Proceedings of 25th Conference
on Decision and Control
Athens, Greece • December 1986

FP11 - 6:00

CAUSAL 2D COMPENSATORS: STABILIZATION ALGORITHMS FOR MULTIVARIABLE 2D SYSTEMS

M. Bisiacco, E. Fornasini, G. Marchesini

Istituto di Elettrotecnica e di Elettronica
Via Gradenigo, 6/A,35131 PADOVA Italy

Abstract

This contribution is concerned with the synthesis of feedback stabilizing 2D compensators obtained by using the Gröbner basis algorithm. A constructive procedure is given to obtain the coefficients of a stable closed loop characteristic polynomial. Since the operations involved are rational, the synthesis algorithm leads to polynomial matrices whose entries are polynomials with coefficients in the same field as the plant.

1. Introduction

A peculiar aspect of the synthesis of a stabilizing 2D compensator is that, in general, even when the plant is given by a factor coprime matrix fraction description ND^{-1}, it is not possible to freely assign the variety of the characteristic closed loop polynomial. In fact this variety is constrained to include the set of points where the minors of maximal order of $[D'\ N']$ vanish simultaneously [1].

This constraint can be satisfied in a direct way when such points are explicitly computed and do not belong to the unit polydisc. In this case it is strightforward to determine a stable separable polynomial vanishing on this set of points, so that a suitable power of this polynomial can be assumed as a closed loop characteristic polynomial.

In the sequel we shall present an approach to the synthesis of a stabilizing compensator which does not require any explicit computation of the set.

2. Feedback compensators and closed loop stable polynomials

Let $W(z,w)$ be a strictly proper transfer matrix of dimension pxm and let

$$N(z,w)D^{-1}(z,w) = W(z,w)$$

be a right factor coprime Matrix Function Description (MFD).

Consider the ideal $\mathcal{J}$ generated by the minors of maximal order $m_1(z,w),\dots,m_\nu(z,w)$ of the matrix

$$\begin{bmatrix} D(z,w) \\ N(z,w) \end{bmatrix}$$

The coprimeness condition on N and D corresponds to assume that the variety $\mathcal{V}(\mathcal{J})$ is a finite subset of $\mathbb{C}\times\mathbb{C}$ or, equivalently, that the quotient $\mathbb{R}[z,w]/\mathcal{J}$ is a finite dimensional vector space over $\mathbb{R}$.

Let $\Sigma = (A_1,A_2,B_1,B_2,C)$ be a 2D realization of $W(z,w)$ [2], where $(I-A_1z,A_2w,\ B_1z+B_2w)$ are left factor coprime and $(I-A_1z-A_2w,\ C)$ are right factor coprime. Under these assumptions, we have [1]

$$\det(I-A_1z-A_2w) = \det D(z,w)$$

Consider an output feedback compensator represented by a proper MFD

$$W_c(z,w) = R^{-1}(z,w)S(z,w)$$

of dimension m x p and let $\Sigma_c = (F_1,F_2,G_1,G_2,H)$ be a realization of W_c satisfying the relation

$$\det(I-F_1z-F_2w) = \det R(z,w)$$

Then the characteristic polynomial Δ of the closed loop system obtained by the output feedback connection of Σ and Σ_c is given by

$$\Delta = \det(RD + SN) \ .$$

Using Binet-Cauchy formula, $\det(RD+SN)$ is expressed as the sum of the products of all possible minors of maximal order, $q_i, i=1,2,\dots,\nu$ of $[R\ S]$ into the corresponding minors of the same order $m_i, i=1,2,\dots,\nu$ of $[D'N']$, that is

$$\det(RD+SN) = \sum_{1}^{\nu}{}_i\ q_i m_i$$

Hence $\det(RD+SN)$ belongs to the ideal $\mathcal{J}$ for any choice of the compensator.

Conversely, given any polynomial $p\in\mathcal{J}$, there exists a compensator $R^{-1}S$ such that

$$\Delta = \det(RD+SN) = p^r$$

for some integer r [1,3].

 CH2344-0/86/0000-2171 $1.00

This implies that $\mathscr{V}(\Delta)$ is freely assignable, except that it must include $\mathscr{V}(\mathscr{I})$ and does not contain (0,0).

Because of these facts, there exists a stabilizing compensator if and only if $\mathscr{V}(\mathscr{I})$ does not intersect the closed unit polydisc $\mathscr{P}_1$ in $\mathbb{C}\times\mathbb{C}$.

If Σ is stabilizable, we can select a stable polynomial p in $\mathscr{I}$ and then solve the polynomial equation

$$RD + SN = pI \tag{1}$$

This gives the MFD of a stabilizing compensator, $R^{-1}S$. The full class of stabilizing compensators is parametrized by a formula whose structure is analogous to the 1D theory [3].

A procedure for determining a stable polynomial p in $\mathscr{I}$ was introduced in [3,4]. This is based on two steps: (i) compute explicitly $\mathscr{V}(\mathscr{I}) = \{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}$;
(ii) construct a 2D stable polynomial q, vanishing on $\mathscr{V}(\mathscr{I})$, and having the structure

$$q(z,w) = \Pi_{1}^{s}{}_{i} (z-\alpha_i) \; \Pi_{s+1}^{t}{}_{j} (w-\beta_j) \tag{2}$$

where we assumed $|\alpha_i| > 1$, $i = 1,\dots,s$ and $|\beta_j| > 1$, $j = s+1,\dots,t$.

Hence $q^r(z,w)$ belongs to $\mathscr{I}$ for some r and in (1) we can assume $p = q^r$.

Once we have computed p, a general method for obtaining R and S that solve (1) resorts to the Gröbner basis algorithm developed in [5].

Remark. The above procedure applies to the design of state feedback compensators. In this case, the matrices D and N have the following form

$$D = I - A_1 z - A_2 w$$

$$N = B_1 z + B_2 w$$

and the solution of (1) can be obtained by a more direct technique, presented in [6].

In the sequel we shall introduce an alternative approach for determining a stable polynomial $p \in \mathscr{I}$ that does not require an explicit computation of $\mathscr{V}(\mathscr{I})$. The procedure ends in a finite number of steps if and only if $\mathscr{V}(\mathscr{I})$ does not intersect the unit polydisc $\mathscr{P}_1$.

Since the Gröbner basis algorithm allows us to compute the polynomials $r_1(z)$ of minimal degree in $\mathbb{R}[z]$ and $r_2(w)$ of minimal degree in $\mathbb{R}[w]$, a first attempt to determine a stable polynomial $p(z,w)\in\mathbb{R}[z,w]$, consists in assuming $p = r_1$ or $p = r_2$ if r_1 or r_2 is stable. If both r_1 and r_2 are unstable, what happens when we have $|\alpha_i| \le 1$ for some i and $|\beta_j| \le 1$ for some j, $\mathscr{I}$ does not contain stable polynomials in one variable. In this case it is necessary to look for polynomials in two variables in order to find a stable $p \in \mathscr{I}$.

Remark. The fact that $r_1(z)$ and $r_2(w)$ are both unstable, does not imply that $\mathscr{V}(\mathscr{I})$ intersects $\mathscr{P}_1$. In fact $\mathscr{V}(\mathscr{I}) \cap \mathscr{P}_1 = \emptyset$ if whenever $|\alpha_i| \le 1$ we have $|\beta_i| > 1$. In this case there exist 2D separable polynomials with structure (2) that are stable. However they can be computed only if we have a prior knowledge of $\{\alpha_i\}$ and $\{\beta_i\}$.

The computing procedure we shall illustrate is based on the following facts:

(i) for each positive integer h, $\mathscr{I}\cap\mathbb{R}[z^h+w^h]$ is a non empty principal ideal in $\mathbb{R}[z^h+w^h]$. Its generator can be explicitly determined by using Gröbner basis;

(ii) if there exist stable polynomials in $\mathscr{I}$, there are positive integers h such that $\mathscr{I}\cap\mathbb{R}[z^h+w^h]$ contains stable polynomials. In particular, the generator of $\mathscr{I}\cap\mathbb{R}[z^h+w^h]$ is stable;

(iii) the 2D stability check of a polynomial in $\mathbb{R}[z^h+w^h]$ reduces to check the stability of a polynomial in one variable.

The proofs of (i) and (ii) are direct consequences of the following Propositions.

Proposition 1. Assume $\mathscr{V}(\mathscr{I})\cap\mathscr{P}_1 = \emptyset$. Then there exist an integer k and a stable polynomial $p\in\mathbb{R}[z^k+w^k]\cap\mathscr{I}$.

proof. For each $(\alpha_i,\beta_i)\in\mathscr{V}(\mathscr{I})$, $i = 1,2,\dots,t$, either $|\alpha_i| > 1$ or $|\beta_i| > 1$. So, for each (α_i,β_i) there exist integers k_i such that $|\alpha_i^{k_i} + \beta_i^{k_i}| > 2$. Then

$$q_i(z^{k_i} + w^{k_i}) \overset{\Delta}{=} z^{k_i} + w^{k_i} - \alpha_i^{k_i} - \beta_i^{k_i}$$

vanishes on (α_i,β_i) and has no zeros on $\mathscr{P}_1$.

It is not difficult to show that we can assume $k_1 = k_2 = \dots = k_t$, provided the common value of $\{k_i\}$ is large enough. Denoting by k this common value, the product

$$q = \Pi\, q_i(z^k + w^k)$$

is a stable 2D polynomial vanishing on $\mathscr{V}(\mathscr{I})$. By Hilbert's Nullstellensatz, there exists an integer r such that

$$p \overset{\Delta}{=} q^r \quad .$$

Proposition 2. For each positive integer k, there exists in $\mathscr{I}$ a monic polynomial $p_k(z^k+w^k)$ of minimal degree in (z^k+w^k). The set $\mathbb{R}[z^k+w^k]\cap\mathscr{I}$ contains stable 2D polynomials if and only if $p_k(z^k+w^k)$ is 2D stable.

proof. Note that for any positive k there exists a polynomial $q_k \in \mathbb{R}[z^k+w^k]$ vanishing on $\mathscr{V}(\mathscr{I})$. Hence, by Hilbert's Nullstellensatz, $q_k^r \in \mathscr{I}$ for some positive integer r and $\mathbb{R}[z^k+w^k] \cap \mathscr{I}$ is a non empty (principal) ideal in $\mathbb{R}[z^k+w^k]$.

In order to determine the polynomial p_k which generates $\mathbb{R}[z^k+w^k] \cap \mathscr{I}$, we use the Gröbner basis algorithm.

Assume that a Gröbner basis $\{g_1,g_2,\ldots,g_h\}$ for $\mathscr{I}$ has been computed. The normal form algorithm [5] with respect to this basis is now applied in the following way:

(0) let $h_o \overset{\Delta}{=} 1$

(1) reduce z^k+w^k to a polynomial h_1 in normal form modulo $\mathscr{I}$. Check if there exists $\alpha_o \in \mathbb{R}$ such that $\alpha_o h_o + h_1 = 0$. If the answer is positive, $\alpha_o+(z^k+w^k)$ is the polynomial of minimal degree in $\mathscr{I}$ we are looking for. Otherwise proceed to (2)

(2) reduce $(z^k+w^k)^2$ to a polynomial h_2 in normal form modulo $\mathscr{I}$. Check if there exist α_o,α_1, in $\mathbb{R}$, such that $\alpha_o h_o + \alpha_1 h_1 + h_2 = 0$. If the answer is positive, $\alpha_o+\alpha_1(z^k+w^k) + (z^k+w^k)^2$ is the polynomial of minimal degree in $\mathbb{R}[z^k+w^k] \cap \mathscr{I}$.
If h_o,h_1,h_2 are linearly independent, proceed to reduce $(z^k+w^k)^3$, et coetera.

Because of the first part of the proof, a finite set of linearly dependent $\{h_i\}$ is eventually obtained. This provides a constructive algorithm to compute p_k.

Since each polynomial in $\mathbb{R}[z^k+w^k] \cap \mathscr{I}$ is a multiple of p_k, then $\mathbb{R}[z^k+w^k] \cap \mathscr{I}$ contains 2D stable polynomials if and only if p_k is stable.

As a corollary of Propositions 1 and 2, we have that if $\mathscr{V}(\mathscr{I}) \cap \mathscr{P}_1 = \varnothing$, there exists a positive integer k such that $p_k(z^k+w^k)$ is 2D stable. In fact, by Proposition 1 there exists an integer k such that $\mathbb{R}[z^k+w^k] \cap \mathscr{I}$ contains 2D stable polynomials. Then the algorithm in the proof of Proposition 2 gives a polynomial p_k which is 2D stable.

Adopting the procedure above, we need to check successively 2D stability of polynomials $p_k(z^k+w^k)$, k=1,2. .., until a stable polynomial is found.

Because of the structure of these polynomials, it is possible to check their stability utilizing 1D stability criteria.

In fact any polynomial $p(z^k+w^k)$ is 2D stable if and only if the corresponding polynomial $p(2v) \in \mathbb{R}[v]$ is 1D stable, in the sense that it has no zeros for $|v| \leq 1$. To see this, assume $p(2\gamma) = 0$ for some γ with $|\gamma| \leq 1$. Then for all α and β such that $\alpha^k = \beta^k = \gamma$, $p(\alpha^k+\beta^k) = 0$ and $(\alpha,\beta) \in \mathscr{P}_1$.

Conversely, if $p(\alpha^k+\beta^k) = 0$ and $|\alpha| \leq 1$, $|\beta| \leq 1$, then $\gamma = (\alpha^k+\beta^k)/2$ satisfies $|\gamma| \leq 1$ and $p(2\gamma) = 0$.

3. Examples

In this section two simple examples are given to illustrate the construction of a stabilizing compensator for scalar 2D systems.

The computation is based on the Gröbner basis method and on the normal form algorithm. The single steps for constructing a Gröbner basis and for computing $h_o, h_1, \ldots$, are not developed in details. The interested reader is referred to [5].

Example 1.

Consider the scalar transfer function $n(z,w)/d(z,w)$ with

$$n(z,w) = z^2+2z-1, \quad d(z,w) = -w+z+2$$

Note that $\{n,d\}$ is already a Gröbner basis for $\mathscr{I} = (n,d)$.

Using the normal form algorithm, the polynomials $r_1(z)$ and $r_2(w)$ of minimal degrees in $\mathscr{I}$ are expressed as

$$r_1(z) = z^2+2z-1, \quad r_2(w) = w^2-2w-1$$

They are both (1D and 2D) unstable. This corresponds to the fact that

$$\mathscr{V}(\mathscr{I}) = \{(\alpha_1,\beta_1), (\alpha_2,\beta_2)\} = (-1-\sqrt{2}, 1-\sqrt{2}), (-1+\sqrt{2}, 1+\sqrt{2})$$

and $|\beta_1| < 1$, $|\alpha_2| < 1$.

Then we proceed to consider the minimal degree polynomial in $\mathbb{R}[z+w] \cap \mathscr{I}$.

By using the normal form algorithm, we have:

$$\begin{aligned} 1 &= h_o \\ z+w &= -d+h_1 , \quad & h_1 = 2z+2 \\ (z+w)^2 &= d(-w-3z-2) + 4n + h_2, \quad & h_2 = 8 \end{aligned}$$

Hence

$$0 = -8h_o + 0 \cdot h_1 + h_2 = -8+(z+w)^2 - d(-w-3z-2) - 4n$$

so that

$$p_1(z+w) = (z+w)^2-8 = d(-w-3z-2) + 4n$$

is the minimal degree polynomial in $\mathbb{R}[z+w] \cap \mathscr{I}$. Since the polynomial $p_1(2v) = 4v^2-8$ has both zeros outside the unit circle, $p_1(z+w)$ is 2D stable and a stabilizing compensator is given by

$$W_c = s(z,w)/r(z,w) = 4/(-w-3z-2)$$

Example 2.

Consider the scalar transfer function $n(z,w)/d(z,w)$, with

$$n(z,w) = z^2 - \frac{3}{2} z + \frac{1}{4}, \qquad d(z,w) = w + z - \frac{3}{2}$$

Note that n,d is already a Gröbner basis for $\mathcal{J} = (n,d)$.

Using the normal form algorithm, we obtain $r_1 z = z^2 - \frac{3}{2} z + \frac{1}{4}$ and $r_2(w) = w^2 - \frac{3}{2} w + \frac{1}{4}$ which are both (1D and 2D) unstable.

The polynomial $p_1(z+w) \in \mathbb{R}[z+w] \cap \mathcal{J}$ coincides with $d(z,w)$ and is 2D unstable.

So, proceed to compute $p_2(z^2+w^2)$. We have

$$1 = h_o$$

$$z^2+w^2 = d(-z+w+\frac{3}{2}) + 2n + h_1, \quad h_1 = 7/4$$

Hence

$$0 = -\frac{7}{4} h_o + h_1 = -\frac{7}{4} + z^2 + w^2 - d(-z+w+\frac{3}{2}) - 2n$$

so that

$$p_2(z^2+w^2) = z^2+w^2 - \frac{7}{4} = d(-z+w+\frac{3}{2}) + 2n$$

is the minimal degree polynomial in $\mathbb{R}[z^2+w^2] \cap \mathcal{J}$. This polynomial is 2D unstable since the zero of $p_2(2v) = 2v - \frac{7}{4}$ is $\frac{7}{8} < 1$.

Proceed now to compute $p_3(z^3+w^3)$. We have

$$1 = h_o$$

$$z^3+w^3 = d(w^2 - zw + \frac{3}{2} w - \frac{3}{2} z + 2) + n(w+z+3) + h_1, \quad h_1 = \frac{9}{4}$$

Hence

$$0 = -\frac{9}{4} h_o + h_1 = -\frac{9}{4} + z^3 + w^3 - d(w^2 - zw + \frac{3}{2} w - \frac{3}{2} z + 2) - n(w+z+3)$$

that gives

$$p_3(z^3+w^3) = z^3 + w^3 - \frac{9}{4} = d(w^3 - zw + \frac{3}{2} w - \frac{3}{2} z + 2) + n(w+z+3)$$

Since $p_3(2v) = 2v - \frac{9}{4}$ vanishes in $v = \frac{9}{8} > 1$, $p_3(z^3+w^3)$ is 2D stable and a stabilizing compensator is given by

$$W_c(z,w) = s(z,w)/r(z,w) = (w+z+3)/(w^3 - zw + \frac{3}{2} w - \frac{3}{2} z + 2) .$$

Remark. When n and d are polynomials with rational coefficients, the algorithm given above provides a stable polynomial in $\mathcal{J}$ still having rational coefficients. This is due to the fact that the construction of Gröbner basis and the normal form algorithm involve only rational operations on the coefficients of n and d. Obviously, the same remark holds for the multivariable case, where N and D are polynomial matrices with rational coefficients.

In general any procedure which requires an explicit computation of $\mathcal{V}(\mathcal{J})$, as those introduced in [3] and [4], leads to polynomials having non rational coefficients.

References

1. M. Bisiacco, E. Fornasini, G. Marchesini: "Controller design for 2D systems", in Frequency Domain and State Space Methods for Linear Systems" C.I. Byrnes and A. Lindquist (ed.): pp. 99-113, Elsevier (1986).
2. E. Fornasini, G. Marchesini: "Doubly indexed dynamical systems: state space models and structural properties", Math. Systems Theory, 12, 1, pp. 59-72, (1978).
3. J.P. Guiver and N.K. Bose: "Causal and weakly Causal 2-D filters with applications in stabilization", in "Multidimensional Systems Theory", N.K. Bose (ed.): pp. 52-100, Reidel (1985).
4. M. Bisiacco, E. Fornasini, G. Marchesini: "On some connections between BIBO and internal stability of two-dimensional filters", IEEE Trans. on Circuits and Systems, vol. CAS-32, n. 9, pp. 948-53, (1985).
5. B. Buchberger: "Gröbner bases: an algorithmic method in polynomial ideal theory", in "Multidimensional Systems Theory", N.K. Bose (ed.): pp. 184-232, Reidel (1985).
6. M. Sebek, M. Bisiacco, E. Fornasini: "Controllability and reconstructibility conditions for 2D systems", submitted for publication.