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STRUCTURE AND PROPERTIES OF 2D SYSTEMS

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1. INTRODUCTION

The first contributions [1-5] takling the problem of defining dynamical systems with input, output and state functions depending on two independent variables appeared nearly ten years ago.

In principle, they were motivated by the necessity of investigating recursive structures for processing two dimensional data, such images, seismic signals, etc.

This processing has been essentially performed since long time using discrete filters, given by ratios of polynomials in two indeterminates, or algorithms assigned via difference equations. Thus the idea of input--output description of systems in two indeterminates as well as the design and analysis techniques based on the frequency response and on the two-dimensional z-transform have been well known since many years.

The new idea that originated the research area on 2D systems consisted in considering these algorithms (i.e. transfer functions and difference equations in two indeterminates) as external representations of dynamical systems and hence in introducing for such systems the concept of state and its updating equations.

From the beginning, very deep and substantial differences from the theory of dynamical systems in one variable have been evidentiated. These are due to the mathematical tools to be used and, above all, to the concept of state itself and to the structure of state updating equations.

In this case, there is no "canonical" algebraic construction that pro vides an intrinsic meaning to a finite dimensional state. Thus a few state models have been introduced with different recursive structures, although they are generated by the same underlying idea that a recursive computation is made possible by a finite dimensional "local state" and that the complete information on the past is kept by an infinite sequence of local states ("global state").

As it can be expected, dealing simultaneously with two notions of state causes several problems, which are further involved by the formalism required by the different models.

In this chapter we try to give a fairly complete account of the structural properties and realization of 2D systems. In order to keep the size of the contribution within reasonable bounds, only one local state model is fully analyzed. Nevertheless other models are considered whenever the problems raised have different solutions depending on the structure of the model.

2. 2D STATE SPACE MODELS

In this section we introduce the external and internal representations of 2D systems and a formal description of the realization problem. A solution to this problem is given in terms of Nerode equivalence, it however provides an infinite dimensional state model. As we shall see, the solution in terms of finite dimensional updating equations is provided by local state models.

2.1 2D Input-Output Maps and Nerode Equivalence

Let T denote the cartesian product $\mathbb{Z} \times \mathbb{Z}$, partially ordered by the product of the orderings, and introduce the notions of "past" and "future" of a point (h,k) in T. We call *past* of (h,k) the set

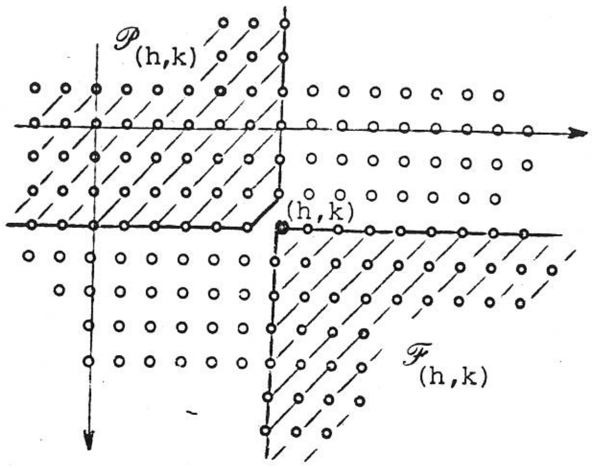

fig. 1

$$\mathscr{P}_{(h,k)} = \{(i,j) : i \le h, j \le k, (i,j) \ne (h,k)\}$$

and *future* of (h,k) the set

$$\mathscr{F}_{(h,k)} = \{(i,j) : h \le i, k \le j\}$$

A function $g : T \to V$ with values in a vector space V is *past finite* if the intersection of the support of g and $\mathscr{P}_{(h,k)}$ is a finite set for any (h,k) in T.

Definition 1. *A 2D system in input-output form* is defined as $\mathscr{S} = \{T, K, \mathscr{U}, \mathscr{Y}, F\}$ where

- T is the *time set* defined above
- $\mathscr{U}$ and $\mathscr{Y}$ are the spaces of *input and output functions*. Their elements are the past finite functions with values on the field K. In formal power series notation, an element of $\mathscr{U}$ or $\mathscr{Y}$ is written

$$r = \sum_{i,j \in \mathbb{Z}} (r, z_1^i z_2^j) z_1^i z_2^j .$$

where $(r, z_1^i z_2^j)$ denotes the coefficient of $z_1^i z_2^j$.

- $F : \mathcal{U} \to \mathcal{Y}$ is the *input-output map*. F is assumed to satisfy the following axioms

1. *linearity*
2. *two dimensional shift invariance:* for any (h,k) in T,

$$F(z_1^h z_2^k u) = z_1^h z_2^k F(u)$$

3. *two dimensional strict causality:* for any (h,k) in T

$$(u_1, z_1^i z_2^j) = (u_2, z_1^i z_2^j) \; , \qquad \forall (i,j) \in \mathscr{P}_{(h,k)}$$

implies

$$(F\, u_1, z_1^h z_2^k) = (F\, u_2, z_1^h z_2^k)$$

Under assumptions 1-3 it is easy to verify that the impulse response $F(1)$ is a *strictly causal power series*, i.e.

$$s \triangleq F(1) = \sum_{\substack{i,j=0 \\ i+j>0}}^{\infty} (F(1), z_1^i z_2^j) z_1^i z_2^j \tag{1}$$

and that

$$F(u) = s\,u \; , \qquad \forall u \in \mathcal{U} \tag{2}$$

so that s constitutes the *transfer function* of the 2D system.

In this way, 2D systems in input-output form are in one-to-one correspondence with formal power series in z_1 and z_2 with zero constant term, called *strictly causal formal power series* and denoted by $K_c[[z_1,z_2]]$. This result generalizes the well known connection existing between input-output representations of one-dimensional systems and formal power series in one indeterminate.

The realization of one-dimensional systems is done by introducing a time vector function $x(\cdot)$, called the state of the system, which has a separation property with respect to the past, in the sense that the knowledge of this vector at any instant τ is sufficient to evaluate the output at $t \geq \tau$. When one deals with 2D input-output maps it is no longer possible to attach a vector having a separation property to a point (h,k) in $\mathbb{Z} \times \mathbb{Z}$ in such a way that the knowledge of this vector and of the input in the future of (h,k) makes possible to compute the output in the future of (h,k). Clearly this fact is intrinsic to the structure of partial ordering of $\mathbb{Z} \times \mathbb{Z}$, since a separation property must interest an infinite set of points.

It is worth while to recall that the structure of 1D models in state space form, i.e.

$$\begin{aligned} x(t+1) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,u(t) \end{aligned} \tag{3}$$

can be axiomatically derived from Nerode equivalence on the input space. With the aim of singling out state space models which realize 2D input-output maps we are naturally led to extend Nerode equivalence to the input space $\mathcal{U}$.

For this, consider in $\mathbb{Z} \times \mathbb{Z}$ the sets

$$\mathscr{C}_i = \{(h,k),\ h+k = i\}, \qquad i = 0, \pm 1, \ldots$$

called *separation sets*. Thus, given $\mathscr{C}_i$, $Z \times Z$ is partitioned in two subsets

$$\mathscr{P}_{\mathscr{C}_i} = \bigcup_{(h,k)\in \mathscr{C}_i} \mathscr{P}_{(h,k)} \qquad \textit{(past region w.r. to } \mathscr{C}_i\textit{)}$$

$$\mathscr{F}_{\mathscr{C}_i} = \bigcup_{(h,k)\in \mathscr{C}_i} \mathscr{F}_{(h,k)} \qquad \textit{(future region w.r. to } \mathscr{C}_i\textit{)}$$

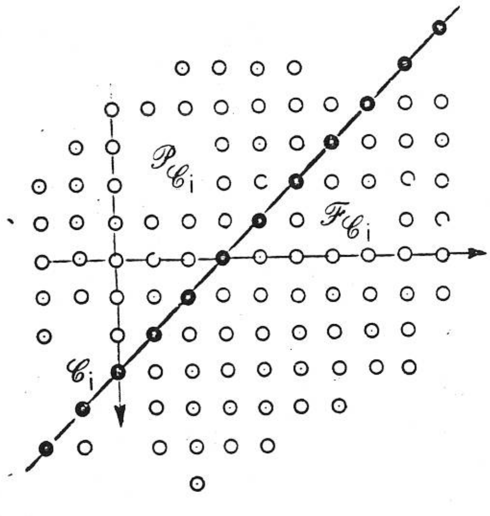

fig. 2

The sets $\mathscr{C}_i$ introduced above are not the unique subsets of $Z \times Z$ having the property of splitting the discrete plane into a past region and a future region. In particular, image processing usually refers to subsets as in fig. 3.

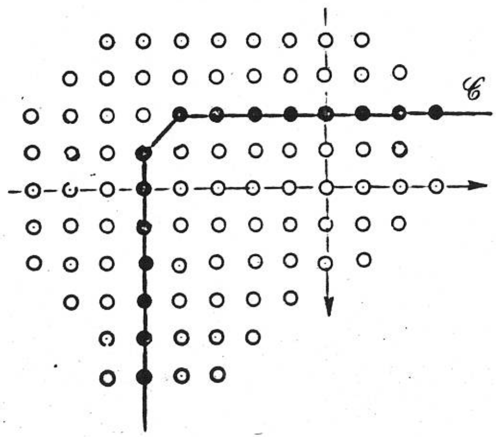

fig. 3

In the following we refer to the separation sets $\mathscr{C}_i$, even if most of the 2D properties can be derived for any subset which partitions $\mathbb{Z}\times\mathbb{Z}$ in the previous sense.

Let $\mathscr{U}^*$ denote the set of functions $u\in\mathscr{U}$ with support in $\mathscr{P}_{\mathscr{C}_1}$ and $\mathscr{Y}^*$ the set of functions with support in $\mathscr{F}_{\mathscr{C}_1}$. For every $u\in\mathscr{U}^*$ let $f(u)$ denote the restriction of $F(u)$ to $\mathscr{F}_{\mathscr{C}_1}$. This defines a linear map

$$f : \mathscr{U}^* \to \mathscr{Y}^* : u \mapsto \sum_{\substack{i,j \\ i+j>0}} (F(u), z_1^i z_2^j) z_1^i z_2^j$$

which characterizes the 2D input-output map in the same sense as F does.

We follow the usual Nerode philosophy that two inputs $u_1, u_2 \in \mathscr{U}^*$ are equivalent $(u_1 \sim u_2)$ iff

$$f(u_1) = f(u_2)$$

— The *Nerode equivalence classes* are then the cosets of $\mathscr{U}^*$ in $\ker f = \{u : u\in\mathscr{U}^*, u\sim 0\}$ and the quotient set

$$X_N \triangleq \mathscr{U}^*/\sim \; = \mathscr{U}^*/\ker f$$

is endowed in a canonical way with a linear structure, as represented by the following commutative diagram

$$\begin{tikzcd} \mathscr{U}^* \arrow[r, "f"] \arrow[d, "\nu"'] & \mathscr{Y}^* \\ X_N = \mathscr{U}^*/\ker f \arrow[ur, "\bar{f}"'] & \end{tikzcd} \tag{4}$$

The space X_N displays the memory function of the map F and can be assumed as the state space of a dynamical system which realizes F.

The construction of X_N is clearly canonical, but suffers from the drawback that the dimension of X_N is infinite even if the input-output map is given by a rational power series (*). In fact, consider the commutative diagram (4) and restrict the input space $\mathcal{U}^*$ to the ring $K[z_1^{-1}z_2]$ of polynomials in the indeterminate $z_1^{-1}z_2$. Since $K[z_1^{-1}z_2]$ is a subring of the integral domain of *truncated Laurent formal power series* $K((z_1,z_2))$, the assumption $f(u) = su = 0$ implies $u = 0$ for all u in $K[z_1^{-1}z_2]$. Since the restriction of f to $K[z_1^{-1}z_2]$, which is an infinite dimensional K-vector space, is one-to-one, then $f(K[z_1^{-1}z_2])$ is infinite dimensional. Hence, $\dim X_N = \dim f(\mathcal{U}^*) \geq \dim f(K[z_1^{-1}z_2]) = \infty$.

This fact shows that the situation for 2D systems is not the same as for discrete-time linear 1D systems. Actually, in the latter case the dimension of the canonical state space X_N is finite if and only if the input-output map is a rational power series.

Remark 1: In the 1D linear case the rationality of the input-output map is equivalent to the existence of nonzero inputs of compact support such that the corresponding outputs are of compact support. This is also true for 2D systems. If $s = p(z_1^{-1},z_2^{-1})/q(z_1^{-1},z_2^{-1})$ and p and q have no com-

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(*) Recall that a formal power series $s \in K[[z_1,z_2]]$ is *rational* if there exist polynomials $\bar{p}$ and $\bar{q}$ in $K[z_1,z_2]$ such that $\bar{q}(0,0) \neq 0$ and $s\bar{q} = \bar{p}$. Equivalently, s is rational if there exist polynomials p and q in $K[z_1^{-1},z_2^{-1}]$, $p = \sum_{0i}^{h} \sum_{0j}^{k} p_{ij} z_1^{-i} z_2^{-j}$, $q = \sum_{0i}^{m} \sum_{0j}^{n} q_{ij} z_1^{-i} z_2^{-j}$ such that $q_{mn} \neq 0$, $h \leq m$, $k \leq n$ and $sq = p$. A *strictly causal rational power series* s satisfies the additional condition $\bar{p}(0,0) = 0$ (or equivalently $p_{mn} = 0$).

mon factors, the class of inputs with compact support giving outputs with compact support is the principal ideal (q), modulo the shift group generated by z_1^{-1} and z_2^{-1}.

Remark 2: If the input space is restricted to $K[z_1^{-1}, z_2^{-1}]$ (and the output space to $K_c[[z_1, z_2]]$) then the dimension of X_N is the rank of the Hankel matrix $\mathscr{H}(s)$ associated with the series s:

$$\mathscr{H}(s) = \begin{bmatrix} (s,1) & (s,z_1) & (s,z_2) & (s,z_1^2) & (s,z_1 z_2) & (s,z_2^2) \dots \\ (s,z_1) & (s,z_1^2) & (s,z_1 z_2) & (s,z_1^3) & \dots\dots\dots\dots & \\ (s,z_2) & (s,z_1 z_2) & \dots\dots\dots\dots & & & \\ \dots\dots\dots\dots & & & & & \end{bmatrix}$$

The rank of $\mathscr{H}(s)$ is finite if and only if s is rational and a denominator q of s can be factorized as $q = q_1 q_2$ with $q_1 \in K[z_1^{-1}]$, $q_2 \in K[z_2^{-1}]$. The series satisfying this property are the elements of the ring $K[(z_1)] \otimes K[(z_2)] = K^{rec}[(z_1, z_2)]$, called the ring of *recognizable series*. The ratios of polynomials representing recognizable power series are commonly called *separable rational functions*.

2.2 Local State Models

The Nerode representation X_N of a 2D input-output map is infinite dimensional, and therefore it is impossible to describe the dynamics of X_N in terms of appropriate finite dimensional updating equations.

This difficulty can be overcome to some extent by introducing the notion of *local state space*. We will show that under suitable conditions there exist a finite-dimensional vector space X, and matrices $A_1, A_2 \in K^{n\times n}$, $C \in K^{1\times n}$, $B_1, B_2 \in K^{n\times 1}$ such that the input-output behavior is descri bed by the updating equations

$$x(h+1,k+1) = A_1 x(h+1,k) + A_2 x(h,k+1) + B_1 u(h+1,k) + B_2 u(h,k+1) \tag{5}$$

$$y(h,k) = C\,x(h,k)$$

The form of the updating equations (5) follows from an axiomatic framework [7] which we describe later. The axioms are derived from the following intuitive picture: a finite dimensional local state space X is attached to each point (h,k) of the plane. If $(h',k') > (h,k)$, then the local state $x(h',k')$ depends not only on $x(h,k)$ but also on the local states $x(i,j)$, $(i,j) \in \mathscr{P}_{(h',k')} \cap \mathscr{C}_{h+k}$ (see fig. 4). Since (h',k') is arbitrary, it is necessary to introduce a *global state space* X consisting of all local state spaces on a separation set.

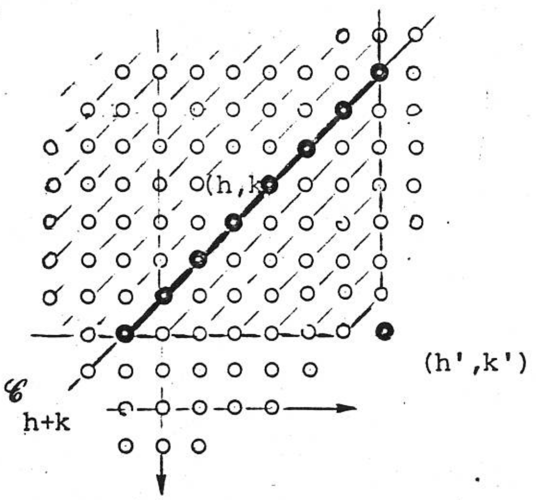

fig. 4.

Formally, we define a *2D system in state space form* as follows:

$$\Sigma \stackrel{\Delta}{=} (T, K, \mathscr{U}, \mathscr{Y}, X, \mathsf{X}, \phi, r)$$

where

- $T, K, \mathcal{U}, \mathcal{Y}$ are as in the definition of $\mathcal{S}$
- $X = K^n$ is the *local state space*
- $\mathbb{X}$ is the *global state space*. Its elements are the sequences $\{x(h,k),\ h+k = \text{cost},\ x(h,k) \in X\}$, namely sequences of local states on a separation set. A global state on the separation set $\mathcal{C}_i$ will be denoted as

$$\mathcal{X}_i = \sum_{h \in \mathbb{Z}} x(h, i-h) z_1^h z_2^{i-h} \tag{6}$$

- $\phi: T \times \mathbb{Z} \times \mathbb{X} \times \mathcal{U} \to X : ((h,k), i, \mathcal{X}_i, u) \mapsto x(h,k),\ i \leq h+k$, is the *state transition function* and $r: X \to K$ is the *readout map*.

We assume that r is a linear functional on X and that ϕ satisfies the following axioms:

1. *2D determinism*: let $u = \hat{u}$ in $\mathcal{P}_{(h,k)} - \mathcal{P}_{\mathcal{C}_i}$ and $\mathcal{X}_i = \bar{\mathcal{X}}_i$ in $\mathcal{P}_{(h,k)} \cap \mathcal{C}_i$ (see fig. 5). Then $\phi((h,k), i, \mathcal{X}_i, u) = \phi((h,k), i, \bar{\mathcal{X}}_i, \hat{u})$
2. *consistency*: let $(h,k) \in \mathcal{C}_i$ and let $\bar{x}$ be the value of the local state at (h,k). Then $\phi((h,k), i, \mathcal{X}_i, u) = \bar{x}$, $\forall u \in \mathcal{U}$.

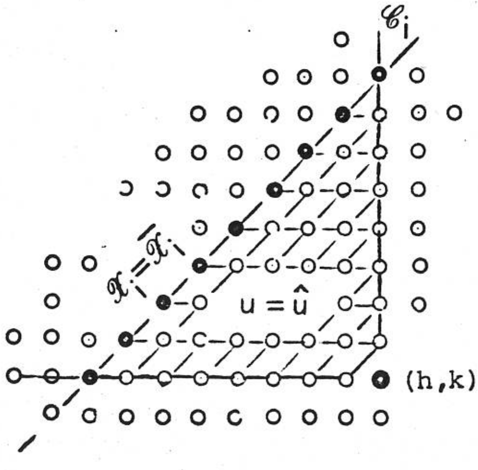

fig. 5

3. *shift invariance*: for any $\sigma_1, \sigma_2 \in \mathbb{Z}$

$$\phi((h,k),i,\mathcal{X}_i,u) = \phi((h+\sigma_1,k+\sigma_2),j,\hat{\mathcal{X}}_j,u\,z_1^{\sigma_1}z_2^{\sigma_2})$$

where

$$j = i+\sigma_1+\sigma_2\,,$$

$$\hat{\mathcal{X}}_j = \sum_s x(s-\sigma_1,j-s-\sigma_2)\,z_1^s z_2^{j-s}$$

4. *composition*. Let $i \leq j \leq h+k$. Then

$$\phi((h,k),i,\mathcal{X}_i,u) = \phi((h,k),i,\hat{\mathcal{X}}_j,u)$$

where

$$\hat{\mathcal{X}}_j = \sum_r \phi((r,j-r),i,\mathcal{X}_i,u)\,z_1^r z_2^{j-r}$$

5. *linearity*. Let $u, \bar{u} \in \mathcal{U}$, and $\mathcal{X}_i$, $\bar{\mathcal{X}}_i \in X$. Then

$$\phi((h,k),i,\mathcal{X}_i,u) + \phi((h,k),i,\bar{\mathcal{X}}_i,\bar{u})$$
$$= \phi((h,k),i,\mathcal{X}_i+\bar{\mathcal{X}}_i,u+\bar{u})$$

Next lemma justifies the matrix form of the updating equations (5).

<u>Lemma 1.</u> *Given a 2D system* Σ *in state space form, there exist an integer* n *and matrices* $A_1, A_2 \in K^{n\times n}$, $B_1, B_2 \in K^{n\times 1}$, $C \in K^{1\times n}$ *such that* ϕ *and* r *are given by (5).*

proof. 2D determinism implies that $x(h+1,k+1)$ depends only on $x(h+1,k)$, $x(h,k+1)$, $u(h+1,k)$ and $u(h,k+1)$. Equations (5) follow by linearity assumptions on ϕ and r .

Throughout the chapter, we adopt the notation (A_1, A_2, B_1, B_2, C) to designate a 2D system Σ represented by the updating equations (5).

As mentioned in the Introduction, the family of 2D state models [1, 4,7] does not include only 2D systems with structure (5). Actually some of these models (i.e. Roesser's model) can be obtained from (5) by imposing structural restrictions on the matrices A_1 and A_2; others have second order state updating equations of the following type

$$\begin{aligned} x(h+1,k+1) &= A_o x(h,k) + A_1 x(h+1,k) + A_2 x(h,k+1) + B\,u(h,k) \\ y(h,k) &= C\,x(h,k) \end{aligned} \tag{7}$$

Usually, 2D systems (7) have been considered in connection with initial conditions assignments on subsets of $\mathbb{Z} \times \mathbb{Z}$ as in fig. 3.

Here we give an account of the structure of the updating equations and leave to the next section the investigation of the relationships between 2D state models and the transfer functions they realize.

Let first consider the *Roesser's model*. Each point (h,k) in $\mathbb{Z} \times \mathbb{Z}$ is tied to a *local state* vector that is the direct sum of an *horizontal state* $x^h(h,k)$ and a *vertical state* $x^v(h,k)$. The horizontal state in (h,k) is assumed to depend only on the values of x^v and x^h at $(h-1,k)$ while the vertical state to depend on the values of x^v and x^h at $(h,k-1)$. Following are the equations of this model:

$$\begin{aligned} \begin{bmatrix} x^h(h+1,k) \\ x^v(h,k+1) \end{bmatrix} &= \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix} \begin{bmatrix} x^h(h,k) \\ x^v(h,k) \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} u(h,k) \\ y(h,k) &= \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} x^h(h,k) \\ x^v(h,k) \end{bmatrix} \end{aligned} \tag{8}$$

where $\hat{A}_i$, $\hat{B}_i$ and $\hat{C}_i$ are matrices of suitable dimensions.

Putting

$$x(h,k) = \begin{bmatrix} x^h(h,k) \\ x^v(h,k) \end{bmatrix}$$

Roesser's equations assume the following form

$$x(h+1,k+1) = \begin{bmatrix} 0 & 0 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix} x(h+1,k) + \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & 0 \end{bmatrix} x(h,k+1) +$$

$$+ \begin{bmatrix} 0 \\ \hat{B}_2 \end{bmatrix} u(h+1,k) + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u(h,k+1) \qquad (9)$$

$$y(h,k) = [\hat{C}_1 \quad \hat{C}_2]\, x(h,k)$$

Then it is clear that, starting from models (5) and imposing on A_i and B_i the following restrictive structural assumptions

$$A_1 = \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \quad A_2 = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ * \end{bmatrix} \qquad B_2 = \begin{bmatrix} * \\ 0 \end{bmatrix} \qquad (10)$$

we obtain all Roesser's models.

A 2D model that exhibits second order updating equations is the *Attasi's model* :

$$\begin{aligned} x(h+1,k+1) &= \bar{A}_1 x(h+1,k) + \bar{A}_2 x(h,k+1) - \\ &\quad - \bar{A}_1\bar{A}_2 x(h,k) + \bar{B}\,u(h,k) \\ y(h,k) &= \bar{C}\,x(h,k) \end{aligned} \tag{11}$$

with $\bar{A}_1\bar{A}_2 = \bar{A}_2\bar{A}_1$.

Clearly the Attasi's model can be derived from (7) assuming $A_1A_2 = A_2A_1$ and $A_o = -A_1A_2$.

2.3 2D Realization

Using the axiomatic framework introduced in the previous sections, the realization problem of a 2D input-output map is formalized by the following definition.

Definition 2. A 2D system Σ in state space form is a realization of a 2D input-output map $\mathscr{S}$ if

$$(F(u), z_1^h z_2^k) = r(\phi((h,k), i, 0, u)$$

for any i,h,k with $i \leq h+k$ and any u in $\mathscr{U}$ with $u(\ell,m) = 0$ for (ℓ,m) in $\mathscr{P}_{\mathscr{C}_i}$.

As in the 1D case, a realization consists in a system Σ whose input-output behaviour, resulting from zero initial global state, coincides with that given by $\mathscr{S}$.

The global state vectors of Σ play the same role of *memory function* as the Nerode equivalence classes in the input-output map $\mathscr{S}$. So we are interested in investigating how X and X_N are related.

Actually, as we shall show, the canonical state space X_N of $\mathscr{S}$ can be embedded in the space X which characterizes a realization Σ and this embedding preserves the system theoretic properties of X_N .

<u>Proposition 1.</u> *Let Σ be a realization of a given $\mathscr{S}$. Then there exists a 1:1 linear map e such that the diagram*

$$\begin{tikzcd}
\mathscr{U}^* \arrow[r, "\nu"] \arrow[d, "\lambda"'] \arrow[dr, "f"] & X_N \arrow[d, dashed, "\bar{f}"] \arrow[dl, dashed, "e"] \\
X \arrow[r, dashed, "\mu"'] & \mathscr{Y}^*
\end{tikzcd}$$

commutes along the dashed arrows (the maps λ and μ are built up in natu ral way from ϕ and r in Σ).

proof. Consider the following diagram

$$\begin{tikzcd}
\mathscr{U}^* \arrow[r, "\lambda"] \arrow[d, "\nu"'] \arrow[dr, "f"] & X \arrow[r, "\pi"] \arrow[d, "\mu"] & X/\ker\mu \arrow[dl, "\bar{\mu}"] \\
X_N \arrow[r, "\bar{f}"'] \arrow[urr, dashed, "h"] & \mathscr{Y}^* &
\end{tikzcd}$$

Since ν is onto and $\bar{\mu}$ is one-to-one, by Zeiger's lemma [8] there exists a unique linear map $h : X_N \to X/\ker\mu$ which makes the diagram commutative, and is one-to-one.

Denote by $\mathscr{T}$ a complement of $\ker\mu$ in X and by $\pi_{\mathscr{T}}$ the restriction of π to $\mathscr{T}$. Since $\pi_{\mathscr{T}} : \mathscr{T} \to X/\ker\mu$ is an isomorphism, there exists a one-to-one linear map $e = \pi_{\mathscr{T}}^{-1} \circ h$ such that $h = \pi \circ e$. The map e depends on the choice of $\mathscr{T}$.

Obviously, the possibility of embedding the Nerode state space X_N in X does not depend on the dimension of the local state space.

As we can expect, not all transfer functions in $K_c[[z_1,z_2]]$ can be realized by 2D state models. In fact, let $\Sigma = (A_1, A_2, B_1, B_2, C)$, $\mathscr{X}_o = 0$, and associate the monomial $x(h,k) z_1^h z_2^k$ with the local state in (h,k). For any input u with support in $\mathscr{F}_{\mathscr{C}_o}$, we have

$$\sum_{h+k\geq 0} x(h,k) z_i^h z_2^k = A_1 \Big(\sum_{h+k\geq 0} x(h,k) z_1^h z_2^k \Big) z_1 +$$

$$+ A_2 \Big(\sum_{h+k\geq 0} x(h,k) z_1^h z_2^k \Big) z_2 + B_1 u z_1 + B_2 u z_2 .$$

Then

$$(I - A_1 z_1 - A_2 z_2) \sum_{h+k\geq 0} x(h,k) z_1^h z_2^k = (B_1 z_1 + B_2 z_2) u$$

The polynomial matrix $(I - A_1 z_1 - A_2 z_2)$ has an inverse in the ring of formal power series $K^{n\times n}[[z_1, z_2]]$ given by

$$(I - A_1 z_1 - A_2 z_2)^{-1} = \sum_{i=0}^{\infty} (A_1 z_1 + A_2 z_2)^i .$$

So we have

$$\sum_{h+k\geq 0} x(h,k) z_1^h z_2^k = (I - A_1 z_1 - A_2 z_2)^{-1} (B_1 z_1 + B_2 z_2) u$$

and

$$y = C \sum_{h+k\geq 0} x(h,k) z_1^h z_2^k =$$

$$= C (I - A_1 z_1 - A_2 z_2)^{-1} (B_1 z_1 + B_2 z_2) u$$

Then, the connection between input and output functions established by a 2D system $\Sigma = (A_1, A_2, B_1, B_2 C)$ is the 2D input-output map characteri zed by the following transfer function

$$s_\Sigma = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) \tag{12}$$

So, the following definition of realization is equivalent to Definition 2.

<u>Definition 2'.</u> A 2D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ is a realization of a series $s \in K_c[[z_1, z_2]]$ if $s = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2)$.

The *dimension of a realization* Σ is defined as the dimension of the local state space X and we say that a realization Σ of s is *minimal* when $\dim \Sigma \leq \dim \Sigma'$ for any Σ' realizing s.

Two problems naturally arise: the first is to specify the subclass of the class of 2D input-output maps which can be realized by 2D systems; the second consists in setting up some techniques for obtaining the most "efficient" realizations in the sense of the dimension of the local state.

In this section we deal with the solution to the first problem. It recalls very closely the solution of the realization problem for 1D discrete linear systems and is given by the following proposition [5,9].

<u>Proposition 2:</u> *Let $s \in K[[z_1, z_2]]$. Then there exists a 2D system (5) which realizes s if and only if s belongs to the ring $K_c[(z_1, z_2)]$ of rational power series with zero constant term.*

proof. The necessity is a direct consequence of (12).

Conversely let $s \in K_c[(z_1, z_2)]$. This means that $s = p(z_1, z_2)/q(z_1, z_2)$, p and q in $K[z_1, z_2]$, $p(0,0) = 0$ and $q(0,0) = 1$. Consider two polynomials π and χ in the ring $K\langle\omega_1, \omega_2\rangle$ of non commutative polynomials such that their commutative images are p and q respectively.

The commutative image of the non commutative series $\sigma = \pi\chi^{-1}$ is the series s. Since σ is rational [10], there exist an integer n and matrices $A_1, A_2 \in K^{n\times n}$, $B \in K^{n\times 1}$ and $C \in K^{1\times n}$ such that

$$\begin{aligned}\sigma &= C(I - A_1\omega_1 - A_2\omega_2)^{-1}B = C \sum_{o}^{\infty}{}_k (A_1\omega_1 + A_2\omega_2)^k B = \\ &= C \sum_{1}^{\infty}{}_k (A_1\omega_1 + A_2\omega_2)^k B = \\ &= C(I - A_1\omega_1 - A_2\omega_2)^{-1}(B_1\omega_1 + B_2\omega_2)\end{aligned} \tag{13}$$

where we put $B_1 = A_1B$, $B_2 = A_2B$.

Since the projection map from the algebra of non commutative power series $K\langle\langle \omega_1, \omega_2\rangle\rangle$ onto $K[[z_1, z_2]]$ is an algebra homomorphism, the series s can be expressed as

$$s = C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)$$

Then $\Sigma = (A_1, A_2, B_1, B_2, C)$ is a zero-state realization of s.

Remark. If (A_1, A_2, B_1, B_2, C) is a realization of dimension n of s, and $T \in K^{n\times n}$ is nonsingular, then $(TA_1T^{-1}, TA_2T^{-1}, TB_1, TB_2, CT^{-1})$ is still a realization of s. The matrix T is associated with a change of basis in the local state space.

For explicit constructions of matrices A_1, A_2, B_1, B_2, C starting from the coefficients of the numerator and denominator of s, see [9, 11-13].

Let's now consider the Roesser and Attasi models given in the previous section and examine their properties in connection with the realization problem.

As far as Roesser's model is concerned, it can be easily proved [11] that Proposition 2 still holds. So, these models realize the whole class of proper rational functions as models (5) do. However, in general, if we decide to realize a transfer function by a Roesser's model, we pay its particular structure by a higher dimension of matrices C, A_i, B_i. This is easily seen, for instance, by considering the rational function $(z_1+z_2)(1-z_1-z_2)^{-1}$ which is realized by the one dimensional 2D system $\Sigma = (1,1,1,1,1)$ having structure (5), but requires a higher dimension of the local state space if we want to realize it in the class of Roesser's models.

Furthermore, due to the particular structure of A_i and B_i matrices in Roesser's model (see (10)), the transfer function assumes the following structure:

$$s \triangleq C \left[\begin{bmatrix} z_1^{-1} I & 0 \\ 0 & z_2^{-1} I \end{bmatrix} - A \right]^{-1} B \tag{14}$$

where

$$A \triangleq A_1 + A_2 \,, \quad B \triangleq B_1 + B_2$$

It is also worth while to notice that Roesser's model is not invariant under a generic change of basis in the state space and keeps its structure only if we use block diagonal transformation matrices which leave invariant the subspaces of vertical and horizontal local states.

As we pointed out earlier, the Attasi's models constitute a subclass of models (7). While the latter realize [7] all rational functions in $(z_1 z_2) K[(z_1, z_2)]$, the assumptions $A_1 A_2 = A_2 A_1$ and $A_o = -A_1 A_2$ constrain the transfer functions realized by the Attasi's models to be separable .

Conversely, any separable transfer function in $(z_1 z_2)K[(z_1, z_2)]$ is realizable by an Attasi's model.

The commutativity hypothesis $A_1A_2 = A_2A_1$, can be adopted also for state equations with structure (5). It turns out that in this case realizable transfer functions have denominators which factorize, over the complex field, as products of linear factors [14]. In fact, commutativity implies that there exists a (complex) similarity transformation that reduces simultaneously A_1 and A_2 to lower (upper) triangular form [15]. Hence the denominator of the transfer function, which is a factor of $\det(I - A_1z_1 - A_2z_2)$, splits into linear factors as

$$\prod_i (1 - a_i^{(1)} z_1 - a_i^{(2)} z_2)$$

3. REACHABILITY AND OBSERVABILITY

The notions of reachability and observability for 2D systems have been introduced following mainly two different approaches. These correspond to two standard methodologies used in linear system theory, that is the geometric approach, where reachable and observable subspaces are naturally defined starting from the system definition of reachable and observable states, and the modal approach, based on polynomial matrices, which exploits the notion of coprimeness. In 1D system theory both methodologies are quite equivalent (Rosenbrock's results), while for 2D systems they are not.

The geometric notions of reachability and observability have been introduced for local and global states. As we shall show, local reachabili ty and observability are too weak to guarantee the minimality of the realization, while the corresponding global properties impose on the model some constraints which are too strong to be fulfilled in the generic case.

As far as the modal approach is concerned, the most serious question is whether coprime realizations do exist. This is a crucial point, since coprime realizations are minimal in the class of Roesser's models.

3.1 Local Reachability and Observability

We say that a local state $\bar{x} \in X$ is reachable if there exist an input $u \in K[[z_1, z_2]]$ and integers $i > 0$, $j > 0$ such that $x(i,j) = \bar{x}$, when the system starts from $\mathcal{X}_o$ identically zero.

By shift invariance we have the following definition:

Definition 3 A state $x \in X$ is *reachable* if $x = ((I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 - B_2 z_2)u, 1)$ for some $u \in K[z_1^{-1}, z_2^{-1}]$.

Then, the *reachable local state space* is

$$X^R = \{x : x = ((I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2)u, 1),\ u \in K[z_1^{-1} z_2^{-1}]\}$$

Let define the following matrices

$$A_1^{r \sqcup\!\sqcup 0} A_2 = A_1^r, \quad A_1^{0 \sqcup\!\sqcup s} A_2 = A_2^s$$

$$A_1^{r \sqcup\!\sqcup s} A_2 = A_1(A_1^{r-1 \sqcup\!\sqcup s} A_2) + A_2(A_1^{r \sqcup\!\sqcup s-1} A_2), \quad r, s \geq 1$$

Then

$$X^R = \mathrm{span}\{B_1, B_2, (A_1{}^{1}\sqcup\!\sqcup{}^{0} A_2)B_1, (A_1{}^{1}\sqcup\!\sqcup{}^{0} A_2)B_2 +$$

$$+ (A_1{}^{0}\sqcup\!\sqcup{}^{1} A_2)B_1, \ldots, (A_1{}^{i}\sqcup\!\sqcup{}^{j-1} A_2)B_2 +$$

$$+ (A_1{}^{i-1}\sqcup\!\sqcup{}^{j} A_2)B_1, \ldots, \} =$$

$$= \mathrm{range}\left[B_1 \; B_2 \ldots (A_1{}^{i}\sqcup\!\sqcup{}^{j-1} A_2)B_2 + (A_1{}^{i-1}\sqcup\!\sqcup{}^{j} A_2)B_1 \ldots\right.$$

$$\left.\ldots (A_1{}^{0}\sqcup\!\sqcup{}^{n-1} A_2)B_2 \ldots\right]$$

As a consequence of a generalization of Cayley-Hamilton theorem [9], X^R is the range of the $n \times (n+2)(n-1)/2$ *local reachability matrix*

$$\mathscr{R}_L = \left[B_1 \; B_2 \; \ldots \; (A_1{}^{i}\sqcup\!\sqcup{}^{j-1} A_2)B_2 + \right.$$
$$\left. + (A_1{}^{i-1}\sqcup\!\sqcup{}^{j} A_2)B_1 \; \ldots \; (A_1{}^{0}\sqcup\!\sqcup{}^{n-1} A_2)B_2\right]. \tag{15}$$

The system is *locally reachable* if $X^R = X$, i.e. if $\mathscr{R}_L$ has full rank.

A state x is indistinguishable from the state $0 \in X$ if the free output evolution resulting from it is identically zero. Then we have the following definition

<u>Definition 4.</u> A state $x \in X$ is *indistinguishable* from the state $0 \in X$ if

$$C(I - A_1 z_1 - A_2 z_2)^{-1} x = 0$$

The *indistinguishable local state space* X^I is defined as

$$X^I = \{x : x \in X,\ C(I - A_1 z_1 - A_2 z_2)^{-1} x = 0\} =$$

$$= \bigcap_{i,j} \ker C(A_1{}^{i}\sqcup\!\sqcup^{j} A_2) = \ker \mathcal{O}_L$$

where the $(n(n-1)/2) \times n$ matrix

$$\mathcal{O}_L \triangleq \begin{bmatrix} C \\ \vdots \\ C(A_1{}^{i}\sqcup\!\sqcup^{j} A_2) \end{bmatrix}_{i+j<n} \tag{16}$$

is the *local observability matrix.*

The system is *locally-observable* if $X^I = \{0\}$, i.e. if $\mathcal{O}_L$ is full rank.

<u>Remark</u> As well known, the reachability and indistinguishability subspaces X^R and X^I of a 1D system (3) are A-invariant. For the 2D case, we don't have an analogous property and X^R and X^I are not in general A_1 and A_2 invariant. This can be checked directly on the following 2D system

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C = [\,0\ 1\ 1\ 0\,].$$

3.2 <u>Global Reachability and Observability</u> [17]

Local reachability condition ensures that, given any local state $x \in X$, there exists an input function which drives the 2D system from $\mathcal{X}_o = 0$ to $x(i,j) = x$, for some positive, finite integers i,j. In general there is no information either on the values assumed by the states at other points of $\mathscr{C}_{i+j}$, or on the possibility of selecting a different u which allows the 2D system to reach additional assigned local states on the same separation set. Intuitively, given more local states on the same separation set, the simultaneous reachability of these states de - pends on the possibility that the inputs which produce each given local state, assume the same values on the intersection of their supports. In general this imposes some constraints on the values of the local states we want to be reached. So, when we deal with simultaneous reachability of local states a stronger reachability notion, that is the so called glo - bal reachability, needs to be introduced.

In the analysis of global properties, it is convenient to introduce the indeterminates

$$\xi = z_1^{-1} z_2 , \quad \eta = z_1$$

and to use bilateral Laurent formal power series in the indeterminate ξ for representing global states and restrictions of input and output func tions to the separations sets:

$$\mathscr{X}_i = \sum_{j=-\infty}^{+\infty} x(i-j,j)\xi^j, \quad \mathscr{U}_i = \sum_{j=-\infty}^{+\infty} u(i-j,j)\xi^j, \quad \mathscr{Y}_i = \sum_{j=-\infty}^{+\infty} y(i-j,j)\xi^j \tag{17}$$

Then, input and output functions can be written as

$$u = \sum_{i=h}^{+\infty} \mathscr{U}_i \eta^i , \quad y = \sum_{i=k}^{\infty} \mathscr{Y}_i \eta^i$$

where h and k are integers. The set $K_b^m((\xi))$ of (bilateral) Laurent formal power series with coefficients in K^m can be naturally endowed with the structure of a $K[\xi,\xi^{-1}]$ - module, where $K[\xi,\xi^{-1}]$ is the subring of

$K(\xi)$ generated by K, ξ and ξ^{-1}. As a consequence of the module structure, the global state updating equations

$$\mathcal{X}_{i+1} = (A_1 + A_2\xi)\,\mathcal{X}_i + (B_1 + B_2\xi)\,\mathcal{U}_i\,, \quad \mathcal{Y}_i = C\,\mathcal{X}_i \tag{18}$$

are easily derived from (5).

A global state $\bar{\mathcal{X}} \in X$ is reachable if there exist an integer i and an input $u = \sum_{j=0}^{\infty} \mathcal{U}_j \eta^j$ such that $\mathcal{X}_i = \bar{\mathcal{X}}$ when $\mathcal{X}_o = 0$. By shift invariance, we therefore have the following definition

Definition 5. A global state $\mathcal{X} \in X$ is *reachable* if

$$\mathcal{X} = \sum_j (A_1+A_2\xi)^j (B_1+B_2\xi)\,\mathcal{U}_j$$

for some finite sequence $\mathcal{U}_{-1}, \mathcal{U}_{-2}, \ldots, \mathcal{U}_{-k}$ with elements in $K_b((\xi))$.

The 2D system is *globally reachable* if any global state $\mathcal{X}$ in X is reachable.

Proposition 3. *The 2D system (5) is globally reachable if and only if the matrix*

$$\mathcal{R} = \left[B_1+B_2\xi \quad (A_1+A_2\xi)(B_1+B_2\xi) \quad \ldots \quad (A_1+A_2\xi)^{n-1}(B_1+B_2\xi) \right] \tag{19}$$

is full rank, i.e. $\det \mathcal{R} \neq 0$.

proof. Recalling the Cayley-Hamilton theorem, the system is globally reachable if an only if the equation

$$\mathcal{X}_o = \mathcal{R} \begin{bmatrix} \mathcal{U}_{-1} \\ \mathcal{U}_{-2} \\ \vdots \\ \mathcal{U}_{-n} \end{bmatrix},$$

where the unknowns $\mathcal{U}_{-1}, \mathcal{U}_{-2}, \dots, \mathcal{U}_{-n}$ are Laurent formal power series, has a solution for any global state $\mathcal{X}_o$. Denoting by V the unimodular polynomial matrix which reduces $\mathcal{R}$ to its row Hermite form, we have

$$\mathcal{X}_o = (\mathcal{R}V)V^{-1}\begin{bmatrix} \mathcal{U}_{-1} \\ \mathcal{U}_{-2} \\ \vdots \\ \mathcal{U}_{-n} \end{bmatrix} = \begin{bmatrix} r_{11}(\xi) & 0 & \dots & 0 \\ r_{21}(\xi) & r_{22}(\xi) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ r_{n1}(\xi) & r_{n2}(\xi) & \dots & r_{nn}(\xi) \end{bmatrix} \begin{bmatrix} \mathcal{U}'_{-1} \\ \mathcal{U}'_{-2} \\ \vdots \\ \mathcal{U}'_{-n} \end{bmatrix} \qquad (20)$$

where

$$\begin{bmatrix} \mathcal{U}'_{-1} \\ \mathcal{U}'_{-2} \\ \vdots \\ \mathcal{U}'_{-n} \end{bmatrix} = V^{-1} \begin{bmatrix} \mathcal{U}_{-1} \\ \mathcal{U}_{-2} \\ \vdots \\ \mathcal{U}_{-n} \end{bmatrix}$$

If $x(\xi) \in K_b((\xi))$ and $r(\xi) \in K[\xi]$ there exist non-zero Laurent formal power series which satisfy the equation $x(\xi) = r(\xi)u(\xi)$ if and only if $r(\xi) \neq 0$. Hence equation (20) is solvable for any $\mathcal{X}_o$ if and only if the diagonal polynomials $r_{ii}(\xi)$ are all different from 0, that is if and only if $\det \mathcal{R} \neq 0$.

The simultaneous observation of initial local states on a separation set raises a problem which is - in some way - analogous to that we encounter in simultaneous reachability. In this case to estimate more local states on the same separation set we use output functions whose supports partially overlap, so that the same output values in the intersection may result from different sequences of local states on the separation set.

Definition 6. A global state $\mathscr{X} \in X$ is *indistinguishable* from the zero global state if the free output evolution resulting from $\mathscr{X}$ is identically zero, i.e. if

$$C(A_1 + A_2\xi)^j \mathscr{X} = 0 \qquad j = 0,1,\dots,$$

A 2D system is *globally observable* if any nonzero global state is distinguishable from zero.

Proposition 4. *The 2D system (5) is globally observable if and only if the matrix*

$$\mathcal{O} = \begin{bmatrix} C \\ C(A_1+A_2\xi) \\ \vdots \\ C(A_1+A_2\xi)^{n-1} \end{bmatrix} \tag{21}$$

has an inverse in $K[\xi,\xi^{-1}]$, *i.e.* $\det \mathcal{O} = k\xi^m$ *for some integer* m *and some nonzero element* k *in* K.

proof. The global observability condition is equivalent, by definition, to require that $\mathcal{O}\mathscr{X} = 0$ implies $\mathscr{X} = 0$.

Suppose now that W is a unimodular polynomial matrix which reduces $\mathcal{O}$ to its row Hermite form. Then

$$\mathcal{O} W W^{-1}\mathscr{X} = \begin{bmatrix} \omega_{11}(\xi) & 0 & \dots & 0 \\ \omega_{21}(\xi) & \omega_{22}(\xi) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \omega_{n1}(\xi) & \omega_{n2}(\xi) & \dots & \omega_{nn}(\xi) \end{bmatrix} \mathscr{X}' = 0,$$

with $\mathscr{X}' = W^{-1}\mathscr{X}$, implies $\mathscr{X}' = 0$.

Given a polynomial $\omega(\xi)$ in $K[\xi,\xi^{-1}]$, there exists a unique scalar Laurent formal power series $x(\xi)$ satisfying $\omega(\xi)x(\xi) = 0$ if and only if $\omega(\xi) = k\xi^h$ for some integer h and non-zero k in K. Since the unit elements in $K[\xi,\xi^{-1}]$ have the form $k\xi^h$, $k \neq 0$, the above condition corresponds to assume $\omega(\xi)$ be invertible in $K[\xi,\xi^{-1}]$. Hence a necessary and sufficieny condition for global observability is that all polynomials ω_{ii} - and consequently det $\mathcal{O}$ - have an inverse over $K[\xi,\xi^{-1}]$.

Remark. If the matrix $\mathcal{O}$ is full rank but det $\mathcal{O}$ is not invertible in $K[\xi,\xi^{-1}]$, the subspace of global states which are indistinguishable from zero is finite dimensional over K. In fact, let $\alpha_{ij}\xi^j$ be the monomial of minimal degree in $\omega_{ii}(\xi)$ and assume $\nu_i = \deg \omega_{ii}(\xi) - j$. Then $\Sigma_{i=1}^n \nu_i$ gives the dimension of the subspace of zero indistinguishable global states. This situation is illustrated in the following example. Assume

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = [1 \quad 0].$$

Since the observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1+\xi \end{bmatrix},$$

all states having the structure

$$\mathcal{X} = \begin{bmatrix} 0 \\ k \sum_{i=-\infty}^{+\infty} (-1)^i \xi^i \end{bmatrix}, \qquad k \in K.$$

are indistinguishable from zero.

Notice that zero indistinguishable global states cannot be represented by truncated Laurent formal power series.

An intermediate problem between global and local reachability and observability consists in reachability and observability of finite sequences of local states on a separation set. In a first approach to this problem, system (18) is considered over the ring $K[\xi,\xi^{-1}]$: inputs and outputs have compact supports on every separation set and states are elements of $K[\xi,\xi^{-1}]^n$. So, by applying the theory of linear systems over rings the reachability conditions becomes

$$\det \mathscr{R} = k\xi^h , \quad k \neq 0 , \tag{22}$$

and the observability condition

$$\det \mathscr{O} \neq 0 . \tag{23}$$

A different approach is based on the fact that global reachability and observability conditions of Propositions 3 and 4 apply to finite sequences of local states on $\mathscr{C}_o$.

— The fact that global reachability condition $\det \mathscr{R} \neq 0$ is w e a k e r than (22) may be given an interpretation in terms of finite sequences of local states.

Proposition 5. *The 2D system (5) is globally reachable if and only if there esists an integer* $N(\geq n^2)$ *such that any set of local states*

$$x(0,0), \quad x(-1,1), \; \ldots , \quad x(-N+1, N-1) \tag{24}$$

on $\mathscr{L} := \{0,0 \; , \; (-1,1), \ldots, (-N+1, N-1)\}$ *is the restriction to* $\mathscr{L}$ *of a global state on* $\mathscr{C}_o$ *produced by some input function with compact support on* $Z \times Z$ (*).

(*) Note that this condition does not impose any constraint on the values assumed by the local states on $\mathscr{C}_o - \mathscr{L}$.

proof. Clearly global reachability implies reachability of (24). Conversely, assume that the following sequences of length $N = n^2$,

$$e_i + 0\xi + 0\xi^2 + \ldots + e_i\xi^{N-1}, \qquad i = 1,2,\ldots,n,$$

where e_i denotes the i-th column of the $n \times n$ identity matrix, are reachable. Then there exist scalars a_{ij} and b_{ij} such that the global states

$$\sum_{j=1}^{n} a_{ij}\xi^{-j} + e_i + e_i\xi^{N-1} + \sum_{j=1}^{n} b_{ij}\xi^{j+N}, \qquad i = 1,2,\ldots,n, \qquad (25)$$

are reachable. Since the n global states in (25) are linearly independent over $K((\xi))$, $\det \mathscr{R} \neq 0$.

Let us now consider the observability case. Here the global condition $\det \mathscr{O} = k\xi^h$, $k \neq 0$, is stronger than (23) and corresponds to the following observability condition on finite sequences of local states. Let $\mathscr{Q} = \{(0,0), (-1,1), \ldots, (-n,n)\}$ be a subset of $\mathscr{C}_0$. A sequence of local states on $\mathscr{Q}$ is observable if the free output values on some finite subset $\mathscr{F} \subseteq \mathbb{Z} \times \mathbb{Z}$ uniquely determine $x(0,0), x(-1,1), \ldots, x(-n,n)$. In fact we have the following proposition:

Proposition 6. *The following facts are equivalent:*

(i) *the 2D system (5) is globally observable;*

(ii) *there exists a finite subset* $\mathscr{F} \subseteq \mathbb{Z} \times \mathbb{Z}$ *such that* $x(0,0)$ *is uniquely determined by the free output values on* $\mathscr{F}$*, whatever local states may be given on* $\mathscr{C}_0 \setminus \{(0,0)\}$;

(iii) *any finite sequence of local states is observable.*

Proof. (i)$\Leftrightarrow$(ii). Obviously, non-zero local states belonging to an unobservable global state cannot be determined from the output values. Conversely, let the 2D system (5) be globally observable. Then $\mathcal{O}$ is invertible over $K[\xi,\xi^{-1}]$ and from

$$\mathcal{X} = \sum_{i=-\infty}^{+\infty} x_i \xi^i = \mathcal{O}^{-1} \begin{bmatrix} \mathcal{Y}_o \\ \mathcal{Y}_1 \\ \vdots \\ \mathcal{Y}_{n-1} \end{bmatrix}$$

we have

$$x(0,0) = \left(\mathcal{O}^{-1} \begin{bmatrix} \mathcal{Y}_o \\ \mathcal{Y}_1 \\ \vdots \\ \mathcal{Y}_{n-1} \end{bmatrix} , \xi^o \right) . \tag{26}$$

Since the entries of $\mathcal{O}^{-1}$ are in $K[\xi,\xi^{-1}]$, the computation of $x(0,0)$ in (26) involves finitely many coefficients of $\mathcal{Y}_o, \mathcal{Y}_1, \dots, \mathcal{Y}_{n-1}$, that is the output values on a finite subset $\mathcal{F}$ of $\mathbb{Z} \times \mathbb{Z}$.

(ii)$\Leftrightarrow$(iii) Using (26) and the shifting property, any finite sequence of local states on a separation set can be computed from a finite set of output values.

Remark. When we deal with the local observability condition, all local states on a separation set are assumed to be zero, except the local state to be determined. At point (ii) of Proposition 6, the assumptions are more general, since no restrictions are imposed on the local states $x(i,-i)$, $i \neq 0$. This explains why global observability, which has been proved to be equivalent to (ii), is a stronger property than local observability.

3.3 Modal Reachability and Observability

It is well known in 1D system theory that a pair (A,B) is reachable if and only if (sI-A) and B are left-coprime and a pair (C,A) is observable if and only if C and (sI-A) are right-coprime.

For 2D systems, whatever local state model we use, there is not immediate equivalence between coprimeness and either local or global reachability and observability.

The notions of modal reachability and observability have been introduced for Roesser's models and then used mostly for this type of state equations [11]. There isn't any difficulty in using these notions for other 2D structures but also any advantage, so we shall confine our attention to Roesser's structures, where the particular form of matrices (10) requires a simpler formalism.

Recall that two matrices $P(z_1,z_2)$ and $Q(z_1,z_2)$ over $K[z_1,z_2]$ with the same number of rows, are left coprime if for every left common factor $D(z_1 z_2)$, such that

$$P(z_1,z_2) = D(z_1,z_2)\bar{P}(z_1,z_2)$$

$$Q(z_1,z_2) = D(z_1,z_2)\bar{Q}(z_1,z_2)$$

det $D(z_1,z_2)$ is a non zero element of K . A dual definition holds for right coprimeness.

<u>Definition 7.</u> System (8) is *modally reachable* if

$$\begin{bmatrix} z_1^{-1}I & 0 \\ 0 & z_2^{-1}I \end{bmatrix} - \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}$$

are left coprime and is *modally observable* if

$$\begin{bmatrix} z_1^{-1}I & 0 \\ 0 & z_2^{-1}I \end{bmatrix} - \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, \quad [\hat{C}_1 \quad \hat{C}_2]$$

are right coprime.

The coprimeness of polynomial matrices in two indeterminates can be checked using the following result [11]:

<u>Proposition 2.</u> *Assume* $P(z_1,z_2)$ *and* $Q(z_1,z_2)$ *be polynomial matrices of size* $n \times n$ *and* $n \times m$. *Then* P *and* Q *are left coprime if and only if*

$$\operatorname{rank}[P(\theta_1,\theta_2) \quad Q(\theta_1,\theta_2)] = n$$

for any generic point (θ_1,θ_2) *of any irreducible algebraic curve* W *of* Θ^2, *where* Θ *is a universal extension field of* K.

The connections among local, global and modal reachability and observability are rather weak, and mutual implications do not exist.

The next example gives us an idea of what the situation is. The following matrices

$$C = [2 \quad 1] \text{ and } \begin{bmatrix} z_1^{-1}I & 0 \\ 0 & z_2^{-1}I \end{bmatrix} - A =$$

$$= \begin{bmatrix} z_1^{-1} & 0 \\ 0 & z_2^{-1} \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

are right coprime. However the corresponding Roesser's model is not globally observable since the global state

$$\mathscr{X}_o = x(i,-i) = \left(-\frac{2}{3}\right)^i \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

is indistinguishable from the zero global state. Thus, while globally reachable (observable) Roesser's models are modally reachable (observable), the viceversa is not true [11].

A similar picture of implications may be given for model (5), when left coprimeness of matrices $(z_1^{-1}z_2^{-1}I - A_2z_1^{-1} - A_1z_2^{-1})$, $(B_1z_2^{-1} + B_2z_1^{-1})$ and right coprimeness of C and $(z_1^{-1}z_2^{-1}I - A_2z_1^{-1} - A_1z_2^{-1})$ are considered.

3.4 Duality

In this section global reachability and observability of 2D systems are related to observability and reachability of systems defined over rings [17,18]. The basic idea of connecting these properties descends from the remark that for 2D systems global reachability is equivalent to the nonsingularity of $\mathscr{R}$ and global observability to the unimodularity of $\mathscr{O}$, while for systems over rings the conditions on $\mathscr{R}$ and $\mathscr{O}$ are in some way exchanged.

Consider the system

$$w(t+1) = F(\xi)w(t) + G(\xi)v(t) \quad , \quad z(t) = H(\xi)w(t) \quad , \tag{27}$$

defined over the ring of polynomials $K[\xi,\xi^{-1}]$. Here the input set is the ring $K[\xi,\xi^{-1}][\eta^{-1}]$, the output set is the ring $K[\xi,\xi^{-1}][[\eta]]$, the states are elements of the free module $K[\xi,\xi^{-1}]^n$ and the matrices $F(\xi)$, $G(\xi)$, $H(\xi)$ have entries in $K[\xi,\xi^{-1}]$.

Denote by $\mathscr{R}_p$ and $\mathscr{O}_p$ the reachability and observability matrices of (27):

$$\mathscr{R}_p = \left[G(\xi) \quad F(\xi)G(\xi) \quad \dots \; F(\xi)^{n-1}G(\xi) \right] \tag{28}$$

$$\mathscr{O}_p = \begin{bmatrix} H(\xi) \\ H(\xi)F(\xi) \\ \vdots \\ H(\xi)F(\xi)^{n-1} \end{bmatrix} \tag{29}$$

It is well known from the theory of systems over rings [16], that reachability of (27) is equivalent to unimodularity of $\mathscr{R}_p$ as a matrix with elements in $K[\xi,\xi^{-1}]$ and observability of (27) is equivalent to assume $\mathscr{O}_p$ to be full rank.

We therefore have that structural properties of systems over rings (27) as well as global properties of 2D systems are expressed in terms of polynomial matrices with elements in $K[\xi,\xi^{-1}]$. The following table shows that the conditions on $\mathscr{R}$ ($\mathscr{O}$) which guarantee global reachability (observability) of a 2D systems are formally the same conditions on $\mathscr{O}_p$ ($\mathscr{R}_p$) which guarantee the observability (reachability) of systems over rings.

	reachability	observability
System (27)	$\mathscr{R}_p$ unimodular	$\mathscr{O}_p$ full rank
2D System	$\mathscr{R}$ full rank	$\mathscr{O}$ unimodular

This suggests to interprete 2D systems as dual of suitable systems over rings.

Since the input and output alphabets and the state set of (27) are K-linear vector spaces, system (27) can be viewed as an infinite dimensional linear system over K.

The construction of the dual system is based on the following facts:

1. The space $K_b^m((\xi))$ of Laurent formal power series with coefficients in K^m is the algebraic dual of $K^m[\xi,\xi^{-1}]$:

$$(K^m[\xi,\xi^{-1}])^* = K_b^m((\xi)) .$$

In fact, let $s = \Sigma_{i=-\infty}^{\infty} s_i \xi^i$ be in $K_b^m((\xi))$ and $r = \Sigma_j r_j \xi^j$ by any element of $K^m[\xi,\xi^{-1}]$. Then the series s defines a linear functional on $K^m[\xi,\xi^{-1}]$ by assuming

$$< s,r > \overset{\Delta}{=} \Big(\sum_{i=-\infty}^{\infty} s_i^T \xi^{-i} \sum_j r_j \xi^j , 0 \Big) \tag{30}$$

where s_i^T is the transpose of s_i. Conversely, every linear functional on $K^m[\xi,\xi^{-1}]$ can be represented by a Laurent power series.

Then assuming in (30) $s = \mathscr{X}$ and $r = w$, it turns out that the global state space of (5) is the algebraic dual of the state space of (27).

Similarly, the input sequences $\mathscr{U} = \Sigma_{i=-\infty}^{+\infty} u_i \xi^i$ and the output sequences $\mathscr{Y} = \Sigma_{i=-\infty}^{+\infty} y_i \xi^i$ over a separation set can be viewed as linear functionals on the spaces of the input and output values of (27)

2. The output space $K_b((\xi))[[\eta]]$ of the 2D system (5), that is the space whose elements are the series

$$\sum_{i=0}^{\infty} \mathscr{Y}_i \eta^i ,$$

where $\mathscr{Y}_i = \Sigma_{j=-\infty}^{+\infty} y(i-j,j)\xi^j$, is the algebraic dual of the input space of system (27):

$$(\eta^{-1}K[\xi,\xi^{-1}][\eta^{-1}])^* = K_b((\xi))[[\eta]] .$$

This follows by assuming

$$< \sum_{i=0}^{\infty} \sum_{j=-\infty}^{+\infty} y(i-j,j)\xi^j \eta^i \; , \; \sum_{q=1}^{Q} \sum_{p=-P}^{P} v(-q-p,p)\xi^p \eta^{-q} >$$

$$= \sum_{q=0}^{Q} < \sum_{j=-\infty}^{\infty} y(-q-j,j)\xi^j \; , \; \sum_{p} v(-q-1-p,p)\xi^p > .$$

Similarly, the space $\eta^{-1}K_b((\xi))[\eta^{-1}]_n$ of inputs of system (5) with length at most n, i.e. the space whose elements are the series

$$\sum_{i=1}^{n} \mathcal{U}_{-i}\eta^{-i}$$

is the algebraic dual of the space of the output restrictions to $[0,n-1]$ of system (27)

$$(K[\xi,\xi^{-1}][\eta]_n)^* = \eta^{-1}K_b((\xi))[\eta^{-1}]_n .$$

With these two facts we are now ready to show that the global rea - chability and observability maps in system (5) are the algebraic dual of the observability and reachability maps of system (27).

Let ρ and ω be the reachability and observability maps of (27):

$$\rho: \; \eta^{-1}K[\xi,\xi^{-1}][\eta^{-1}] \to K[\xi,\xi^{-1}]^n :$$

$$\sum_{q=1}^{Q} \sum_{p} v(-q-p,p)\xi^p \eta^{-q} \mapsto \sum_{q=1}^{Q} F(\xi)^{q-1} G(\xi) \sum_{p} v(-q-p,p)\xi^p ,$$

$$\omega: \; K[\xi,\xi^{-1}]^n \to K[\xi,\xi^{-1}][[\eta]] : \; \sum_{i} x_i \xi^i \mapsto \sum_{j=0}^{\infty} F(\xi)^j H(\xi) \sum_{i} x_i \xi^i \eta^j ,$$

and denote by π_n the projection map of $K[\xi,\xi^{-1}][[\eta]]$ onto the sub - space $K[\xi,\xi^{-1}][\eta]_n$ of polynomials with degree less than n .

In the diagram

$$\begin{tikzcd} \eta^{-1}K[\xi,\xi^{-1}][\eta^{-1}] \arrow[r, "\rho"'] & K[\xi,\xi^{-1}]^n \arrow[rr, "\omega_n"] \arrow[dr, "\omega"'] & & K[\xi,\xi^{-1}][\eta]_n \\ & & K[\xi,\xi^{-1}][[\eta]] \arrow[ur, "\pi_n"'] & \end{tikzcd}$$

ρ is injective if and only if system (27) is reachable, ω and (by the Cayley-Hamilton theorem) $\omega_n := \pi_n \circ \omega$ are surjective if and only if system (27) is observable.

In the dual sequence

$$K_b((\xi))[[\eta]] \xleftarrow{\rho^*} K_b((\xi))^n \xleftarrow{\omega_n^*} \eta^{-1}K_b((\xi))[\eta^{-1}]_n$$

ρ^* and ω_n^* are the dual maps of ρ and ω_n and provide the observability and the n-steps reachability maps of the dual system of (27) given by

$$\bar{w}(t+1) = F^T(\xi)\bar{w}(t) + H^T(\xi)\bar{z}(t) \ , \quad \bar{v}(t) = G^T(\xi)\bar{w}(t) \ , \tag{31}$$

where the input and output alphabets are Laurent formal power series and the state space is $K_b((\xi))^n$.

Then, from the theory of dual spaces we have that ρ^* is injective if and only if ρ is surjective and ω_n^* is surjective if and only if ω_n is injective.

Since the reachability map of system (31)

$$\omega' : \eta^{-1}K_b((\xi))[\eta^{-1}] \longrightarrow K_b((\xi))^n$$

is surjective if and only if ω_n^* is surjective, reachability (observability) of system (31) is equivalent to observability (reachability) of system (27). In particular, assuming

$$F(\xi) = A_1^T + A_2^T\xi, \quad G(\xi) = C^T \ , \quad H(\xi) = B_1^T + B_2^T\xi \ ,$$

we have proved the following proposition:

Proposition 8. *The 2D system (5) is globally reachable (observable) if and only if the system*

$$w(t+1) = (A_1^T + A_2^T\xi)w(t) + C^T v(t) \;, \qquad z(t) = (B_1^T + B_2^T\xi)w(t) \;, \quad (32)$$

defined over the ring $K[\xi,\xi^{-1}]$ *is observable (reachable).*

This result provides also an alternative approach to global reachability and observability criteria, based on the theory of systems over rings. In fact, system (32) is reachable if and only if its reachability matrix

$$\mathscr{R}' = \left[C^T \,\middle|\, (A_1^T + A_2^T\xi)C^T \,\middle|\, \dots \;\; (A_1^T + A_2^T\xi)^{n-1} C^T \right]$$

has an inverse in the ring $K[\xi,\xi^{-1}]$, i.e. if and only if $\det \mathscr{R}' = k\xi^h$, for some non-zero k in K and some integer h. Then, since $\mathcal{O} = \mathscr{R}'^T$, the duality theorem provides the global observability condition given in Proposition 4.

Analogously, the 2D global reachability condition of Proposition 3 follows by duality from the non singularity of the observability matrix of system (32)

$$\mathcal{O}' = \begin{bmatrix} (B_1^T + B_2^T\xi) \\ (B_1^T + B_2^T\xi)(A_1^T + A_2^T\xi) \\ \vdots \\ (B_1^T + B_2^T\xi)(A_1^T + A_2^T\xi)^{n-1} \end{bmatrix} .$$

4. STRUCTURE OF 2D REALIZATIONS

Given a rational transfer function in two indeterminates, we know by Proposition 2, that there exists a 2D system which constitutes a state space realization of it.

As with 1D systems, the problem that naturally arises is how to obtain an "efficient realization". It is well known that for 1D systems there is a complete solution to this problem in terms of minimal dimension realizations which are fully characterized as reachable and observable and, for a given transfer function, are unique modulo similarity. Also, there exist linear techniques based on a finite number of steps which give such minimal realizations.

The same problem appears to be much more involved for 2D systems and still far from being solved. In the following we shall give a short list of questions ordered according to an increasing degree of theoretical interest and of mathematical difficulty:

i) are there available computational techniques for an effective construction of low order 2D systems which realize a rational transfer function?

ii) what relevance could be assigned to structural properties (reachability and observability) in computing efficient realizations and in reducing the dimension of a given 2D state model?

iii) how minimal realizations can be computed and what is the algebraic structure of their set?

In this section we examine what answers can be given using the results so far available.

4.1 Low Dimensional Realizations

At this moment there isn't any general technique for constructing 2D realizations with minimal dimension. This makes interesting to look for linear procedures which provide at least low dimensional realizations. There are mainly two approaches, leading to realizations with Roesser's structure. In both cases the underlying idea consists in considering the transfer function in two indeterminates as a transfer function in one indeterminate (for example z_1) over the field of rational functions in z_2)..

The first realization technique [19,11] reduces essentially to an implementation method for a 2D rational transfer function

$$s = \sum_{i=o}^{M} \sum_{j=o}^{N} p_{ij} z_1^i z_2^j \Big/ \sum_{i=o}^{M} \sum_{j=o}^{N} q_{ij} z_1^i z_2^j , \qquad q_{oo} \neq 0 ,$$

using some delay elements z_1 and z_2. The circuit implementation corresponds directly to a state space model of Roesser's type. The outputs of the z_1 delays are the horizontal states and the outputs of the z_2 delays are the vertical states. The realization thus obtained has dimension $N+2M$ (or $M+2N$).

The second method [12,13] is based on two steps: the "first level realization" provides a 1D system with matrices having entries on the field of rational functions in one variable, and the "second level realization" gives the system matrices in Roesser's form over the ground field K. This method applies also to multi input-output 2D transfer functions, however we shall only sketch the procedure in the single input-single output case.

Consider a rational transfer function

$$s = \frac{p(z_1^{-1}, z_2^{-1})}{q(z_1^{-1}, z_2^{-1})}$$

and let $m = \deg_{z_1^{-1}} q$, $n = \deg_{z_2^{-1}} q$. The strict properness of s is equivalent to assume that

i) the coefficient of $z_1^{-m} z_2^{-n}$ in q is different from zero

ii) $\deg_{z_1^{-1}} p \leq m$, $\deg_{z_2^{-1}} p \leq n$, and the coefficient of $z_1^{-m} z_2^{-n}$ in p is zero.

Now, rewrite the polynomials p and q as polynomials in z_1^{-1} with coefficients in $K(z_2^{-1})$

$$s = \frac{\sum_{i=0}^{m} a_i(z_2^{-1}) z_1^{-i}}{\sum_{i=0}^{m} b_i(z_2^{-1}) z_1^{-i}} = \frac{\sum_{i=0}^{m} \alpha_i(z_2) z_1^{-i}}{\sum_{i=0}^{m} \beta_i(z_2) z_1^{-i}}$$

where $a_i(z_2^{-1})$, $b_i(z_2^{-1})$ belong to $K[z_2^{-1}]$ and $\alpha_i(z_2)$, $\beta_i(z_2)$ are defined by

$$\alpha_i(z_2) = \frac{a_i(z_2^{-1})}{b_m(z_2^{-1})}, \quad \beta_i(z_2) = \frac{b_i(z_2^{-1})}{b_m(z_2^{-1})} \tag{33}$$

Assume - for simplicity - $\alpha_m(z_2) = 0$, and construct the *first level realization* of s in controllability canonical form

$$F(z_2) = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ -\beta_o(z_2) & -\beta_1(z_2) & \cdots & -\beta_{m-1}(z_2) \end{bmatrix}.$$

$$G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$H(z_2) = \left[\alpha_o(z_2)\ \alpha_1(z_2) \ldots \alpha_{m-1}(z_2) \right]$$

Since $\alpha_i(z_2)$ and $\beta_i(z_2)$ are proper rational functions, matrices $F(z_2)$ and $H(z_2)$ are realizable by 1D systems over the field K, that is

$$\begin{aligned} F(z_2) &= C_F(z_2^{-1} I - A_F)^{-1} B_F + D_F \\ H(z_2) &= C_H(z_2^{-1} I - A_H)^{-1} B_H + D_H \end{aligned} \tag{34}$$

If realizations (34) are reachable and observable, the assumptions (33) imply that matrices A_F and A_H have dimension less than or equal to n.

The sequence of matrices A_F, B_F, C_F, D_F, G, A_H, B_H, C_H, D_H is called the *second level realization* of s and gives the following system in Roesser's form

$$\begin{bmatrix} x^h(h+1,k) \\ \hline x_1^v(h,k+1) \\ x_2^v(h,k+1) \end{bmatrix} = \begin{bmatrix} D_F & C_F & 0 \\ \hline B_F & A_F & 0 \\ B_H & 0 & A_H \end{bmatrix} \begin{bmatrix} x^h(h,k) \\ \hline x_1^v(h,k) \\ x_2^v(h,k) \end{bmatrix} + \begin{bmatrix} G \\ \hline 0 \\ 0 \end{bmatrix} u(h,k)$$

$$y(h,k) = \left[D_H \;\vdots\; 0 \quad C_H \right] \begin{bmatrix} x^h(h,k) \\ \hline x_1^v(h,k) \\ x_2^v(h,k) \end{bmatrix}$$

which realizes the transfer function s with dimension $m+2n$.

4.2 Reachable and Observable Realizations

Given a realization of a 2D transfer function, we may ask whether it is possible to obtain a more efficient one, by reducing the dimension of the local state space.

In 1D case, a standard technique consists in constructing Kalman reachability and/or observability canonical forms and then in suppressing the unreachable and/or unobservable parts.

A similar procedure, referring to local reachability and observability, can be applied to 2D state models [20]. However, in this case the power of the method is quite reduced, since it does not necessarily lead to minimal realizations. On the other hand if we try to extend the 1D method in connection with 2D global (or modal) properties, we get into severe troubles, since the existence of globally (or modally) reachable and observable realizations is not guaranteed. Thus we are in some way constrained to look for reduction procedures based on local properties.

To this purpose, a computational procedure developed in [20] is presented in the remainder of this section.

Since locally reachable and indistinguishable subspaces are not

A_1 and A_2 invariant, the proof of the reduction algorithm cannot be derived by direct extension from 1D theory and looks more involved, though the algorithm itself has the same features.

Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of dimension n of a transfer function, and let $\mathscr{R}_L$ be the local reachability matrix of Σ. Assume rank $\mathscr{R}_L = r < n$. The algorithm for constructing a locally reachable realization of s of dimension r is based on the following two steps.

Step 1. Construct a matrix $T \in Gl(n,K)$ having the last $n-r$ rows orthogonal to the space spanned by the columns of $\mathscr{R}_L$.

Since the matrix T induces a change of basis in X, the system $\hat{\Sigma} = (\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2, \hat{C})$ defined by

$$
\begin{aligned}
\hat{A}_1 &= T A_1 T^{-1} \\
\hat{A}_2 &= T A_2 T^{-1} \\
\hat{B}_1 &= T B_1 \\
\hat{B}_2 &= T B_2 \\
\hat{C} &= C T^{-1}
\end{aligned}
$$

still realizes s. Obviously the first r elements of the new basis in X are a basis for X^R.

Step 2. Write $\hat{A}_1, \hat{A}_2$ in partitioned form:

$$\hat{A}_k = \begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ \hat{A}_{21}^{(k)} & \hat{A}_{22}^{(k)} \end{bmatrix}, \quad \hat{A}_{11}^{(k)} \in K^{r \times r}, \quad k = 1,2$$

and partition $\hat{B}_1$, $\hat{B}_2$ and $\hat{C}$ conformably:

$$\hat{B}_k = \left[\begin{array}{c} \hat{B}_1^{(k)} \\ \hline 0 \end{array}\right], \quad k = 1,2, \quad \hat{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}, \qquad \hat{B}_1^{(k)}, \hat{C}_1^T \in K^{r\times 1}$$

Then $(\hat{A}_{11}^{(1)}, \hat{A}_{11}^{(2)}, \hat{B}_1^{(1)}, \hat{B}_1^{(2)}, \hat{C}_1)$ is a locally reachable realization of s. In fact, let $x(h,k)$ be the state produced by an input $u \in \mathcal{U}$, and assume as a basis in $\underset{\sim}{X}$ the basis corresponding to $(\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2, \hat{C})$. With respect to this basis, the last n-r components of $x(h,k)$ are zero and the system

$$\begin{bmatrix} x_1(h+1,k+1) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11}^{(1)} & \hat{A}_{12}^{(1)} \\ \hat{A}_{21}^{(1)} & \hat{A}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_1(h+1,k) \\ 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} \hat{A}_{11}^{(2)} & \hat{A}_{12}^{(2)} \\ \hat{A}_{21}^{(2)} & \hat{A}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1(h,k+1) \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{B}_1^{(1)} \\ 0 \end{bmatrix} u(h+1,k) + \begin{bmatrix} \hat{B}_1^{(2)} \\ 0 \end{bmatrix} u(h,k+1)$$

$$y(h,k) = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} x_1(h,k) \\ 0 \end{bmatrix}$$

realizes the same input-output map as

$$x_1(h+1,k+1) = \hat{A}_{11}^{(1)} x_1(h+1,k) + \hat{A}_{11}^{(2)} x_1(h,k+1) +$$

$$+ \hat{B}_1^{(1)} u(h+1,k) + \hat{B}_1^{(2)} u(h,k+1)$$

$$y(h,k) = \hat{C}_1 x_1(h,k)$$

Assume now rank $\mathcal{O}_L = r' < n$. The algorithm for obtaining a locally ob servable realization of s is also based on two steps and is substantially the same as the above reachability algorithm, although the proof of the second step is based on two steps and is substantially the same as the above reachability algorithm, although the proof of the second step is based on somehow different reasonings.

<u>Step I.</u> Construct a matrix $Q^{-1} \in G\ell(n,K)$ having the last $n-r'$ columns orthogonal to the space spanned by the rows of $\mathcal{O}_L$. The 2D system $\Sigma = (\tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C})$ defined by

$$\begin{aligned}
\tilde{A}_1 &= Q A_1 Q^{-1} \\
\tilde{A}_2 &= Q A_2 Q^{-1} \\
\tilde{B}_1 &= Q B_1 \\
\tilde{B}_2 &= Q B_2 \\
\tilde{C} &= C Q^{-1}
\end{aligned}$$

realizes s and the elements in the last $n-r'$ columns of its local observability matrix

$$\tilde{\mathcal{O}}_L = \begin{bmatrix} \tilde{C} \\ \vdots \\ \tilde{C}(\tilde{A}_1 \; {}^i\sqcup\!\sqcup^j \; \tilde{A}_2) \end{bmatrix}_{i+j<n}$$

are zeros.

<u>Step II.</u> Write $\tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}$ in partitioned form as in step 2 above, and notice that $\tilde{C} = [\tilde{C}_1 \;\; 0]$, $\tilde{C}_1 \in K^{1\times r'}$. To show that $(\tilde{A}_{11}^{(1)}, \tilde{A}_{11}^{(2)}, \tilde{B}_1^{(1)}, \tilde{B}_1^{(2)}, \tilde{C}_1)$ is a realization of s we have to prove that

$$\begin{aligned}
s &= \tilde{C}(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2)^{-1}(\tilde{B}_1 z_1 + \tilde{B}_2 z_2) = \\
&= \tilde{C}_1(I - \tilde{A}_{11}^{(1)} z_1 - \tilde{A}_{11}^{(2)} z_2)^{-1}(\tilde{B}_1^{(1)} z_1 + \tilde{B}_1^{(2)} z_2)
\end{aligned}$$

For brevity the notation $\tilde{M}_{ij}$ will be used to denote the matrix $\tilde{A}_1 \; {}_{i}\!\sqcup\!\sqcup\!{}_{j} \; \tilde{A}_2$ and M_{ij} the matrix $\tilde{A}_{11}^{(1)} \; {}_{i}\!\sqcup\!\sqcup\!{}_{j} \; \tilde{A}_{11}^{(2)}$. Observe that for $i,j > 0$:

$$\tilde{C}\,\tilde{M}_{ij} = \tilde{C}\,\tilde{M}_{i-1,j}\tilde{A}_1 + \tilde{C}\,\tilde{M}_{i,j-1}\tilde{A}_2$$

$$\tilde{C}_1 M_{ij} = \tilde{C}_1 M_{i-1,j}\tilde{A}_{11}^{(1)} + \tilde{C}_1 M_{i,j-1}\tilde{A}_{11}^{(2)}$$

and assume by induction that the first r' elements of $\tilde{C}\tilde{M}_{i-1,j}$, $\tilde{C}\tilde{M}_{i,j-1}$ coincide with $\tilde{C}_1 M_{i-1,j}$, $\tilde{C}_1 M_{i,j-1}$, respectively. Then

$$\tilde{C}\,\tilde{M}_{i-1,j}\tilde{A}_1 = \begin{bmatrix} \tilde{C}_1 M_{i-1,j} & 0 \end{bmatrix} \begin{bmatrix} \tilde{A}_{11}^{(1)} & \tilde{A}_{12}^{(1)} \\ \tilde{A}_{21}^{(1)} & \tilde{A}_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} \tilde{C}_1 M_{i-1,j}\tilde{A}_{11}^{(1)} & * \end{bmatrix}$$

and similarly

$$\tilde{C}\,\tilde{M}_{i,j-1}\tilde{A}_2 = \begin{bmatrix} \tilde{C}_1 M_{i,j-1}\tilde{A}_{11}^{(2)} & * \end{bmatrix}$$

from which it is clear that the first r' elements in $\tilde{C}\,\tilde{M}_{ij}$ are the same as in $\tilde{C}_1 M_{ij}$. Of course the last $n-r'$ elements are zero because of the structure of $\mathcal{O}_L$.

The alternate application if the above algorithms leads eventually to a locally reachable and observable realization of s. So we have established the following result.

<u>Proposition 9.</u> *Let* $\Sigma = (A_1,A_2,B_1,B_2,C)$ *be a realization of* s. *Then a locally reachable and locally observable realization can be obtained by linear algorithms from* Σ *in a finite number of steps.*

Corollary 1. *Every minimal realization is locally-reachable and locally--observable.*

In general the converse of the above corollary does not hold. This can be easily seen from the following example.

Example: Let K be a field of characteristic 0 and consider

$$s = \frac{-z_2}{1-z_1^2-z_2^2}$$

The following 2D systems $\Sigma = (A_1, A_2, B_1, B_2, C)$:

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C = [\,0 \;\; 1\,]$$

and $\bar{\Sigma} = (\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C})$:

$$\bar{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{C} = [\,0 \;\; -1 \;\; 0\,]$$

are both locally reachable and observable realizations of s over the field K . Clearly $\bar{\Sigma}$ is not a minimal realization.

Remark The above construction is based on a change of basis in the local state space, so, if we start with a Roesser's structure, in general the system we eventually obtain does not keep the same structure. If we want to obtain a system which still belongs to the class of Roesser models, we can not use a general reduction procedure leading to locally reachable and ob servable realizations. In fact, as we shall see later, there exist mini - mal Roesser's realizations which are not locally reachable and observable.

4.3 Minimal Realizations

When we takle the problem of constructing 2D minimal realizations, we are faced by a series of facts which make the picture much more involved than in the 1D case.

We now consider some of the most important of these.

i) The dimension of minimal realizations of a 2D transfer function $\tilde{s}$ depends on the field in which the elements of the matrices are chosen. Let us illustrate the above concept by an example.

Example. Consider the transfer function

$$s = \frac{2\, z_1 z_2}{1+z_1^2+z_2^2} \qquad (35)$$

It admits the following complex realization of dimension 2:

$$A_1 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} i \\ -i \end{bmatrix}, \; C = [1 \;\; 1]$$

On the other side, the identity $1+z_1^2+z_2^2 = \det(1-A_1 z_1 - A_2 z_2)$ cannot be satisfied if A_1, A_2 belong to $\mathbb{R}^{2\times 2}$. For the contrary, assume

$$1 + z_1^2 + z_2^2 = \det \begin{bmatrix} 1+p & r \\ t & 1+q \end{bmatrix}$$

with p,q,r,t homogeneous polynomials of degree one in $\mathbb{R}[z_1, z_2]$. Since the left hand side does not include monomials of degree one, we get $p=-q$. Letting $p = \alpha z_1 + \beta z_2$, we have

$$(\alpha^2+1) z_1^2 + 2\alpha\beta\, z_1 z_2 + (\beta^2+1) z_2^2 = -rt$$

which would imply that a positive definite quadratic form admits a pro per factorization on $\mathbb{R}[z_1,z_2]$.

The dependence of the minimal dimension on the ground field suggests that a solution to the problem of computing minimal realizations should resort to mathematical tools which are beyond linear algebra. This peculiar aspect of 2D realization was first noticed in [20] and, later, in [21,22].

It is significant to point out that a tentative for obtaining minimal realizations in the class of Roesser's models was made in [11] and it led to a non linear algebraic problem as we mentioned above.

ii) Local reachability and observability are necessary but not sufficient to guarantee the minimality of a realization (see Example in sec. 4.2). The relevance of local properties is even weaker if we consider Roesser's models. Actually in this case, minimality (w.r. to the subclass of Roesser's models) does not imply local reachability and observability, as we can see by the following system [11]:

$$\begin{bmatrix} x^v(h+1,k) \\ x^h(h,k+1) \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x^v(h,k) \\ x^h(h,k) \end{bmatrix} + \left[\begin{array}{c} 1 \\ \hline 0 \\ 1 \end{array}\right] u(h,k)$$

$$y(h,k) = \left[\begin{array}{c|cc} 1 & 1 & 1 \end{array}\right] \begin{bmatrix} x^v(h,k) \\ x^h(h,k) \end{bmatrix}$$

This system is a minimal Roesser's realization of the transfer function

$$s = \frac{z_1^{-1}(z_1^{-1}+z_2^{-1}-1)}{(z_1^{-1}-1)(z_1^{-1}z_2^{-1}-z_2^{-1}-z_1^{-1})}$$

in spite of the fact that it is not locally observable.

Remark In chapter 10.2 of [23] it is claimed that all local state models suffer of the disadvantage that minimal realizations are not (locally) reachable and observable. It must be pointed out that this is false for models (5) and (7).

iii) Global reachability and observability conditions are sufficient for minimality [24]. However they are not necessary, because for a given transfer function globally reachable and observable realizations may not exist. In order to prove that global properties imply minimality, let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a globally-reachable and observable realization of dimension n of a transfer function s . Then, assuming $\xi = z_1^{-1} z_2$, and $\mathscr{A} \triangleq A_1 + A_2\xi \in K(\xi)^{n\times n}$, $\mathscr{B} \triangleq B_1 + B_2\xi \in K(\xi)^{n\times 1}$, $C \in K(\xi)^{1\times n}$ we have

$$s = C(z_1^{-1} I - (A_1 + A_2\xi))^{-1}(B_1 + B_2\xi) = C(z_1^{-1} I - \mathscr{A})^{-1}\mathscr{B}$$

Since the reachability and observability matrices of the system

$$\begin{aligned} w(t+1) &= \mathscr{A}\, w(t) + \mathscr{B}\, v(t) \\ r(t) &= C\, w(t) \end{aligned} \qquad (36)$$

defined on the field $K(\xi)$ are the global reachability and observability matrices of Σ , system (36) is a minimal realization of s on $K(\xi)$.

Now, if Σ were not a 2D minimal realization, we would be able to find a new 2D realization $\bar{\Sigma} = (\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C})$ of dimension lower than n and to express the series s as

$$s = \bar{C}(z_1^{-1}I - (\bar{A}_1+\bar{A}_2\xi))^{-1}(\bar{B}_1+\bar{B}_2\xi) = \bar{C}(z_1^{-1}I-\mathscr{A})^{-1}\mathscr{B} .$$

Hence the following 1D system:

$$\bar{w}(t+1) = \mathscr{A}\,\bar{w}(t) + \mathscr{B}\,v(t)$$

$$r(t) = \bar{C}\,\bar{w}(t)$$

would be a realization of s on $K(\xi)$ of dimension lower than n, which is a contradiction.

On the other side, the transfer function $s = z_1^2+z_2^2$ does not admit [25] globally reachable and observable realizations over any field K. In fact, assume $\Sigma = (A_1,A_2,B_1,B_2,C)$ be a globally reachable and observable realization of

$$s = z_1^2 + z_2^2 = \frac{1+\xi^2}{z_1^{-2}} \tag{37}$$

Since $(A_1+A_2\xi)$, $(B_1+B_2\xi)$, C is a minimal 1D realization of s over $K(\xi)$, its dimension is 2 (i.e. the degree of z_1^{-1} in the denominator of (37)) and $(A_1+A_2\xi)$ is a nilpotent matrix with an index of nilpotency 2. The conditions

$$(A_1+A_2\xi)^2 = A_1^2 + (A_1A_2+A_2A_1)\xi + A_2^2 = 0$$

and

$$C\,A_1B_1 = C\,A_2B_2 = 1$$

impose that A_1 and A_2 are nilpotent matrices too with the same index.

Then, we can assume without loss of generality

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and, from $A_1A_2 + A_2A_1 = 0$ we obtain

$$A_2 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad a \neq 0$$

Therefore, there isn't any matrix $C = [c_1 \; c_2]$, that makes the global observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ C(A_1 + A_2\xi) \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ 0 & c_1(1+a\xi) \end{bmatrix}$$

unimodular over $K[\xi, \xi^{-1}]$, contrary to the assumption.

iv) Any 2D realization $\Sigma = (A_1, A_2, B_1, B_2, C)$ of a 2D transfer function $s = p(z_1^{-1}, z_2^{-1})/q(z_1^{-1}, z_2^{-1})$ gives a 1D realization $\tilde{\Sigma} = (A_1 + A_2\xi,\ B_1 + B_2\xi,\ C)$ — of the 1D transfer function

$$p(z_1^{-1}, z_1^{-1}\xi)/q(z_1^{-1}, z_1^{-1}\xi) \tag{38}$$

with coefficients in $K(\xi)$, obtained from s assuming $\xi = z_1^{-1} z_2$.
We may ask if a minimal 1D realization of (38) over $K(\xi)$ can be always reduced by a similarity transformation on $K(\xi)$ to have the form

$$\begin{aligned} A(\xi) &= A_1 + A_2\xi \\ B(\xi) &= B_1 + B_2\xi \\ C(\xi) &= C \end{aligned} \tag{39}$$

where the entries of A_1, A_2, B_1, B_2, C are on K, and hence to be inter -

preted as a 2D realization, which -at that point- should be globally reachable and observable.

The answer is negative, as shown by the following 1D minimal realization over $K(\xi)$ of the transfer function (37)

$$A(\xi) = \begin{bmatrix} 0 & 1 \\ -(\xi^2+1) & 0 \end{bmatrix}, \quad B(\xi) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C(\xi) = [2\xi \quad 0]$$

It is strightforward to see that there isn't any change of basis in $\mathbb{R}^2(\xi)$ which reduces the matrices $A(\xi)$, $B(\xi)$, $C(\xi)$ to have the structure (39). This agrees with the fact that there are no 2D systems of dimension 2 which realize (37) over $\mathbb{R}$.

v) As well known, 1D minimal realizations of a transfer function are algebraically equivalent, i.e. unique modulo a change of basis in the state space. This does not extend to 2D systems, where there exist 2D minimal realizations of the same transfer function which are not algebraically equivalent. The following example illustrates the above concept.

<u>Example</u>:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0]$$

and

$$\bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{C} = [1 \quad 0]$$

are minimal realizations of the same transfer function, but they are not algebraically equivalent.

A partial explanation of the differences between 1D and 2D situation we mentioned above, is given by the following considerations. The classical realization procedure of a 1D rational series s is based on Ho's algorithm, which essentially consists on a finite set of linear operations on the finite rank Hankel matrix $\mathscr{H}(s)$. The algorithm produces a minimal realization (A,B,C) of s, whose dimension is the rank of $\mathscr{H}(s)$.

The Hankel matrix associated with a rational series in two indeterminates in general is not finite rank, and this makes impossible to use directly $\mathscr{H}(s)$ to generalize Ho's algorithm. However, we can get over this difficulty by resorting to some properties of non commutative power series. This enables us to attack the 2D realization problem following the 1D philosophy.

We mentioned in the proof of Proposition 2, that a power series σ in the non commutative variables ω_1 and ω_2 is rational if and only if there exist an integer n and matrices $A_1, A_2 \in K^{n\times n}$, $B \in K^{n\times 1}$, $C \in K^{1\times n}$ such that σ can be represented in the form

$$\sigma = C(I - A_1\omega_1 - A_2\omega_2)^{-1}B \tag{40}$$

The rank of the Hankel matrix

$$\mathscr{H}(\sigma) = \begin{bmatrix} (\sigma,1) & (\sigma,\omega_1) & (\sigma,\omega_2) & (\sigma,\omega_1^2) & (\sigma,\omega_1\omega_2) & (\sigma,\omega_2,\omega_1) & \cdots \\ (\sigma,\omega_1) & (\sigma,\omega_1^2) & (\sigma,\omega_1\omega_2) & (\sigma,\omega_1^3) & (\sigma,\omega_1^2\omega_2) & (\sigma,\omega_1\omega_2\omega_1) & \cdots \\ (\sigma,\omega_2) & (\sigma,\omega_2\omega_1) & (\sigma,\omega_2^2) & (\sigma,\omega_2\omega_1^2) & (\sigma,\omega_2\omega_1\omega_2) & \cdots & \\ (\sigma,\omega_1^2) & (\sigma,\omega_1^3) & (\sigma,\omega_1^2\omega_2) & & \cdots & & \\ & \cdots & & & & & \end{bmatrix}$$

is finite whenever σ is rational, and gives the minimal dimension of the representations of σ in the form (40).

When σ has zero constant term, the construction introduced in the proof of Proposition 2 provides the following representation,

$$\sigma = C(I - A_1\omega_1 - A_2\omega_2)^{-1}(B_1\omega_1 + B_2\omega_2) \tag{41}$$

which is slightly different from (40). The dimension of minimal representations with structure (41) is now given by the rank of the matrix $\mathscr{H}_o(\sigma)$ which is obtained from $\mathscr{H}(\sigma)$ by deleting the first column. The generalization of Ho's algorithm, given in [26] for computing minimal representations (40) still works, with minor adjustments, for computing minimal representations with structure (41) from $\mathscr{H}_o(\sigma)$.

We shall now show how the techniques for computing minimal representations of noncommutative power series can be used for tackling the problem of constructing 2D minimal realizations. The basic idea is the following: first, associate the commutative power series s with the set of noncommutative power series having s as commutative image, then construct minimal realizations of s using minimal representations of the corresponding noncommutative rational power series.

In order to do this in a precise way, let $\phi : K\langle\langle\omega_1,\omega_2\rangle\rangle \to K[[z_1,z_2]]$ be the algebra morphism defined by $\phi(k) = k$, $\forall k \in K$, $\phi(\omega_1) = z_1$, $\phi(\omega_2) = z_2$. Consider a 2D transfer function $s \in K_c[(z_1,z_2)]$ and pick any noncommutative rational series σ such that $\phi(\sigma) = s$. Then, for any representation (A_1, A_2, B_1, B_2, C) of σ, we have

$$\sigma = C(I-A_1\omega_1-A_2\omega_2)^{-1}(B_1\omega_1+B_2\omega_2) \xrightarrow[\phi]{} C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2) = s$$

Thus each representation (A_1,A_2,B_1,B_2,C) of σ is a realization of s. Conversely, if $\Sigma = (A_1,A_2,B_1,B_2,C)$ is a realization of s, Σ is a representation of the noncommutative rational power series $\sigma_\Sigma = \sum_{i=0}^{\infty} C(A_1\omega_1+A_2\omega_2)^i (B_1\omega_1+B_2\omega_2)$, which satisfies $\phi(\sigma_\Sigma) = s$. Hence, the class of realizations of s coincides with the class of representations of noncommutative rational power series σ satisfying

$$\phi(\sigma) = s .$$

Obviously, there are infinitely many noncommutative rational power series which satisfy the above equality.

As an interesting consequence, we point out that two minimal realizations of a given s are not necessarily similar: that happens whenever these minimal realizations come from minimal representations of two different noncommutative power series which have both s as commutative image.

In order to obtain all minimal realizations we shall therefore go through the following steps [22,26]:

Step 1. Select in the set of rational noncommutative power series having s as ϕ-image, the subset $\mathcal{M} \subseteq K<(\omega_1,\omega_2)>$ of those series σ whose Hankel matrix $\mathcal{H}_o(\sigma)$ has minimal rank;

Step 2. For each series $\sigma \in \mathcal{M}$ obtain, via Ho's algorithm, a minimal representation (which is unique modulo a similarity transformation).

The first step looks difficult to be implemented, because it involves the solution of several nonlinear algebraic equations which arise when minimizing the rank of a finite submatrix of $\mathcal{H}_o(\sigma)$ with the constraint $\phi(\sigma) = s$.

Other approaches to the minimal realization problem have been pursued in [11,1,26]. These differ from the situation considered above because at least one of the two terms of the realization problem, namely, the nature of the input-output map and the structure of the local state updating equation, has been changed.

The first attempt to be considered refers to Roesser's models and exploits the connections between minimality and modal reachability and observability [11]. We now summarize the results obtained in this context.

i) Modally reachable and observable Roesser's realizations are minimal in their own class, i.e. there are no Roesser's models of lower dimension which realize the same transfer function.

ii) Consider a transfer function

$$s = \frac{p(z_1^{-1}, z_2^{-1})}{q(z_1^{-1}, z_2^{-1})}$$

with $\deg_{z_1^{-1}} q = m$, $\deg_{z_2^{-1}} q = n$. Modally reachable and observable realizations of s have dimension $n+m$.

The point to note is that we don't know whether modally reachable and observable realizations exist and how to compute them even when they exist. As a matter of fact, the existence of such realizations cannot be guaranteed on the real field and their existence on the complex field has only been proved in the case $m=n=1$ [11](*). To our knowledge, no general con-

(*) The relationship between minimality and modal controllability and observability is "natural", as claimed in [23], as long as tautologies are natural, since in [11] minimality is only defined in terms of mo-

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structive procedure has been given for obtaining modally reachable and observable realizations either on the real or on the complex field.

Remark It is interesting to notice that the dimension $n+m$, which is the lowest bound if one looks for minimal realizations in the class of Roesser's models, is not necessarily minimal in the class of models (5). In order to see this, consider the transfer function $(z_1+z_2)/(1-z_1-z_2)$ whose minimal realizations in Roesser's form have dimension 2 while it is clearly realized by a system in form (5) of dimension 1.

The second approach [1] refers to Attasi's models which realize separable transfer functions (see sec. 2.3).

In this case, given a separable transfer function s, the rank of the Hankel matrix $\mathscr{H}(z_1^{-1} z_2^{-1} s)$ is finite and gives the dimension of minimal (Attasi) realizations of s.

It is important to point out that for this class of realizations the following properties hold, namely:

i) the minimal realization is unique, modulo similarity transformations;

ii) there exists a linear procedure - analogous to Ho's algorithm — which provides the matrices of the realization.

In the third approach [27] an attempt is made to assign the input-output behaviour via a noncommutative power series, the "generating series", which should substitute for the usual impulse response. This would enable us to exploit directly the established representation theory for noncommu-

./. dal properties. On the other hand, as far as the existence of modally reachable and observable realizations is not proved, a "modal" definition of minimality seems to be be meaningless.

tative power series. However, we remark that in this way we loose a great deal of system significance since the coefficient associated with the non-commutative word $w = \omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_r}$ in the generating series does not show any intrinsic meaning deducible from input-output experiments. In fact, input-output experiments allow us to single out only the coefficients of the impulse response, and these are the sums of the coefficients of the words in the generating series which have the same commutative image.

5. STABILITY

We know that in general the internal dynamics of 2D systems is not completely determined by their i/o behaviour. When we deal with stability problems, a major consequence of this fact is that external (=BIBO) stability analysis, based on the transfer function, is not sufficient for a complete characterization of internal stability properties.

These facts are not peculiar to the 2D situation: it is well known that internal asymptotic stability of 1D linear systems guarantees, but is not guaranteed by, the BIBO stability. However in 2D systems some new problems arise, mainly because the BIBO stability criterion (see Proposition 12) is in some sense unsatisfactory and BIBO stability cannot be guaranteed even for minimal realizations.

We begin the study of 2D stability by considering some necessary and sufficient conditions of polynomial type for internal stability and then establishing their connections with BIBO stability. Later in this section, we give some further internal stability conditions based on the extension of Lyapunov ideas.

In the last part, we are concerned with the state feedback stabilization problem. This very attractive control problem has been attacked very recently in 2D system theory and we only have preliminary results.

5.1 Polynomial Stability Criteria

The notion of internal stability of a 2D system given on the real field refers to the behaviour of the free evolution of local states resulting from a bounded global state assignment on the separation set $\mathscr{C}_o$ [28,29].

With regard to global states we introduce the following notation

$$\|\mathscr{X}_i\| = \sup_{(h,k)\in\mathscr{C}_i} \|x(h,k)\|$$

where $\|x\|$ denotes the Euclidean norm of x.

Definition 8: Let $u=0$. The 2-D system (5) is internally stable if given $\varepsilon \geq 0$, for every $\mathscr{X}_o$ with $\|\mathscr{X}_o\| < \infty$ there exists a positive integer m such that $\|\mathscr{X}_i\| < \varepsilon$ for $i \geq m$.

As we know, internal stability of a discrete linear system given by equations (3) can be checked either by evaluating the distribution of the roots of the characteristic polynomial of A with respect to the unit circle in the Gauss plane, or by solving the Lyapunov equation.

By looking at 2D systems, we now establish some results concerning polynomial stability criteria and we devote next section to investigate 2D Lyapunov theory.

The following proposition provides the 2D counterpart of the 1D stability criterion based on the characteristic polynomial.

<u>Proposition 10.</u> [9,28]: *A 2D system* $\Sigma = (A_1, A_2, B_1, B_2, C)$ *is internally stable if and only if the polynomial* $\det(I - A_1 z_1 - A_2 z_2)$ *is devoid of zeros in the closed polydisc*

$$\mathscr{P}_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \leq 1, |z_2| \leq 1\} .$$

proof. *sufficiency.* Let $\det(I - z_1 A_1 - z_2 A_2) \neq 0$ in $\mathscr{P}_1$ and call V the algebraic curve defined by $\det(I - A_1 z_1 - A_2 z_2) = 0$.

Since V and $\mathscr{P}_1$ are closed, $V \cap \mathscr{P}_1 = \emptyset$ implies that there exists $\varepsilon > 0$ such that the polydisk

$$\mathscr{P}_{1+\varepsilon} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \leq 1 + \varepsilon, |z_2| \leq 1 + \varepsilon\}$$

does not intersect V.

Then the rational matrix $(I - A_1 z_1 - A_2 z_2)$ can be inverted in $\mathscr{P}_{1+\varepsilon}$ and its McLaurin series expansion, given by

$$(I - A_1 z_1 - A_2 z_2)^{-1} = \Sigma_{ij} M_{ij} z_1^i z_2^j$$

converges normally in the interior of $\mathscr{P}_{1+\varepsilon}$ [29].

It follows that the series $\Sigma_{ij} \| M_{ij} \|$ converges and $\sum_{i+j=r} \| M_{ij} \| \to 0$ as $r \to \infty$ [31]. This implies the asymptotic stability of Σ. For, assume $\| \mathscr{X}_0 \|$ finite and pick in $\mathscr{X}_r$, $r > 0$, any local state $x(m, r-m)$. Then

$$\| x(m, r-m) \| = \| \sum_{i+j=r} M_{ij} x(m-i, r-m-j) \|$$

$$\leq \sum_{i+j=r} \| M_{ij} \| \, \| x(m-i, r-m-j) \| \leq \| \mathscr{X}_0 \| \sum_{i+j=r} \| M_{ij} \|$$

necessity. Assume Σ be asymptotically stable. Then for any $x \in X$, $M_{ij}x \to 0$ as $i+j \to \infty$. This fact and

$$\| M_{ij} \| \leq \sum_{1}^{n} {}_k \| M_{ij} e_k \| ,$$

(where $\{e_k\}_i^n$ is the standard basis in $X = \mathbb{R}^n$) imply

$$\lim_{i+j \to \infty} \| M_{ij} \| \leq \lim_{i+j \to \infty} \sum_{1}^{n} \| M_{ij} e_k \| = 0 .$$

By Abel's lemma, the series $\Sigma_{ij} M_{ij} z_1^i z_2^j$ converges in the interior of $\mathscr{P}_1$. Then $(I - A_1 z_1 - A_2 z_2)$ is invertible in the interior of $\mathscr{P}_1$.

The proof will be complete by showing that $\det(I - A_1 z_1 - A_2 z_2) \neq 0$ on the boundary $\rho \mathscr{P}_1$ of $\mathscr{P}_1$. For, let (a_1, a_2) belong to $\rho \mathscr{P}_1$ and assume

$$\det(I - A_1 a_1 - A_2 a_2) = 0.$$

Hence, there exists a nonzero vector $v \in \mathbb{C}^n$ which satisfies $v = A_1 a_1 v + A_2 a_2 v$. It is not restrictive to assume $|a_1| = 1$, so that it makes sense to consider $\mathscr{X}_o = \{x_{n,-n}\}$ with

$$x_{n,-n} \begin{cases} = 0, & \text{if } n < 0 \\ = \alpha a_1^{-n} a_2^{n} v + \breve{\alpha} \breve{a}_1^{-n} \breve{a}_2^{n} \breve{v}, & \text{if } n \geq 0,\ \alpha \in \mathbb{C} . \end{cases}$$

Assume now $\alpha = e^{j\psi}$ and write $a_1 = e^{-j\phi}$ and $v = r + jw$. Then the state values on $(0,k)$, $k = 0,1,2,\ldots$ are given by the sequence

$$x(k,0) = 2r \cos(k\phi + \psi) - 2w \sin(k\phi + \psi) , \qquad k = 0,1,2,\ldots$$

and it is always possible to select a phase ψ which makes the sequences not infinitesimal as k goes to infinity.

In the literature (see [32] for a detailed review), several tests have been proposed to check if $\mathcal{P}_1$ intersects the complex variety of a polynomial $q \in \mathbb{R}[z_1, z_2]$. The original field of application of these tests is BIBO stability analysis. As it will be clarified later, however, they seem to fit better to internal stability, since necessary BIBO stability conditions cannot be derived by simply checking the intersection between $\mathcal{P}_1$ and the variety associated with the denominator of the transfer function.

Now, we briefly present some facts about 2D BIBO stability.

Definition 9. Assume $\mathcal{X}_0 = 0$. A 2D system is BIBO stable if any bounded input u in $\mathcal{F}_{\mathcal{C}_0}$ produces a bounded output y.

As a direct consequence of the above definition, we have the following Proposition.

Proposition 11. *Let* $s = \Sigma s_{ij} z_1^i z_2^j$ *be the transfer function of a 2D system* Σ. *Then* Σ *is BIBO stable if and only if* $\sum_{ij} |s_{ij}| < \infty$.

In 1D linear systems, the condition that the coefficients of the impulse response $\Sigma s_i z^i$ constitute a ℓ_1 sequence is equivalent to the condition that the zeros of the denominator of the transfer function

$$s = \frac{p(z)}{q(z)} \qquad (p,q) = 1$$

lie outside the closed unit disc in the complex plane.

Here, the equivalence is not complete. In fact Shanks theorem [33] states that a 2D system with transfer function $1/q$ is BIBO stable if and only if $q(z_1,z_2) \neq 0$ in $\mathscr{P}_1$. Of course, the condition $q \neq 0$ in $\mathscr{P}_1$ guarantees also BIBO stability when the transfer function is p/q, with p and q coprime.

However, in [34] it is shown that a 2D system with transfer function p/q can be BIBO stable even when the denominator q vanishes in some points of the torus

$$T_1 = \{(z_1,z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| = |z_2| = 1\} .$$

This is formalized in the following Proposition

Proposition 12. *Let* $s = p(z_1,z_2)/q(z_1,z_2)$ *denote the transfer function of a BIBO stable 2D system. Then* s *has no poles in the closed unit polydisc* $\mathscr{P}_1$ *and no nonessential singularities of the second kind on* $\mathscr{P}_1$ *except possibly on* T_1.

Then the mutual implications between root distribution of q and BIBO stability are as follows [32]

$$\text{BIBO stability} \Rightarrow q(z_1,z_2) \neq 0 \quad \text{in } \mathscr{P}_1 - T_1$$
$$\text{BIBO stability} \Leftarrow q(z_1,z_2) \neq 0 \quad \text{in } \mathscr{P}_1 .$$

This shows that the absence of intersections between $\mathscr{P}_1$ and the variety of q, which can be checked by the above mentioned tests, ensures BIBO stability. Nevertheless we can have BIBO stability even when the intersection is nonempty, that is, when the tests would give a negative answer.

Remark: Assume that $s = p(z_1,z_2)/q(z_1,z_2)$ is a BIBO stable transfer function and has some nonessential singularities of the second kind on T_1. Thus $q(z_1,z_2)$ vanishes at some points of T_1. Since for any realization

(A_1,A_2,B_1,B_2,C), $q(z_1,z_2)$ divides $\det(I-A_1z_1-A_2z_2)$, there are not internally stable realizations of s, even if s is BIBO stable.

5.2 2D Lyapunov Theory

The problem of deriving a Lyapunov equation for 2D linear systems seems to be a difficult one and as yet no satisfactory solutions are available.

As we shall see, a first insight into this area is provided by a Lyapunov equation depending on the real parameter ω. Its solution shows us a way for defining a quadratic form which can be viewed as an energy function connected with the global state.

In order to do this, we need a preliminary result, which expresses 2D stability of a pair (A_1,A_2) in terms of 1D stability of the family of matrices $A_1+e^{j\omega}A_2$.

Proposition 13. *A 2D system* $\Sigma = (A_1,A_2,B_1,B_2,C)$ *is internally stable if and only if the complex matrix*

$$A_1 + e^{j\omega}A_2 \tag{42}$$

is stable (i.e., the magnitude of its eigenvalues is less than 1) for any real ω.

proof. First recall that, by Huang's criterion [35], a polynomial $q \in \mathbb{R}[z_1,z_2]$ is devoid of zeros in $\mathscr{P}_1$ if and only if i) $q(z_1,0) \neq 0$ for $|z_1| \leq 1$ and ii) $q(z_1,z_2) \neq 0$ for $|z_1| = 1$ and $|z_2| \leq 1$.

Assume now the matrix $A_1+e^{j\omega}A_2$ to be stable for any real ω. The images of the unit polydisc given by the polynomial functions

$$q_1(z_1,\eta) = \det(I-A_1z_1-A_2z_1\eta)$$

and

$$q_2(\eta,z_2) = \det(I-A_1z_2\eta - A_2z_2)$$

coincide with the images of the polynomial function $\det(I-A_1z_1-A_2z_2)$, when acting on the sets

$$\mathscr{P}_1 \cap \{(z_1,z_2) : |z_1| \geq |z_2|\}$$

and

$$\mathscr{P}_1 \cap \{(z_1,z_2) : |z_2| \geq |z_1|\} .$$

Since $q_1(0,\eta) \neq 0$ for $|\eta| \leq 1$ and, by stability assumption on $A_1 + e^{j\omega}A_2$, $q_1(z_1,e^{j\omega}) \neq 0$ for $|z_1| \leq 1$, $q_1(z_1,\eta)$ is devoid of zeros in $\mathscr{P}_1$ by Huang's criterion. The same property holds for $q_2(\eta,z_2)$. Therefore, $\det(I-A_1z_1-A_2z_2) \neq 0$ in $\mathscr{P}_1$, and Σ is internally stable.

Conversely, internal stability implies $\det(I-A_1z_1-A_2z_2) \neq 0$ in $\mathscr{P}_1$. Hence $\det(I-A_1z_1-A_2z_2e^{j\omega}) \neq 0$ for $|z_1| \leq 1$ which gives the stability of $A_1+e^{j\omega}A_2$.

<u>Remark</u> Assuming in (42) $\omega=0$ or $\omega=\pi$, we obtain that the stability of the matrices A_1+A_2 and A_1-A_2 is a necessary condition for 2D internal stability.

As a corollary of Proposition 13 we have that *a 2D system* $\Sigma = (A_1,A_2,B_1,B_2,C)$ *is internally stable if and only if the Lyapunov equation*

$$P(\omega) = I + (A_1+e^{-j\omega}A_2)^T P(\omega)(A_1+e^{j\omega}A_2) \qquad (43)$$

admits a positive definite Hermitian solution $P(\omega)$ *for every real* ω.

If Σ is internally stable, then $P(\omega)$ is a $n\times n$ rational matrix in the indeterminate $e^{j\omega}$, continuous w.r. to ω and periodic with period 2π.

The Fourier coefficients P_k of the expansion

$$P(\omega) = \sum_{k=-\infty}^{+\infty} P_k e^{j\omega k}$$

satisfy the following properties:

i) *the sequence* $\{P_k\}$ *belongs to* $\ell^2(\mathbb{R}^{n\times n})$;

ii) *for any integer* k, $P_k = P_{-k}^T$, *and the following set of equalities holds:*

$$\begin{aligned}
P_o &= I + A_1^T P_0 A_1 + A_2^T P_0 A_2 + A_1^T P_1^T A_2 + A_2^T P_1 A_1 \\
P_1 &= A_1^T P_1 A_1 + A_2^T P_1 A_2 + A_1^T P_0 A_2 + A_2^T P_2 A_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
P_k &= A_1^T P_k A_1 + A_2^T P_k A_2 + A_1^T P_{k-1} A_2 + A_2^T P_{k+1} A_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{44}$$

iii) *the doubly infinite block Toeplitz matrix*

$$\mathscr{P} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & P_{-1} & P_0 & P_1 & \ddots & \\ & \ddots & P_{-1} & P_0 & P_1 & \ddots \\ & & \ddots & P_{-1} & P_0 & P_1 \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \tag{45}$$

induces a positive definite scalar product in the space $\ell^2(\mathbb{C}^n)$.

Property iii) relies on the fact that for every nonzero vector function $v(\cdot): [-\pi,\pi] \to \mathbb{C}^n$, with elements in $L^2[-\pi,\pi]$ we have

$$\int_{-\pi}^{\pi} \check{v}^T(\omega)P(\omega)v(\omega)d\omega > 0 .$$

Then

$$\sum_{h,k} \check{v}_h^T P_{h-k} v_k > 0$$

where v_k are the Fourier coefficients of $v(\cdot)$:

$$v(\omega) = \sum_{k=-\infty}^{+\infty} v_k e^{j\omega k} .$$

By Riesz-Fischer theorem, there is a bijection between $\ell^2(\mathbb{C}^n)$ and the space of n-tuples with elements in $L^2[-\pi,\pi]$. Therefore, the relation

$$(u,w) = \sum_{h,k} \check{u}_h^T P_{h-k} w_k$$

defines a positive definite scalar product in $\ell^2(\mathbb{C}^n)$.

The converse is also true. For assume that there exists a sequence $\{P_k\}$ of matrices in $\mathbb{R}^{n\times n}$ satisfying conditions i), ii), iii). Then the series $\sum_{k=-\infty}^{+\infty} P_k e^{j\omega k}$ defines a.e. a matrix function $P(\omega)$ with elements in $L^2[-\pi,\pi]$ which solves (43). In fact, using (44) we obtain

$$I + A_1^T P(\omega)A_1 + A_2^T P(\omega)A_2 + A_1^T e^{j\omega} P(\omega)A_2 + A_2^T e^{-j\omega} P(\omega)A_1 =$$

$$= I + \sum_k A_1^T P_k A_1 e^{j\omega k} + \sum_k A_2^T P_k A_2 e^{j\omega k} + \sum_k A_1^T P_k A_2 e^{j\omega(k+1)} +$$

$$+ \sum_k A_2^T P_k A_1 e^{j\omega(k-1)} =$$

$$= \sum_{k<0} e^{j\omega k}(A_1^T P_k A_1 + A_2^T P_k A_2 + A_1^T P_{k-1} A_2 + A_2^T P_{k+1} A_1)$$

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$$+ (I + A_1^T P_0 A_1 + A_2^T P_0 A_2 + A_1^T P_{-1} A_2 + A_2^T P_1 A_1) + \sum_{k>0} e^{j\omega k} (A_1^T P_k A_1 + A_2^T P_k A_2$$

$$+ A_1^T P_{k-1} A_2 + A_2^T P_{k+1} A_1)$$

$$= \sum_k P_k e^{j\omega k} = P(\omega) .$$

Hence the entries of $P(\omega)$ are a.e. real rational functions of $e^{j\omega}$. Since

$$P_0 = \int_{-\pi}^{\pi} P(\omega) d\omega$$

is finite, and $P(\omega)$ is a.e. positive definite, the analytic extension $Q(z)$ of $Q(e^{j\omega}) \overset{\Delta}{=} P(\omega)$ has no poles on the unit circle.

Since by continuity $P(\omega)$ is at least positive semidefinite and satisfies (44) for all real ω, then it is positive definite for all real ω.

<u>Example.</u> (Scalar Case): If the local state space is one dimensional, the matrices A_1 and A_2 become scalars a_1 and a_2, and a closed form computation of the Fourier coefficients P_k is possible when the 2D system is internally stable.

The solution of (43), given by

$$P(\omega) = (1 - a_1^2 - a_2^2 - 2a_1 a_2 \cos \omega)^{-1}$$

is positive for every real ω if and only if

$$1 - a_1^2 - a_2^2 < |2a_1 a_2| . \tag{46}$$

Incidentally note that condition (46) can be restated in the follo -

wing equivalent forms:

i) $|a_1| + |a_2| < 1$;

ii) $q(z_1,z_2) = 1 - a_1 z_1 - a_2 z_2 \neq 0$ in $\mathscr{P}_1$;

iii) $\begin{bmatrix} 1 - a_1^2 - a_2^2 & 2a_1 a_2 \\ 2a_1 a_2 & 1 - a_1^2 - a_2^2 \end{bmatrix} > 0$.

When condition (46) is satisfied, we obtain

$$P_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1-a_1^2-a_2^2-2a_1 a_2 \cos\omega)^{-1} d\omega = \left[(1-a_1^2-a_2^2)^2 - 4a_1^2 a_2^2\right]^{-1/2}.$$

Once P_0 has been computed, the coefficients $P_k, k = 1,2,\ldots$, are obtained recursively from (44).

The basic idea of the direct method of Lyapunov for verifying stability is to seek an "energy function" that decreases toward a minimum as the system evolves. Quadratic functions have a special role in linear systems. Now, we show how these can be introduced in the 2D framework and interpreted as Lyapunov functions.

Represent the global state $\mathscr{X}_i = \sum_h x(h+i,-h)\xi^h$ as an infinite column vector

$$\mathscr{X}_i = \begin{bmatrix} \vdots \\ x(i-1,1) \\ x(i,0) \\ x(i+1,-1) \\ \vdots \end{bmatrix}$$

and denote by $\mathscr{A}$ and $\mathscr{A}^*$ the doubly infinite block Toeplitz matrices

$$\mathscr{A} = \begin{bmatrix} \ddots & \ddots & & \\ \ddots & A_1 & A_2 & 0 \\ \ddots & 0 & A_1 & A_2 \ddots \\ & 0 & 0 & A_1 \ddots \\ & & & \ddots \end{bmatrix}, \quad \mathscr{A}^* = \begin{bmatrix} \ddots & \ddots & & \\ \ddots & A_1^T & 0 & 0 \\ \ddots & A_2^T & A_1^T & 0 \\ & 0 & A_2^T & A_1^T \ddots \\ & & & \ddots \end{bmatrix}$$

Using the above notations, the free evolution of the global state is expressed by

$$\mathscr{X}_{i+1} = \mathscr{A}\,\mathscr{X}_i$$

Then, assuming the system Σ be internally stable, we have

i) the matrix $\mathscr{P}$ satisfies the equation

$$\mathscr{P} = \mathscr{I} + \mathscr{A}^*\mathscr{P}\mathscr{A}$$

where $\mathscr{I}$ is the (infinite) identity matrix

ii) for any $\mathscr{X}_i$ in ℓ_2 the quadratic form

$$\mathscr{V}(\mathscr{X}_i) = \mathscr{X}_i^T\mathscr{P}\mathscr{X}_i$$

can be viewed as a Lyapunov function associated with the global state.

It is easily seen that satisfies the following relation

$$\mathscr{V}(\mathscr{X}_{i+1}) = \mathscr{V}(\mathscr{X}_i) - \mathscr{X}_i^T\mathscr{X}_i \tag{47}$$

and decreases as i increases.

5.3 State Feedback Stabilizability

The structure of partial ordering which underlies 2D systems, makes possible to consider state-feedback schemes which cannot be derived by simply extending 1D concepts. It is clear that, following 1D philosophy, we can adopt inputs which depend only on the states at the same "time" and/or inputs whose dynamic dependence on the states fits with the causality induced by the partial ordering in $\mathbb{Z} \times \mathbb{Z}$. In this way, we obtain systems which still keep the state updating structure of a 2D system.

However, as it will be formally stated later, it is possible to take into account inputs which depend dynamically on the states but do not respect the causality induced by the partial ordering of $\mathbb{Z} \times \mathbb{Z}$. So doing, in general we destroy the 2D structure of the original system and we obtain the so called "weakly causal" 2D systems [13].

In the following we shall give a first insight into the application of different state-feedback to the stabilization of 2D systems given on the real field.

Referring to the state feedback structures, we can essentially deal with two situations [36], namely

a) *state feedback preserving 2D causality*

a.1) static state feedback

$$u(h,k) = K\, x(h,k) , \qquad K \in \mathbb{R}^{1\times n} \tag{48}$$

a.2) 2D recursive state feedback

$$u(h,k) = \sum_{i+j=1}^{q} \alpha_{ij} u(h-i,k-j) + \sum_{i,j=0}^{p} K_{ij} x(h-i,k-j) , \tag{49}$$

$$\alpha_{ij} \in \mathbb{R} , \quad K_{ij} \in \mathbb{R}^{1\times n}$$

b) *state feedback producing 2D weakly causal systems:*

$$u(h,k) = \sum_{i=-m}^{m} K_i x(h-i,k+i) \qquad K_i \in \mathbb{R}^{1\times n} \tag{50}$$

or, in formal power series notation,

$$\mathcal{U}_t(\xi) = \left(\sum_{i=-m}^{m} K_i \xi^i \right) \mathcal{X}_t(\xi) = K(\xi)\, \mathcal{X}_t(\xi) ,$$

$$K(\xi) \in \mathbb{R}[\xi,\xi^{-1}]^{1\times n}$$

Remark It is easy to see that the state feedback b) does not preserve the structure (5) of the state updating equation. In fact the computation of the local state at some point (h,k) belonging to the separation set $\mathscr{C}_t$, involves not only data at points of $\mathscr{C}_{t-1}$ which are in the past of (h,k) but also data at points of $\mathscr{C}_{t-1}$ which are not causally related to (h,k).

Consider a 2D system $\Sigma = (A_1,A_2,B_1,B_2,C)$ and assume $u(h,k)=Kx(h,k)$ (static state feedback). We obtain a new 2D system $\bar{\Sigma} = (\bar{A}_1,\bar{A}_2,B_1,B_2,C)$ where

$$\bar{A}_1 = A_1+B_1K , \qquad \bar{A}_2 = A_2+B_2K$$

By Proposition 10, it is strightforward to see that *if the 2D system* Σ *is stabilizable by means of static state feedback, then the pairs* (A_1,B_1) *and* (A_2,B_2) *are simultaneously stabilizable.*

However, as we show in the following example, the converse of the above result does not hold.

Example. Consider the following 2D system

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3/4 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3/4 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since the polynomial

$$p(z_1,z_2) = \det(I-(A_1+B_1K)z_1-(A_2+B_2K)z_2)$$

$$= 1 + \frac{3}{4} z_1 - \frac{3}{4} z_2 + k_0(z_1+z_2)^3 + k_1(z_1+z_2)^2 + k_2(z_1+z_2)$$

vanishes at $(-2/3, 2/3)$ for any $K = [k_0 \; k_1 \; k_2]$, $(A_1,A_2,B_1,B_2,-)$ is not stabilizable by state feedback. Nevertheless (A_1,B_1) and (A_2,B_2) are simultaneously stabilizable.

Consider again a 2D system and assume a state feedback having structure (50), i.e.

$$\mathcal{U}_t(\xi) = K(\xi)\mathcal{X}_t(\xi)$$

The global state evolves accordingly to the following equation

$$\mathcal{X}_{t+1} = [\, (A_1+A_2\xi) + (B_1+B_2\xi)K(\xi) \,]\, \mathcal{X}_t$$

It is interesting to examine the following two cases

(i) $\det \mathscr{R} = k\xi^m$, $k \neq 0$. This condition corresponds to assume that

$$w(t+1) = (A_1+A_2\xi)w(t) + (B_1+B_2\xi)v(t) \tag{51}$$

is reachable as a system over the ring $\mathbb{R}[\xi,\xi^{-1}]$. Consequently the polynomial

$$\det((\eta I-A_1-A_2\xi) - (B_1+B_2\xi)K(\xi)) \tag{52}$$

in the indeterminate η is coefficient assignable and hence the 2D system is stabilizable.

Remark It is important to point out that in this case the stabilization can be achieved even when the pairs (A_1,B_1) and (A_2,B_2) are not simultaneously stabilizable, as we can easily see from the following system

$$(A_1,A_2,B_1,B_2,C) = (2,3,1,0,-) .$$

(ii) $\det \mathscr{R} \neq k\xi^m$. This condition corresponds to assume that the 2D system (5) is globally reachable but $(A_1+A_2\xi,\ B_1+B_2\xi,\ C)$ is not reachable as a system over $\mathbb{R}[\xi,\xi^{-1}]$. This implies that the coefficients of the polynomial (52) cannot be arbitrarily assigned. However this type of state feedback allows us to stabilize 2D systems in cases when static feedback does not give positive results. Actually, the following example shows that this type of state feedback can solve the stabilization problem even in cases when the pairs (A_1,B_1) and (A_2,B_2) are not simultaneously stabilizable.

<u>Example.</u> Consider the 2D system $(A_1, A_2, B_1, B_2, -) = (-4.4, 0, 1, \frac{1}{3}, -)$. The reachability matrix $\mathscr{R} = B_1 + B_2\xi = 1 + \frac{1}{3}\xi$ is not unimodular and the pairs (A_1, B_1) and (A_2, B_2) are not simultaneously stabilizable.

Assuming $K(\xi) = 4.8 - 1.2\xi$, we obtain the following characteristic polynomial

$$(\eta - A_1 - A_2\xi - (B_1 + B_2\xi)K(\xi)) = \eta - 0.4 - 0.4\xi + 0.4\xi^2$$

which shows that the free evolution of the global state converges to zero.

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