$C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D \triangleq W_\Sigma(z_1,z_2) \tag{10}$$

is the transfer matrix of Σ.

A system Σ is strictly causal when D=0 and is finite-memory if for any set of initial conditions $\mathcal{X}_0$, the free state evolution goes to zero in a finite number of steps.

The following matrices

$$\begin{bmatrix} I-A_1z_1-A_2z_2 & B_1z_1+B_2z_2 \end{bmatrix} \tag{11}$$

and

$$\begin{bmatrix} I-A_1z_1-A_2z_2 \\ C \end{bmatrix} \tag{12}$$

are of paramount importance in analyzing the internal structure of Σ and in designing feedback control policies.

Matrices (11) and (12) are related to the notions of controllability and causal reconstructibility [1,2], in the sense that Σ is controllable if and only if (11) is full rank for every (z_1,z_2) and is causally reconstructible if and only if (12) is full rank for every (z_1,z_2).

The role played by the corresponding 1D matrices

$$\begin{bmatrix} I-Az & Bz \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I-Az \\ C \end{bmatrix} \tag{13}$$

in defining how the dynamics of output feedback connected systems can be modified, is well known. Also, we can always realize a proper transfer matrix W(z) so that (13) have both full rank for preassigned values of z, and in particular for every z. In the latter case, the realization is controllable and reconstructible and dynamic output feedback allows to obtain arbitrary characteristic polynomials.

The situation is quite different in the 2D case, since there exist transfer matrices $W(z_1,z_2)$ that cannot be realized by 2D systems having (11) and (12) of full rank for every (z_1,z_2).

This is a direct consequence of the following Theorem.

<u>Theorem 3.</u> Consider a strictly proper transfer matrix W and let $N_RD_R^{-1}$ be a right coprime MFD of W. If $\Sigma=(A_1,A_2,B_1,B_2,C)$ is a realization of $N_RD_R^{-1}$ and $(z_{10},z_{20})\in\mathcal{V}(W)$, then one of the matrices (11) and (12) has not full rank for (z_{10},z_{20}).

proof. Given $(z_{10},z_{20})\in\mathcal{V}(W)$, consider a straight line

$$z_1 = z_{10}+\alpha t \;, \quad z_2 = z_{20}+\beta t \tag{14}$$

where α and β are chosen so that $\det(I-A_1z_1-A_2z_2)$ is not identically zero on this line.

Substituting (14) in the expression of the transfer matrix of Σ, we obtain

$$C(I-(A_1z_{10}+A_2z_{20})-t(\alpha A_1+\beta A_2))^{-1}(B_1z_{10}+B_2z_{20}+t(\alpha B_1+\beta B_2)) \\ = C\,V(t)^{-1}U(t) \overset{\Delta}{=} W_\Sigma(t) \tag{15}$$

where

$$V(t) = I-(A_1z_{10}+A_2z_{20}) - t(\alpha A_1+\beta A_2)$$

$$U(t) = B_1z_{10} + B_2z_{20} + t(\alpha B_1+\beta B_2)$$

Now, letting $R(t)$ be the g c r d of C and $V(t)$, we have

$$C = \hat{C}(t)R(t) \\ V(t) = \tilde{V}(t)R(t) \tag{16}$$

and letting $L(t)$ be the g c l d of $\tilde{V}(t)$ and $U(t)$, we have

$$\tilde{V}(t) = L(t)\hat{V}(t) \\ U(t) = L(t)\hat{U}(t) \tag{17}$$

Using (15), (16) and (17) we obtain

$$W_\Sigma(t) = \hat{C}(t)\hat{V}^{-1}(t)\hat{U}(t)$$

where $\hat{C}(t)$ and $\hat{V}(t)$ are right coprime, $\hat{V}(t)$ and $\hat{U}(t)$ are left coprime and

$$V(t) = L(t)\hat{V}(t)R(t) \tag{18}$$

Considering $N_RD_R^{-1}$ on (14) we obtain

$$N_R(z_{10}+\alpha t,\ z_{20}+\beta t)D_R^{-1}(z_{10}+\alpha t,\ z_{20}+\beta t) = N_R(t)D_R^{-1}(t)$$

where

$$N_R(t) \overset{\Delta}{=} N_R(z_{10}+\alpha t,\ z_{20}+\beta t)$$

$$D_R(t) \overset{\Delta}{=} D_R(z_{10}+\alpha t,\ z_{20}+\beta t)$$

If $S(t)$ is the g c r d of $N_R(t)$ and $D_R(t)$, then

$$N_R(t) = \hat{N}_R(t)S(t) \\ D_R(t) = \hat{D}_R(t)S(t) \tag{19}$$

Using the identity

$$\hat{N}_R(t)\hat{D}_R^{-1}(t) = \hat{C}(t)\hat{V}^{-1}(t)\hat{U}(t)$$

and recalling the coprimeness properties involved, we have

$$\det \hat{D}_R(t) = \det \hat{V}(t) \tag{20}$$

Recalling that

$$N_R D_R^{-1} = C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)$$

and that N_R and D_R are factor coprime, there exists a polynomial $h(z_1,z_2)$ such that:

$$\det(I-A_1z_1-A_2z_2) = h(z_1,z_2)\det D_R$$

which gives

$$\det V(t) = h(t)\det D_R(t) \tag{21}$$

Because of (18) and (20) we obtain

$$\begin{aligned}\det V(t) &= \det L(t)\ \det \hat{V}(t)\ \det R(t) = \\ &= \det L(t)\ \det \hat{D}_R(t)\ \det R(t)\end{aligned}$$

Now, recalling (19) and (21) we have

$$h(t)\ \det S(t)\ \det \hat{D}(t) = \det L(t)\ \det \hat{D}_R(t)\ \det R(t)$$

hence

$$h(t)\ \det S(t) = \det L(t)\ \det R(t) \tag{22}$$

Assume now, by contradiction, that both (11) and (12) have full rank for (z_{10},z_{20}). Then $[V'(0)\ \ C']$ and $[V(0)\ \ U(0)]$ have full rank. Consequently, since the product matrix

$$\begin{bmatrix} \tilde{V}(0) \\ \hat{C}(0) \end{bmatrix} R(0) = \begin{bmatrix} V(0) \\ C \end{bmatrix}$$

has full rank, R(0) has full rank too. Similarly, since

$$\begin{aligned} L(0)[\hat{V}(0)R(0)\ \ \hat{U}(0)] &= [\tilde{V}(0)R(0)\ \ U(0)] = \\ &= [V(0)\ \ U(0)] \end{aligned}$$

has full rank, so L(0) has.

On the other side, since the matrix on the right hand side of the following identity

$$\begin{bmatrix} \hat{N}_R(0) \\ \hat{D}_R(0) \end{bmatrix} S(0) = \begin{bmatrix} N_R(0) \\ D_R(0) \end{bmatrix}$$

has not full rank and $[\hat{N}_R(0)\ \ \hat{D}_R(0)]$ has full rank because of the coprimeness of $\hat{N}_R(t)$ and $\hat{D}_R(t)$, it follows that S(0) is singular.

Now, letting t=0 in (22) we have

$$0 = h(0) \det S(0) = \det L(0) \det R(0)$$

that is a contradiction since $L(0)$ and $R(0)$ are non singular.

We shall now characterize the realizations of the matrix fraction $N_R D_R^{-1}$ having the property that the subset of $\mathbb{C} \times \mathbb{C}$ where at least one of the matrices (11) and (12) has not full rank, coincides with $\mathscr{V}(W)$.

The following Theorem provides the answer to this problem.

Theorem 4. Let $N_R D_R^{-1}$ be a right coprime, proper matrix fraction and let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of it. Then

$$\det D_R \mid \det(I - A_1 z_1 - A_2 z_2) \tag{23}$$

and the following facts are equivalent:

(i) $\det(I - A_1 z_1 - A_2 z_2) = \det D_R$;

(ii) $C(I - A_1 z_1 - A_2 z_2)^{-1}$ and $(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2)$ are respectively right coprime and left coprime matrix fractions;

(iii) the subset of $\mathbb{C} \times \mathbb{C}$ where at least one of the matrices (11) and (12) has not full rank, coincides with $\mathscr{V}(W)$.

proof. (ii) $\rightarrow$ (i) recalling Theorem 5.5 in [3], if in the following equality

$$C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) = N_R D_R^{-1}$$

$(C, (I - A_1 z_1 - A_2 z_2))$ are right coprime and $((I - A_1 z_1 - A_2 z_2), (B_1 z_1 + B_2 z_2))$ are left coprime, then $\det(I - A_1 z_1 - A_2 z_2) = \det D_R$.

In general, if only (N_R, D_R) are right coprime, the matrix product $C(I - A_1 z_1 - A_2 z_2)^{-1}$ $(B_1 z_1 + B_2 z_2)$ can be put in the form $T V^{-1} U$, where (T, V) are right coprime, (V, U) are left coprime and $\det V \mid \det(I - A_1 z_1 - A_2 z_2)$. Now using Theorem 5.5 in [3], we have

$$\det D_R \mid \det(I - A_1 z_1 - A_2 z_2)$$

(i) $\rightarrow$ (iii) denote by $\mathscr{M}$ and $\mathscr{N}$ the sets of points where (11) and (12) have not full rank. Then by Theorem 2 we obtain

$$\mathscr{S} = \mathscr{M} \cup \mathscr{N} \supseteq \mathscr{V}(W).$$

To prove the converse inclusion, assume, by contradiction, $(z_{10}, z_{11}) \in \mathscr{M}$ and $(z_{10}, z_{20}) \notin \mathscr{V}(W)$ and consider a straight line

$$z_1 \triangleq z_{10} + \alpha t , \quad z_2 = z_{20} + \beta t \tag{24}$$

with α and β such that on that line, $\det D_R$ is not identically zero and $[N_R' \ \ D_R']$ has full rank. Then the restriction to (24) of $N_R D_R^{-1}$, given by

$$N_R(z_{10} + \alpha t,\ z_{20} + \beta t) D_R^{-1}(z_{10} + \alpha t,\ z_{20} + \beta t) \triangleq N_R(t) D_R^{-1}(t)$$

is a 1D right coprime matrix fraction. Also, in the expression of the transfer matrix restricted to (24), given by

$$C(I-A_1(z_{10}+\alpha t) - A_2(z_{20}+\beta t))^{-1}(B_1(z_{10}+\alpha t) + B_2(z_{20}+\beta t)) =$$
$$= C\, V^{-1}(t)U(t) \ ,$$

the matrices $V(t)$ and $U(t)$ are not right coprime since by assumption $[V(0) \quad U(0)]$ does not have full rank.

Consequently $\det D_R(t)$ is proper divisor of $\det V(t)$, that contradicts the assumption

$$\det D_R(z_1,z_2) = \det(I-A_1z_1-A_2z_2)$$

(iii) $\rightarrow$ (ii) since $\mathcal{S} = \mathcal{V}(W)$, $\mathcal{S}$ is a finite cardinality set. This implies that $\mathcal{M}$ and $\mathcal{N}$ are also finite cardinality sets and hence $(C,(I-A_1z_1-A_2z_2))$ and $((I-A_1z_1-A_2z_2), (B_1z_1+B_2z_2))$ are factor coprime.

At this point a natural question arises as whether there exist realizations having the property that $\mathcal{S}$ coincides with $\mathcal{V}(W)$. As we shall see, such realizations do exist and the proof of Theorem 5 will also provide an explicit construction for them. In fact it will be shown how to obtain realizations where at least one of matrices (11) and (12) has full rank for every (z_1,z_2) (so Σ turns out to be controllable and/or causally reconstructible) and $\mathcal{V}(W)$ coincides with the subset where the other matrix does not have full rank.

Theorem 5. Let $N_RD_R^{-1}$ be a strictly proper, right coprime MFD. Then there exists a 2D system $\Sigma = (A_1,A_2,B_1,B_2,C)$ that realizes $N_RD_R^{-1}$, i.e. $W_\Sigma = N_RD_R^{-1}$, such that

(i) $\quad [I-A_1z_1-A_2z_2 \quad B_1z_1+B_2z_2]$ has full rank on $\mathbb{C}\times\mathbb{C}$

(ii) $\quad \begin{bmatrix} I-A_1z_1-A_2z_2 \\ C \end{bmatrix}$ has full rank on $\mathbb{C}\times\mathbb{C} \setminus \mathcal{V}(W)$.

proof. there is no restriction in assuming $D_R(0,0) = I_m$. Denote by k_i, $i = 1,2,\ldots$ $\ldots,m$ the column degree of the i-th column of

$$\begin{bmatrix} N_R \\ D_R \end{bmatrix},$$

that is the degree of the polynomial of maximal degree in the i-th column.

We can write

$$D_R = I_m - D_{HT}\Psi$$
$$N_R = N_{HT}\Psi$$

where

$$
\Psi' = \left[\begin{array}{c|c|c}
z_2^{k_1} \quad z_1 z_2^{k_1-1} \quad \dots \quad z_1^{k_1} \dots z_2 \quad z_1 & 0 \dots\dots & 0 \dots\dots\dots\dots\dots\dots \\
\hline
 & \ddots & \\
0 \dots\dots\dots\dots\dots\dots & \dots\dots 0 & z_2^{k_m} \quad z_1 z_2^{k_m-1} \quad \dots \quad z_1^{k_m} \dots z_2 \quad z_1
\end{array}\right]
$$

$$
D_{HT} = \begin{bmatrix} D_{11} & \dots\dots & D_{1m} \\ & & \\ D_{m1} & \dots\dots & D_{mm} \end{bmatrix}, \qquad
N_{HT} = \begin{bmatrix} N_{11} & \dots\dots & N_{1m} \\ & & \\ N_{p1} & \dots\dots & N_{pm} \end{bmatrix}
$$

and D_{ij} and N_{ij} are row-vectors whose elements are the coefficients of the (i,j) indexed polynomial in $-D_R + I_m$ and N_R.

Introduce now the following matrices

$$
A_{10}^{(h)} = \left[\begin{array}{cccc}
M_h & & & \\
 & M_{h-1} & & \\
 & & \ddots & \\
 & & & M_2 \\
\hline
0 & \dots\dots & \dots\dots & 0 \\
0 & \dots\dots & \dots\dots & 0
\end{array}\right], \qquad
B_1^{(h)} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ 1 \end{array}\right]
$$

$$
A_{20}^{(h)} = \left[\begin{array}{cccc}
N_h & & & \\
 & N_{h-1} & & \\
 & & \ddots & \\
 & & & N_2 \\
\hline
0 & \dots\dots & \dots\dots & 0 \\
0 & \dots\dots & \dots\dots & 0
\end{array}\right], \qquad
B_2^{(h)} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \end{array}\right]
$$

with

$$
M_j = \begin{bmatrix} 0 & 0 \dots\dots 0 \\ & I_j \end{bmatrix}
$$

$$
N_j = \begin{bmatrix} 1 & 0 \dots\dots 0 \\ & 0_j \end{bmatrix}
$$

and define

$$A_{10} = \text{diag}\,[A_{10}^{(k_1)} \quad A_{10}^{(k_2)} \quad .... \quad A_{10}^{(k_m)}]$$

$$A_{20} = \text{diag}\,[A_{20}^{(k_1)} \quad A_{20}^{(k_2)} \quad .... \quad A_{20}^{(k_m)}]$$

$$B_1 = \text{diag}\,[B_1^{(k_1)} \quad B_2^{(k_2)} \quad .... \quad B_2^{(k_m)}]$$

$$B_2 = \text{diag}\,[B_2^{(k_1)} \quad B_2^{(k_2)} \quad .... \quad B_2^{(k_m)}]$$

It is a matter of simple computation to show that

$$(I-A_{10}z_1-A_{20}z_2)^{-1}(B_1z_1+B_2z_2) = \Psi$$

Assuming now

$$\mathscr{A}_o = A_{10}z_1 + A_{20}z_2, \qquad \mathscr{B} = B_1z_1 + B_2z_2$$

$$\mathscr{A} = \mathscr{A}_o + \mathscr{B}\, D_{HT}$$

we have

$$(I-\mathscr{A})^{-1}\mathscr{B} = (I-\mathscr{A}_o-\mathscr{B}\, D_{HT})^{-1}\mathscr{B} =$$

$$= ((I-\mathscr{B}\, D_{HT}(I-\mathscr{A}_o)^{-1})(I-\mathscr{A}_o))^{-1}\mathscr{B} =$$

$$= (I-\mathscr{A}_o)^{-1}(I-\mathscr{B}\, D_{HT}(I-\mathscr{A}_o)^{-1})^{-1}\mathscr{B} =$$

$$= (I-\mathscr{A}_o)^{-1}\mathscr{B}\,(I-D_{HT}(I-\mathscr{A}_o)^{-1}\mathscr{B})^{-1} =$$

$$= \Psi(z_1,z_2)(I-D_{HT}\Psi)^{-1} = \Psi\, D_R^{-1}$$

Since

$$N_RD_R^{-1} = N_{HT}\,\Psi\, D_R^{-1} = N_{HT}(I-\mathscr{A})^{-1}\mathscr{B} =$$

$$= N_{HT}\,(I-(A_{10}+B_1D_{HT})z_1 - (A_{20}+B_2D_{HT})z_2)^{-1}(B_1z_1+B_2z_2)$$

the matrices $A_1 = A_{10}+B_1D_{HT}$, $A_2 = A_{20}+B_2D_{HT}$, $B_1, B_2, C = N_{HT}$ furnish a realization of $N_RD_R^{-1}$.

The realization obtained in this way is controllable. In fact

$$\text{rank}[I-\mathscr{A} \mid \mathscr{B}] = \text{rank}[I-\mathscr{A}_o-\mathscr{B}D_{HT} \mid \mathscr{B}] =$$

$$= \text{rank}[I-\mathscr{A}_o \mid \mathscr{B}] =$$

$$= \text{rank}[\text{diag}\{I-A_{10}^{(k_i)}z_1-A_{20}^{(k_i)}z_2,\ i=1...m\} \mid \text{diag}\{B_1^{(k_i)}z_1+B_2^{(k_i)}z_2,\ i=1...m\}]$$

is full for every (z_1,z_2) since

$$[I-A_{10}^{(k_i)}z_1-A_{20}^{(k_i)}z_2 \mid B_1^{(k_i)}z_1+B_2^{(k_i)}z_2], \quad i=1,2,\dots,m$$

has full rank for every (z_1,z_2).

We have still to prove that $\mathscr{V}(W)$ is the set of points where

$$\begin{bmatrix} I-A_1z_1-A_2z_2 \\ C \end{bmatrix}$$

does not have full rank. Because of Theorem 4 it is enough to prove that $\det(I-A_1z_1-A_2z_2)=\det D_R$. This follows from the identities

$$\det D_R = \det(I-D_{HT}\Psi) = \det(I-\Psi D_{HT})$$

and

$$\det(I-\mathscr{A}) = \det(I-\mathscr{A}_0-\mathscr{B}D_{HT}) =$$

$$= \det(I-\mathscr{A}_0)\det[I-(I-\mathscr{A}_0)^{-1}\mathscr{B}D_{HT}] =$$

$$= \det(I-\Psi D_{HT})$$

CONTROLLER DESIGN

Consider the 2D system of fig. 1.

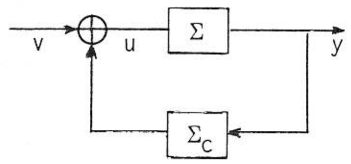

fig. 1

Let $N_RD_R^{-1}$ be a right coprime MFD of the strictly proper transfer matrix of $\Sigma=(A_1,A_2,B_1,B_2,C)$. Hence, by Theorem 4,

$$\det(I-A_1z_1-A_2z_2) = h(z_1,z_2)\det D_R \tag{25}$$

Also, let $R_L^{-1}S_L$ be a left MFD of the proper transfer matrix of $\Sigma_c=(F_1,F_2,G_1,G_2,H,J)$ with

$$\det(I-F_1z_1-F_2z_2) = \det R_L$$

Let x and x_c denote the local state vectors of Σ and Σ_c and assume $\bar{x}' = [x' \ x_c']$. It is easy to check that the dynamics of the local state of the feedback system is given by

$$\bar{x}(h+1,k+1) = \bar{A}_1\bar{x}(h,k+1)+\bar{A}_2\bar{x}(h+1,k)+\bar{B}_1\,v\,(h,k+1)+\bar{B}_2\,v\,(h+1,k)$$

where

$$\bar{A}_1 = \begin{bmatrix} A_1+B_1JC & B_1H \\ G_1C & F_1 \end{bmatrix}, \qquad \bar{A}_2 = \begin{bmatrix} A_2+B_2JC & B_2H \\ G_2C & F_2 \end{bmatrix}$$

$$\bar{B}_1 = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \qquad \bar{B}_2 = \begin{bmatrix} G_2 \\ 0 \end{bmatrix}$$

Then

$$\det(I-\bar{A}_1z_1-\bar{A}_2z_2) = h(z_1,z_2)\det(R_LD_R+S_LN_R) \tag{26}$$

We shall now investigate to what extent it is possible to assign arbitrarily the characteristic polynomial (26) of the feedback system and/or the associated variety, by selecting an appropriate Σ_C.

Obviously, the factor $h(z_1,z_2)$ in (26), that descends from the existence of common factors in at least one of the matrix fractions $C(I-A_1z_1-A_2z_2)^{-1}$ and $(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)$, cannot be influenced by any dynamic output feedback.

It remains to analyze how $\det(R_LD_R+S_LN_R)$ changes as we vary R_L and S_L with the constraint that they characterize a proper transfer matrix. Once we have selected R_L and S_L, it is possible to construct a 2D system that realizes $R_L^{-1}S_L$ (see for instance the construction in Theorem 5).

Using the Binet-Cauchy formula, $\det(R_LD_R+S_LN_R)$ can be expressed as the sum of the products of all possible minors of maximal order, q_i, $i=1,2,\dots,\nu$, of $[S_L\ R_L]$ into the corresponding minors of the same order, m_i, $i=1,2,\dots,\nu$, of $[N_R'\ D_R']$, that is

$$\det(R_LD_R + S_LN_R) = \sum_1^{\nu} q_im_i \tag{27}$$

Clearly all polynomials that are obtained from (27) as R_L and S_L vary, vanish on $\mathscr{V}(W)$, since they belong to the ideal $(m_1,m_2,\dots,m_\nu)$ generated by the minors of maximal degree of $[N_R'\ D_R']$.

<u>Theorem 6.</u> Let $p\in(m_1,m_2,\dots,m_\nu)$. Then there exist R_L and S_L such that

$$R_LD_R + S_LN_R = p\ I_m \tag{28}$$

proof. Denote by M_i, $i=1,2,\dots,\nu$ the submatrix of maximal order of $[N_R'\ D_R']'$ correspondig to m_i. Then there exist constant matrices L_i and K_i such that

$$M_i = L_iD_R + K_iN_R$$

So we have

$$m_iI_m = (\mathrm{Adj}\ M_i)M_i = (\mathrm{Adj}\ M_i)L_iD_R + (\mathrm{Adj}\ M_i)K_iN_R\ , \quad i=1,2,\dots,\nu \tag{29}$$

Now, assuming $p = \sum_{1}^{\nu} q_i m_i$, from (29) we obtain

$$p\, I_m = \sum_{1}^{\nu} q_i m_i I_m = \Big(\sum_{1}^{\nu} q_i (\mathrm{Adj}\ M_i) L_i\Big) D_R + \Big(\sum_{1}^{\nu} q_i (\mathrm{Adj}\ M_i) K_i\Big) N_R$$

that coincides with (28) once we have taken

$$R_L = \sum_{1}^{\nu} q_i (\mathrm{Adj}\ M_i) L_i, \qquad S_L = \sum_{1}^{\nu} q_i (\mathrm{Adj}\ M_i) K_i$$

As a direct consequence of the Theorem above and of Hilbert Nullstellensatz, we have that given any polynomial p vanishing on $\mathcal{V}(W)$, there exist R_L, S_L and a positive integer r such that

$$\det(R_L D_R + S_L N_R) = p^r \tag{30}$$

It is easy to check that there are cases where r cannot be unitary. As an example, consider the following right coprime matrix fraction

$$N_R D_R^{-1} = [(z_1 - z_2)^2 \;\vdots\; 0] \begin{bmatrix} (1-z_1)^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = W$$

We have $m_1 = (1-z_1)^2$, $m_2 = (z_1 - z_2)^2$, so that $(W) = (1,1)$. The polynomial $(1-z_2)^2$ vanishes on (W) but does not belong to (m_1, m_2), so that $(1-z_2)^2 \neq \det(R_L D_R + S_L N_R)$ for every R_L and S_L.

The requirement that (R_L, S_L) characterizes a proper transfer matrix is equivalent to impose that p in (30) satisfies $p(0,0) = 0$. In fact, since $N_R(0,0) = 0$ and $D_R(0,0) = I_m$, we have

$$\det[R_L(0,0) \;\vdots\; S_L(0,0)] \begin{bmatrix} D_R(0,0) \\ N_R(0,0) \end{bmatrix} = \det R_L(0,0)$$

so that $p(0,0) \neq 0$ is equivalent to $\det R_L(0,0) \neq 0$ that is to the possibility of realizing $R_L^{-1} S_L$ by means of a 2D proper system.

In some applications we are interested in designing a compensator Σ_c such that the resulting feedback system is finite memory, i.e. $\det(I - \bar{A}_1 z_1 - \bar{A}_2 z_2) = 1$. Such a compensator is called dead-beat compensator.

The existence of a dead-beat compensator depends on the following facts:

(i) the transfer matrix of Σ should make possible to solve the equation

$$\det(R_L D_R + S_L N_R) = 1 \tag{31}$$

for any right factor coprime MFD $N_R D_R^{-1} = W$.

(ii) The matrices A_1, A_2, B_1, B_2 and C of Σ should be selected so that the polynomial h in (25) is unitary.

Now, equation (31) is solvable if and only if $[N_R' \ \ D_R']$ has full rank for every (z_1,z_2), that is if N_R and D_R are zero coprime. In this case by Theorem 6 the equation

$$R_L D_R + S_L N_R = I_m$$

has solution and hence (31) too.

The condition h=1 implies that $(C,\ I-A_1z_1-A_2z_2)$ are right factor coprime and $(I-A_1z_1-A_2z_2,\ B_1z_1+B_2z_2)$ are left factor coprime.

If we recall the controllability and causal reconstructibility criteria [1, 2], the existence of a dead-beat controller can be related both to internal structural properties of Σ and to external properties of $N_R D_R^{-1}$. To this purpose we have the following Theorem.

Theorem 7. Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of the right coprime matrix fraction $N_R D_R^{-1}$. Then the following properties are equivalent:

(i) there exists a dead-beat compensator

(ii) Σ is controllable and causally reconstructible

(iii) $(C,\ I-A_1z_1-A_2z_2)$ are right factor coprime, $(I-A_1z_1-A_2z_2,\ B_1z_1+B_2z_2)$ are left factor coprime, N_R and D_R are right zero coprime.

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