Systems & Control Letters 10 (1988) 45–50
North-Holland

A note on output feedback stabilizability of multivariable 2D systems

E. FORNASINI
Dipartimento di Elettronica e di Informatica, Via Gradenigo 6A, 35131 Padova, Italy

Received 20 May 1987
Revised 15 September 1987

Abstract: The output feedback stabilizability conditions of 2D systems are expressed in term of structural properties of a pair of commuting linear transformations. An algorithm is given for obtaining a stable closed-loop 2D characteristic polynomial.

Keywords: 2D systems, Gröbner basis, Stabilization, Commutative matrices, Polynomial matrices.

1. Introduction

As in the 1D case, the procedure for synthesizing 2D compensators is based on the solution of a suitable polynomial equation in two variables [1].

For many systems, however, it is not possible to arbitrarily assign the variety of the closed-loop polynomial, even when the plant is given by a factor coprime matrix fraction description (MFD)

$$N(z_1, z_2)D^{-1}(z_1, z_2).$$

In fact, this variety must include the set $\mathcal{S}$ of points where the maximal order minors of

$$[D^{\mathrm{T}}\ N^{\mathrm{T}}]$$

simultaneously vanish.

When faced with this rather common difficulty, there are two avenues of approach. The first is to evaluate approximately the set $\mathcal{S}$ and, in case it does not intersect the closed unit polydisc, to assign a closed-loop polynomial that vanishes on $\mathcal{S}$ [7].

The second approach is to derive finite tests for checking feedback stabilizability without any explicit computation of $\mathcal{S}$ and to assign a stable closed-loop polynomial in such a way that the above constraints are automatically satisfied.

Keeping with the spirit of the second approach, the content of this paper is based on some constructive methods of the polynomial ideal theory and in particular on a matrix version of the Gröbner basis algorithm [4]. Some results are still not complete and in their respect this is a progress report on the state of the art on the subject.

2. Characteristic polynomials of closed-loop 2D systems

Let $W(z_1, z_2)$ be a strictly proper transfer matrix of dimension $p \times m$ and let

$$N(z_1, z_2)D^{-1}(z_1, z_2) = W(z_1, z_2)$$

be a right factor coprime MFD.

Consider the ideal $\mathcal{I}$ generated by the minors of maximal order $m_1(z_1, z_2), \dots, m_\rho(z_1, z_2)$ of the matrix

$$\begin{bmatrix} D(z_1, z_2) \\ N(z_1, z_2) \end{bmatrix}. \tag{1}$$

The coprimeness condition on N and D corresponds to assuming that the variety $\mathcal{V}(\mathcal{I})$ is a finite subset of $\mathbb{C} \times \mathbb{C}$ or, equivalently, that the quotient $\mathbb{R}[z_1, z_2]/\mathcal{I}$ is a finite-dimensional vector space over $\mathbb{R}$.

Let

$$\Sigma = (A_1, A_2, B_1, B_2, C),$$

$$x(h+1, k+1) = A_1 x(h, k+1) + A_2 x(h+1, k) + B_1 u(h, k+1) + B_2 u(h+1, k),$$

$$y(h, k) = Cx(h, k),$$

be a 2D realization of $W(z_1, z_2)$ [6], with

$$(I - A_1 z_1 - A_2 z_2,\ B_1 z_1 + B_2 z_2)$$

left factor coprime and

$$(I - A_1 z_1 - A_2 z_2,\ C)$$

0167-6911/88/$3.50

right factor coprime. Under these assumptions, we have [1]

$$\det(I - A_1 z_1 - A_2 z_2) = \det D(z_1, z_2).$$

Consider an output feedback compensator represented by a proper left MFD

$$W_c(z_1, z_2) = R^{-1}(z_1, z_2) S(z_1, z_2)$$

of dimension $m \times p$ and let

$$\Sigma_c = (F_1, F_2, G_1, G_2, H)$$

be a realization of W_c satisfying the relation

$$\det(I - F_1 z_1 - F_2 z_2) = \det R(z_1, z_2). \tag{2}$$

The characteristic polynomial of the closed-loop system obtained by the output feedback connection of Σ and Σ_c is given by

$$\Delta = \det(RD + SN).$$

Using the Binet–Chauchy formula, this is expressed as the sum of the products of all possible minors of maximal order, q_i, $i = 1, 2, \ldots, \rho$, of $[R \ S]$ into the corresponding minors of the same order m_i, $i = 1, 2, \ldots, \rho$, of $[D^T \ N^T]$, that is

$$\det(RD + SN) = \sum_{i=1}^{\rho} q_i m_i.$$

Hence $\det(RD + SN)$ belongs to the ideal $\mathcal{I}$ for any choice of the compensator.

Conversely, given any polynomial $p \in \mathcal{I}$, there exists a compensator $R^{-1}S$ such that [1,2]

$$RD + SN = pI. \tag{3}$$

Hence the characteristic polynomial Δ is a power of p and $\mathcal{V}(\Delta)$ is freely assignable except that it must include $\mathcal{V}(\mathcal{I})$ and does not contain $(0, 0)$. We summarize our conclusions in:

Proposition 1. *The system Σ admits a stabilizing compensator if and only if $\mathcal{V}(\mathcal{I})$ does not intersect the closed unit polydisc $\mathcal{P}_1 \subset \mathbb{C} \times \mathbb{C}$.*

There is a diversity of issues associated with the synthesis of 2D compensators. There is, first of all, the question of checking feedback stabilizability. That is, how can a particular 2D system be recognized as being stabilizable or not, without an explicit computation of $\mathcal{V}(\mathcal{I})$?

There is also the issue of obtaining stable 2D polynomials in $\mathcal{I}$. That is, assuming that feedback stabilization is possible, how does one actually find explicitly a polynomial to substitute for p in equation (3)?

Finally, there are the issues of computing R and S and of realizing the compensator in state space form.

The first issue is emphasized in this paper. The reasons for this emphasis are that stabilizability has been proved to guarantee the finiteness of some procedures for computing stable polynomials in $\mathcal{I}$; equation (3) has been solved by resorting to Gröbner-basis algorithms [7] or to Diophantine-equations techniques [8]; and, finally, state-space realizations of W_c are allowable satisfying condition (2).

The next section recasts some results concerning 2D polynomial ideals in terms of the structure of a suitable pair of commutative matrices (M_1, M_2). Our main objective is the proof of Proposition 2, where 2D stabilizability is related to the spectral properties of M_1 and M_2.

3. Stabilizability conditions

Let $\mathcal{G} = (g_1, g_2, \ldots, g_h)$ denote a Gröbner basis [4] of the ideal $\mathcal{I}$. Then the set $\{p_1 = 1, p_2, \ldots, p_\nu\}$ of monic monomials in z_1 and z_2 that are not multiple of the leading power products of any of the polynomials in $\mathcal{G}$ is finite. The corresponding residue classes modulo $\mathcal{I}$, $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_\nu$, constitute a basis for the vector space $\mathbb{R}[z_1, z_2]/\mathcal{I}$.

Consider now the following maps:

$$\mathcal{Z}_1 : \mathbb{R}[z_1, z_2]/\mathcal{I} \to \mathbb{R}[z_1, z_2]/\mathcal{I},$$
$$q + \mathcal{I} \to z_1 q + \mathcal{I},$$
$$\mathcal{Z}_2 : \mathbb{R}[z_1, z_2]/\mathcal{I} \to \mathbb{R}[z_1, z_2]/\mathcal{I},$$
$$q + \mathcal{I} \to z_2 q + \mathcal{I}.$$

They are both well defined, commuting linear transformations on the $\mathbb{R}$-vector space $\mathbb{R}[z_1, z_2]/\mathcal{I}$. This implies that when $\mathbb{R}[z_1, z_2]/\mathcal{I}$ is represented onto $\mathbb{R}^\nu$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are represented by a pair of commuting matrices M_1 and M_2 in $\mathbb{R}^{\nu \times \nu}$.

Note that the smallest $\mathcal{Z}_1$- and $\mathcal{Z}_2$-invariant subspace generated by $\bar{p}_1 = \bar{1}$ is the whole space $\mathbb{R}[z_1, z_2]/\mathcal{I}$. Thus $\{M_1^i M_2^j e_1, \ i, \ j \in \mathbb{N}\}$ is a set of generators for the space $\mathbb{R}^\nu$.

The construction of M_1 and M_2 is performed along the following lines:

(i) Associate with $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_\nu$ the standard basis vectors in $\mathbb{R}^\nu$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_\nu = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

(ii) For each $\bar{p}_j$, represent $\overline{z_1 p_j}$ as a linear combination of $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_\nu$. This requires a straightforward application of the so called 'normal form algorithm' [4] with respect to the Gröbner basis $(g_1, g_2, \dots, g_h)$.

(iii) The coefficients m'_{ij} in

$$\overline{z_1 p_j} = \sum_j m'_{ij} \bar{p}_i, \quad i = 1, 2, \dots, \nu,$$

are the entries of the matrix M_1 we are looking for.

$$M_1 = [m'_{ij}].$$

(iv) and (v) refer to the representation of $\overline{z_2 p_j}$ for obtaining the columns of M_2, and are analogous to (ii) and (iii).

Example. Consider the transfer function

$$W(z_1, z_2) = \frac{z_1^3 - \frac{5}{3}z_1^2 - \frac{5}{2}z_1}{z_2 - z_1^2 - \frac{3}{2}z_1 - 3} = D^{-1}N.$$

The ideal $\mathcal{I}$ is generated by the maximal order minors in (1), namely N and D, and it is easy to check that N and D constitute a Gröbner basis with regard to the lexicographical ordering of the power products.

The only monomials that are not multiple of the leading products z_1 and z_2 of N and D are

$$p_1 = 1, \quad p_2 = z_1, \quad p_3 = z_2^2.$$

Hence

$$\{ p_i + \mathcal{I} = \bar{p}_i, \ i = 1, 2, 3\}$$

is a basis of $\mathbb{R}[z_1, z_2]/\mathcal{I}$. Associate e_1 with $\bar{1}$, e_2 with $\bar{z}_1$ and e_3 with $\bar{z}_1^2$. Clearly

$$\mathcal{X}_1 \bar{1} = \bar{z}_1, \qquad \mathcal{X}_1 \bar{z}_1 = \bar{z}_1^2.$$

Moreover,

$$z_1^3 = (z_1^3 - \tfrac{5}{2}z_1^2 - \tfrac{5}{2}z_1) + (\tfrac{5}{2}z_1^2 + \tfrac{5}{2}z_1)$$
$$= \tfrac{5}{2}z_1^2 + \tfrac{5}{2}z_1 \quad (\mathrm{mod}\ \mathcal{I}).$$

so that

$$\mathcal{X}_1 \bar{z}_1^2 = \tfrac{5}{2}\bar{z}_1^2 + \tfrac{5}{2}\bar{z}_1.$$

Hence we have

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{5}{2} \end{bmatrix}.$$

The computation of M_2 is a little bit more involved. Repeated applications of the normal form algorithm give

$$z_2 = -z_1^2 + \tfrac{3}{2}z_1 + 3,$$
$$z_1 z_2 = -z_1^2 + \tfrac{1}{2}z_1,$$
$$z_1^2 z_2 = -2z_1^2 - \tfrac{5}{2}z_1 \quad (\mathrm{mod}\ \mathcal{I}).$$

This implies

$$\mathcal{X}_2 \bar{1} = -\bar{z}_1^2 - \tfrac{3}{2}\bar{z}_1 + 3 \cdot \bar{1},$$
$$\mathcal{X}_2 \bar{z}_1 = -\bar{z}_1^2 + \tfrac{1}{2}\bar{z}_1,$$
$$\mathcal{X}_2 \bar{z}_1^2 = -2\bar{z}_1^2 - \tfrac{5}{2}\bar{z}_1,$$

and finally

$$M_2 = \begin{bmatrix} 3 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -1 & -1 & -2 \end{bmatrix}.$$

It is easy to check that M_1 and M_2 commute.

Several properties of the ideal $\mathcal{I}$ and of the (finite) variety $\mathcal{V}(\mathcal{I})$ are strictly related to the structure of the commuting matrices M_1 and M_2 introduced above.

Property 1. *A polynomial $q \in \mathbb{R}[z_1, z_2]$ belongs to the ideal $\mathcal{I}$ if and only if $q(M_1, M_2) = 0$.*

Proof. Let

$$q(z_1, z_2) = \sum_{ij} q_{ij} z_1^i z_2^j \in \mathcal{I}.$$

This implies

$$0 = \sum_{ij} q_{ij} \bar{z}_1^i \bar{z}_2^j = \sum_{ij} q_{ij} \mathcal{X}_1^i \mathcal{X}_2^j \bar{1} \tag{4}$$

and equivalently

$$0 = \sum_{ij} q_{ij} M_1^i M_2^j e_1. \tag{5}$$

Multiplying (5) on the left by $M_1^r M_2^s$ and recalling the matrix commutativity we have

$$0 = \left(\sum q_{ij} M_1^i M_2^j\right)\left(M_1^r M_2^s e_1\right), \quad r, s = 0, 1, 2, \ldots.$$

This proves that $q(M_1, M_2) = 0$.

The viceversa is easily obtained by following backward the lines of the proof above.

Corollary. *Let $\psi_i(\xi)$, $i = 1, 2$, denote the minimum polynomial of M_i. Then $\psi_i(z_i)$ is the minimum degree polynomial in $\mathbb{R}[z_i] \cap \mathscr{I}$.*

A classical result [5] in the theory of commutative matrices is the existence of a common eigenvector for M_1 and M_2. Property 2 clarifies how the pairs of eigenvalues that correspond to common eigenvectors are related to the structure of the variety $\mathscr{V}(\mathscr{I})$.

Property 2. *Let $(\alpha_1, \alpha_2) \in \mathbb{C} \times \mathbb{C}$. Then $(\alpha_1, \alpha_2) \in \mathscr{V}(\mathscr{I})$ if and only if M_1 and M_2 have a common eigenvector v and*

$$M_1 v = \alpha_1 v, \qquad M_2 v = \alpha_2 v. \tag{6}$$

Proof. Assume that (6) holds and consider any polynomial

$$q(z_1, z_2) = \sum q_{ij} z_1^i z_2^j$$

in $\mathscr{I}$. By Property 1,

$$0 = q(M_1, M_2) = \sum q_{ij} M_1^i M_2^j,$$

$$0 = \sum q_{ij} M_1^i M_2^j v = \sum q_{ij} \alpha_1^i \alpha_2^j v,$$

$$0 = \sum q_{ij} \alpha_1^i \alpha_2^j = q(\alpha_1, \alpha_2).$$

Since $q(z_1, z_2)$ is arbitrary in $\mathscr{I}$, $(\alpha_1, \alpha_2) \in \mathscr{V}(\mathscr{I})$. Viceversa, assume that (α_1, α_2) belongs to $\mathscr{V}(\mathscr{I})$ and denote by k_1 and k_2 the algebraic multiplicities of $z_1 - \alpha_1$ and $z_2 - \alpha_2$ in ψ_1 and ψ_2 respectively,

$$\psi_1(z_1) = h_1(z_1)(z_1 - \alpha_1)^{k_1}, \quad h_1(\alpha_1) \neq 0,$$

$$\psi_2(z_2) = h_2(z_2)(z_2 - \alpha_2)^{k_2}, \quad h_2(\alpha_2) \neq 0.$$

Note that $h_2(z_1)h_2(z_2) \in \mathscr{I}$, since $h_1(\alpha_1)h_2(\alpha_2) \neq 0$. Let t, $0 \leqslant t < k_1$, be the largest integer such that

$$h_1(z_1)h_2(z_2)(z_1 - \alpha_1)^t \notin \mathscr{I}$$

and let r, $0 \leqslant r < v$, be the largest integer such that

$$s(z_1, z_2) = h_1(z_1)h_2(z_2)(z_1 - \alpha_1)^t (z_2 - \alpha_2)^r \notin \mathscr{I}.$$

We then have that

$$s(z_1, z_2) \notin \mathscr{I},$$

$$s(z_1, z_2)(z_1 - \alpha_2) \in \mathscr{I},$$

$$s(z_1, z_2)(z_2 - \alpha_2) \in \mathscr{I}.$$

Hence

$$v \triangleq s(M_1, M_2) e_1 \neq 0 \tag{7}$$

and

$$(M_1 - \alpha_1 I)v = 0, \qquad (M_2 - \alpha_2 I)v = 0.$$

The last two equations show that the vector v defined in (7) is a common eigenvector.

A different characterization of the variety $\mathscr{V}(\mathscr{I})$ is based on the Frobenius theorem [9] on simultaneous triangularization of commutative matrices.

Theorem (Frobenius). *Let M_1 and M_2 be a pair of real commutative matrices. Then M_2 and M_2 can be simultaneously reduced to triangular form over $\mathbb{C}$ by a similarity transformation.*

Property 3. *Let $T_1 = [t_{ij}^1]$ and $T_2 = [t_{ij}^2]$ be a pair of common triangular forms of the matrices M_1 and M_2. Then (α_1, α_2) in $\mathbb{C} \times \mathbb{C}$ belongs to $\mathscr{V}(\mathscr{I})$ if and only if there exists an integer i such that*

$$t_{ii}^1 = \alpha_1, \qquad t_{ii}^2 = \alpha_2.$$

Proof. Since T_1 and M_1 as well as T_2 and M_2 are connected by a common similarity transformation, Property 1 holds for matrices T_1 and T_2 too. Therefore, $q(z_1, z_2)$ belongs to $\mathscr{I}$ if and only if $q(T_1, T_2) = 0$.

Let $q(z_1, z_2) \in \mathscr{I}$. Then

$$0 = q(T_1, T_2)$$

$$= \begin{bmatrix} q(t_{11}^1, t_{11}^2) & & * \\ & q(t_{22}^1, t_{22}^2) & \\ 0 & & q(t_{rr}^1, t_{rr}^2) \end{bmatrix}.$$

Since $q(z_1, z_2)$ is arbitrary in $\mathscr{I}$, $q(t_{ii}^1, t_{ii}^2) = 0$ implies $(t_{ii}^1, t_{ii}^2) \in \mathscr{V}(\mathscr{I})$.

Viceversa, let $(\alpha_1, \alpha_2) \in \mathscr{V}(\mathscr{I})$ and suppose, by contradiction,

$$(\alpha_1, \alpha_2) \neq (t_{ii}^1, t_{ii}^2), \quad i = 1, 2, \ldots, \nu.$$

Then, there exists a polynomial $q(z_1, z_2)$ vanishing in (t_{ii}^1, t_{ii}^2), $i = 1, 2, \ldots, \nu$, and different from zero in (α_1, α_2). We therefore have

$$q(T_1, T_2) = \begin{bmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

so that

$$q^{\nu}(T_1, T_2) = 0 \quad \text{and} \quad q^{\nu}(z_1, z_2) \in \mathscr{I}.$$

Since $q^{\nu}(\alpha_1, \alpha_2)$ is different from zero, $(\alpha_1, \alpha_2) \notin \mathscr{V}(\mathscr{I})$, contrary to the assumption.

The condition for output feedback stabilizability, given in Proposition 1, can be restated in terms of structural properties of the commutative matrices M_1 and M_2. The following proposition is a straightforward consequence of Properties 2 and 3 above:

Proposition 2. *The following facts are equivalent*:
(i) *Σ is output feedback stabilizable*;
(ii) *any common eigenvector of M_1 and M_2 refers to a pair of eigenvalues (α_1, α_2) such that $|\alpha_1| > 1$ and/or $|\alpha_2| > 1$.*
(iii) *any pair (t_{ii}^1, t_{ii}^2) in the triangular form of M_1 and M_2 satisfies $|t_{ii}^1| > 1$ and/or $|t_{ii}^2| > 1$.*

Remark. The proposition above does not provide an efficient algorithm for testing output feedback stabilizability of Σ. In fact, simultaneous triangularization of M_1 and M_2 as well as the computation of common eigenvectors cannot be performed in a finite number of steps. However, Properties 2 and 3 have a theoretical intrinsic interest, in the sense that they could constitute a good starting point for obtaining linear stabilizability criteria in the style of Lyapunov equations.

In some particular cases, stabilizability conditions are easy to check. For instance, all the eigenvalues of M_1 have modulus greater than one if and only if there exists a negative definite matrix P satisfying the linear matrix equation

$$M_1^{\mathrm{T}} P M_1 - P = -Q \quad (Q \text{ positive definite}).$$

In this case $\psi_1(z_1)$, the minimum polynomial of M_1, is devoid of zeros in the closed unit disk $\{z_1: |z_1| \leq 1\}$. Since $\psi_1 \in \mathscr{I}$ by the Corollary of Property 1, there exists a stabilizing compensator such that the closed-loop 2D characteristic polynomial of the system is given by some power of $\psi_1(z_1)$.

Analogous considerations hold if all the eigenvalues of M_2 have modulus greater than one.

A more general situation comes out when some products $M_1^i M_2^j$ are devoid of eigenvalues in the closed unit disk.

This happens if and only if the equation

$$(M_1^i)^{\mathrm{T}} (M_2^j)^{\mathrm{T}} P M_2^j M_1^i - P = -Q \tag{8}$$

(Q positive definite) admits a negative definite solution P.

Referring to triangular forms, it is easy to convince ourselves that condition (iii) in Proposition 2 is satisfied.

In this case the minimum polynomial $\psi_{ij}(\xi)$ of $M_1^i M_2^j$ is devoid of zeros in the closed unit disk, and the variety of

$$\psi_{ij}(z_1^i z_2^j) = (z_1^i z_2^j - \delta_1)(z_1^i z_2^j - \delta_2) \cdots (z_1^i z_2^j - \delta_t)$$

does not intersect $\mathscr{P}_1$, since $|\delta_i| > 1$, $i = 1, 2, \ldots$.

4. Construction of a stable closed-loop polynomial

The previous remark shows that in some cases it is possible to construct directly a 2D stable polynomial in $\mathscr{I}$: whenever equation (8) admits a negative definite solution P, the minimum polynomial of $M_1^i M_2^j$ can be used.

Suppose, on the other hand, we have some finite criterion (possibly based on M_1 and M_2) for deciding whether $\mathscr{I}$ includes a stable polynomial and that the criterion does not provide any constructive technique for obtaining such a polynomial.

In this case an iterative procedure for obtaining a stable closed-loop polynomial has been presented in [3]. Nevertheless, the analysis of the algorithm is simplified by resorting to the matrices M_1 and M_2 that characterize the ideal $\mathscr{I}$. This is fairly clear from the proofs of Propositions 3 and 4 below.

Proposition 3. *For each positive integer k, there exists in $\mathscr{I}$ a monic polynomial $\psi_k(z_1^k + z_2^k)$ of* *minimal degree in* $(z_1^k + z_2^k)$. *The set*

$$\mathbb{R}[z_1^k + z_2^k] \cap \mathscr{I}$$

contains stable 2D *polynomials if and only if* $\psi_k(z_1^k + z_2^k)$ *is* 2D *stable.*

Proof. Consider the matrix $M_1^k + M_2^k$ and denote by $\psi_k(\xi)$ its minimum polynomial. Then $\psi_k(z_1 + z_2) \in \mathscr{I}$, since $\psi_k(M_1^k + M_2^k) = 0$, and minimality follows from the definition of ψ_k.

Since each polynomial in $\mathbb{R}[z_1^k + z_2^k] \cap \mathscr{I}$ is a multiple of ψ_k, it follows that $\mathbb{R}[z_1^k + z_2^k] \cap \mathscr{I}$ contains 2D stable polynomials if and only if ψ_k is stable.

Proposition 4. *Assume* $\mathscr{V}(\mathscr{I}) \cap \mathscr{P}_1 = 0$. *Then there exist an integer* k *and a* 2D *stable polynomial in* $\mathbb{R}[z_1^k + z_2^k] \cap \mathscr{I}$.

Proof. Referring to commutative matrices in triangular form, any (complex) pair (t_{ii}^1, t_{ii}^2) satisfies $|t_{ii}^1| > 1$ and/or $|t_{ii}^2| > 1$. It is not difficult to show that there exists an integer h such that

$$\left|\left(t_{ii}^1\right)^h + \left(t_{ii}^2\right)^h\right| > 2 \quad \text{for } i = 1, 2, \ldots.$$

Let $\psi_h(\xi)$ denote the minimum polynomial of $M_1^h + M_2^h$. The polynomial $\psi_h(z_1^h + z_2^h)$ belongs to $\mathscr{I}$ by Proposition 2 and factorizes as

$$\psi_h\left(z_1^h + z_2^h\right) = \left(z_1^h + z_2^h - \gamma_1\right)\left(z_1^h + z_2^h - \gamma_2\right) \cdots \left(z_1^h + z_2^h - \gamma_t\right).$$

Since $\gamma_h(\xi)$ is devoid of zeros in the disk $\{\xi\colon |\xi| \le 2\}$, it follows that $|\gamma_i| > 2$, $i = 1, 2, \ldots, t$, which in turn implies 2D stability of all factors $z_1^h + z_2^h - \gamma_i$.

The proof of Proposition 4 immediately suggests an algorithm for computing a 2D stable polynomial in $\mathscr{I}$, based on the following steps.

1. Consider the sequence of matrices

$$S_1 = \frac{M_1 + M_2}{2}, \quad S_2 = \frac{M_1^2 + M_2^2}{2}, \quad S_3 = \frac{M_1^3 + M_2^3}{2}, \ldots,$$

and solve the matrix equations

$$S_i^{\mathrm{T}} P_i S_i - P_i = -I, \quad i = 1, 2, 3, \ldots,$$

until a negative definite P_i is found. By Proposition 4, this procedure stops after a finite number of steps if and only if the system is stabilizable. Let h be the first integer such that S_h is negative definite.

2. Construct the minimum polynomial $\psi_h(\xi)$ of $M_1 + M_2$. Then $\psi_h(z_1 + z_2)$ is a stable 2D polynomial in $\mathscr{I}$.

References

[1] M. Bisiacco, E. Fornasini and G. Marchesini, Controller design for 2D systems, in: C.I. Byrnes and A. Lindquist, Eds., *Frequency Domain and State Space Methods for Linear Systems* (Elsevier, Amsterdam, 1986) pp. 99–113.

[2] M. Bisiacco, E. Fornasini and G. Marchesini, On some connections between BIBO and internal stability of two-dimensional filters, *IEEE Trans. Circuits and Systems* **32** (9) (1985) 948–53.

[3] M. Bisiacco, E. Fornasini and G. Marchesini, Causal 2D compensators: Stabilization algorithms for multivariable 2D systems, *Proc. 25th CDC*, Athens (1986).

[4] B. Buchberger, Gröbner bases: An algorithmic method in polynomial ideal theory, in: *Multidimensional Systems Theory* (Reidel, Dordrecht, 1985) pp. 184–232.

[5] M.P. Drazin, J.W. Dungey and K.W. Grüenberg, Some theorems on commutative matrices, *J. London Math. Soc.* **26** (1951) 221–228.

[6] E. Fornasini and G. Marchesini, Doubly indexed dynamical systems: State space models and structural properties, *Math. Systems Theory* **12** (1) (1978) 59–72.

[7] J.P. Guiver and N.K. Bose, Causal and weakly causal 2D filters with applications in stabilization, in: *Multidimensional Systems Theory* (Reidel, Dordrecht, 1985) pp. 52–100.

[8] M. Sebek, M. Bisiacco and E. Fornasini, Controllability and reconstructibility conditions for 2D systems, *IEEE Trans. Automat. Control*, to appear.

[9] D.A. Suprunenko and R.I. Tyshkevich, *Commutative Matrices* (Academic Press, New York, 1968).