DYNAMIC REGULATION OF TWO-DIMENSIONAL SYSTEMS

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ABSTRACT

The possibilities of modifying the dynamical behaviour of 2D state space models by output feedback compensation are investigated, and a characterization of the closed loop polynomial varieties is given. The polynomial matrix approach allows to show that hidden modes and rank singularities are the unique constraints we have to cope with in the compensator synthesis. The proof of this result is based on some algebraic manipulations of 2D MFD's and on a coprime realization algorithm presented in Section 3.

INTRODUCTION

In this paper we shall be concerned with the effects of output feedback compensators on 2D systems, whose dynamical behaviour is represented by state space models. We shall approach this subject from the point of view of classical system theory, by connecting the structural properties of the state variable description with the possibility of assigning the closed loop characteristic polynomial via output feedback.

The analysis will be developed on the basis of 2D polynomial matrix algebra. 2D matrix fraction descriptions (MFD's) provide a very convenient tool to investigate how input/output maps (characteristic of the classical methods in filter theory) are associated with internal representations (universally adopted in control problems) and to obtain the transfer matrices of compensators by solving Bézout polynomial equations in two variables.

A few observations might serve to motivate this detailed reexamination of feedback theory in the 2D context. Recently there has been an increasing interest in studying 2D control problems, which have been tackled using mainly two different approaches.

The first approach is essentially reductionist, in the sense that 2D systems are viewed as 1D systems over the ring of polynomials in one variable, while the second fully exploits the partial ordering of the 2D structure and data processing is not connected with any preferred direction.

By pursuing the first approach [1,2], in the literature have been introduced compensators that preserve the quarter plane causality and compensators that do not. However in the former case the feedback performances that can be obtained are not so good as in case of 2D compensators with unconstrained structure. Moreover most results apply there to Roesser model only.

Following the second approach, some authors [3] dealt with an input/output analysis of 2D systems, based on a factorization of the plant and compensator transfer matrices in two variables, others [4,5] with state-space models and 2D PBH controllability and reconstructibility criteria. The unquestioned success of the i/o and the state space compensation methods in 1D theory mainly relies on the canonical properties of minimal realizations, allowing for a polynomial matrix (i.e. input/output) solution of control problems and for a subsequent synthesis of the compensator transfer function, that does not introduce unwanted hidden modes in the feedback loop. However in the 2D case the classical techniques have presented a lot of difficulties to be extended, since the equivalence between minimal and reachable and observable realization does not longer hold.

One of our objective in this paper is to formulate a realization procedure which leads to closed loop 2D systems free of hidden modes without pursuing the state space minimization. The results are then applied to the analysis of closed loop characteristic polynomials of 2D systems in state space form.

Finally, some algorithms are presented for deciding whether a given polynomial in two variables is assignable as a closed loop characteristic of a 2D system and for computing the compensator transfer matrix which produces such a polynomial.

PRELIMINARY NOTATIONS AND STATEMENT OF THE PROBLEM

A 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ is a dynamical model [6]

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) + \\ &\quad + B_1 u(h,k+1) + B_2 u(h+1,k) \\ y(h,k) &= Cx(h,k) + Du(h,k) \end{aligned} \qquad (1)$$

where the local state x is an n-dimensional vector over the real field $\mathbb{R}$, input and output functions take values in $\mathbb{R}^m$ and $\mathbb{R}^p$, A_1, A_2, B_1, B_2, C and D are matrices of suitable dimensions with entries in $\mathbb{R}$.

Denoting by

$$\mathscr{X}_0 = \sum_{i=-\infty}^{+\infty} x(i,-i) z_1^i z_2^{-i}$$

the global state on the separation set

$$\mathscr{C}_0 = \{(i,j) : i+j=0\}$$

and by

$$X(z_1,z_2) = \sum_{i+j \geq 0} x(i,j) z_1^i z_2^j$$

$$U(z_1,z_2) = \sum_{i+j \geq 0} u(i,j) z_1^i z_2^j$$

$$Y(z_1,z_2) = \sum_{i+j \geq 0} y(i,j) z_1^i z_2^j$$

the state, input and output functions, respectively, one gets from (1)

$$(I-A_1 z_1 - A_2 z_2) X(z_1,z_2) - (B_1 z_1 + B_2 z_2) U(z_1,z_2) = \mathcal{X}_o \qquad (2)$$

and

$$Y(z_1,z_2) = CX(z_1,z_2) + DU(z_1,z_2). \qquad (3)$$

So, assuming zero initial conditions $\mathcal{X}_o = 0$, the rational transfer matrix

$$W(z_1,z_2) = C(I-A_1 z_1 - A_2 z_2)^{-1} (B_1 z_1 + B_2 z_2) + D \qquad (4)$$

gives the input-out map

$$Y(z_1,z_2) = W(z_1,z_2) U(z_1,z_2).$$

A 2D system Σ is strictly proper when D=0 and is finite memory if for any set of initial conditions $\mathcal{X}_o$, the free state evolution goes to zero in a finite number of steps. Denoting by

$$\Delta(z_1,z_2) := \det(I-A_1 z_1 - A_2 z_2) \qquad (5)$$

the characteristic polynomial of Σ, the finite memory property is equivalent [4] to the condition $\Delta(z_1,z_2)=1$.

Suppose now that a 2D strictly proper plant $\Sigma = (A_1, A_2, B_1, B_2, C)$ has been given, and consider the feedback connection (see fig. 1) with a compensator $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$

$$\begin{aligned} \bar{x}(h+1,k+1) &= F_1 \bar{x}(h,k+1) + F_2 \bar{x}(h+1,k) + G_1 y(h,k+1) + G_2 y(h+1,k) \\ \bar{y}(h,k) &= H \bar{x}(h,k) + J\, y(h,k) \\ u(h,k) &= \bar{y}(h,k) + v(h,k) \end{aligned} \qquad (6)$$

where $v(h,k)$ is the external input at (h,k)

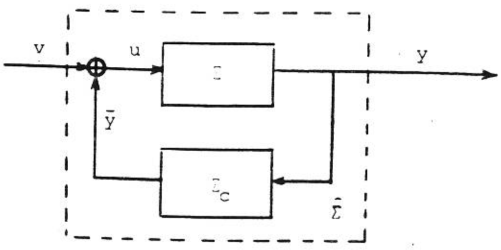

fig. 1

The local state $\hat{x} = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}$ of the resulting closed loop 2D system $\hat{\Sigma}$ updates according to the following transition matrices

$$\hat{A}_1 = \left[\begin{array}{c|c} A_1 + B_1 JC & B_1 H \\ \hline G_1 C & F_1 \end{array}\right] \qquad \hat{A}_2 = \left[\begin{array}{c|c} A_2 + B_2 JC & B_2 H \\ \hline G_2 C & F_2 \end{array}\right] \qquad (7)$$

and the corresponding closed loop characteristic polynomial of $\hat{\Sigma}$

$$\hat{\Delta}(z_1,z_2) := \det(I - \hat{A}_1 z_1 - \hat{A}_2 z_2) \qquad (8)$$

depends on the matrices of the compensator. We say that a polynomial $c(z_1,z_2) \in \mathbb{R}[z_1,z_2]$ is assignable if it can be assumed as the closed loop characteristic polynomial of the feedback connection of Σ and Σ_c, for a suitable compensator Σ_c.

Given Σ, the set of assignable polynomials is a proper subset of $\mathbb{R}[z_1,z_2]$. A first obvious constraint on assignable polynomials is that the constant term must be one. Depending on the structure of Σ, further constraints can arise, relative either to the plant transfer matrix or to the particular state space model that realizes it.

Referring to that, our objectives are the following

i) for a given plant, characterize the subset of assignable polynomials

ii) derive the conditions to be fulfilled in order that the subset above includes all polynomials in two variables with unit constant term

iii) given any specific $c(z_1,z_2)$ in $\mathbb{R}[z_1,z_1]$, decide about its assignability (or at least the assignability of its variety $\mathcal{V}(c)$) and then realize Σ_c.

The 2D matrix fraction description (MFD) approach provides the natural setting for studying the problems above. In section 3, the elementary properties of MFD's will be briefly recalled and some new results will be presented, that will support the feedback analysis and the synthesis procedures of sections 4 and 5.

SOME PROPERTIES OF 2D MFD's

Let $A(z_1,z_2)$ and $B(z_1,z_2)$ be matrices with elements in $\mathbb{R}[z_1,z_2]$, of dimensions kxh and hxk respectively and assume $\det A(z_1,z_2) \neq 0$.

Denote by $m_1, m_2, \ldots, m_\nu$ the maximal order minors of

$$[A(z_1,z_2) \quad B(z_1,z_2)] \qquad (9)$$

and by

$$\mathcal{J}(A,B) \stackrel{\Delta}{=} (m_1, m_2, \ldots, m_\nu)$$

the ideal generated by $m_1, m_2, \ldots, m_\nu$.

Clearly, matrix (9) is full rank except in the points of the complex variety

$$\mathcal{V}(A,B) \stackrel{\Delta}{=} \mathcal{V}(\mathcal{J}(A,B)),$$

where the maximal order minors of (9) simultaneously vanish. When $\mathcal{V}(A,B) \neq \phi$, A and B are called left zero coprime (l.z.c.). A necessary and sufficient condition for left zero coprimeness is that the Bézout equation

$$A\,X + B\,Y = I \qquad (10)$$

admits a 2D polynomial matrix solution in X and Y.

A hxh polynomial matrix $Q(z_1,z_2)$ is called a common left divisor of A and B if

$$A = Q\hat{A} \qquad B = Q\hat{B} \tag{11}$$

where $\hat{A}$ and $\hat{B}$ are polynomial matrices. A and B are left factor coprime (l.f.c.) if det Q is a nonzero constant for all Q satisfying (11).

If A and B are not l.f.c., a greatest common left divisor (GCLD) can be extracted using the primitive factorization algorithm [7]. Different procedures can also be adopted [8]. Left factor coprimeness is implied, but does not imply, left factor coprimeness. In fact, l.f. coprimeness is equivalent to the finite cardinality of $\mathcal{V}(A,B)$.

Let $W(z_1,z_2)$ be a hxk rational matrix in two variables and suppose that the above polynomial matrices A and B satisfy

$$W = A^{-1}B \tag{12}$$

Then $A^{-1}B$ is a left MFD of W. If further A and B are l.f.c., A^{-1} is a left coprime MFD of W.

$W(z_1,z_2)$ is proper if any one of the following equivalent conditions holds:

i) W admits a l.c.MFD $A^{-1}B$, with $A(0,0) = I$

ii) for any coprime l.MFD $A^{-1}B = W$, det $A(0,0) \neq 0$

iii) the entries of W are proper rational functions.

In the sequel, when dealing with proper left-coprime MFD's, we shall always assume $A(0,0) = I$ and, in particular, $A(z_1,z_2) = I$ when $W(z_1,z_2)$ is a FIR filter.

Right MFD's can be introduced with the obvious changes. In particular, given a r.MFD

$$W = C A^{-1}$$

we denote by $\mathcal{I}(C,A)$ the ideal generated by the maximal order minors of $[A^T(z_1,z_2)\ C^T(z_1,z_2)]$.

The following theorem shows that the ideals generated by the maximal order minors of a coprime MFD of $W(z_1,z_2)$ do not depend on the particular representation (left or right).

Theorem 1 Let $N_R D_R^{-1} = D_L^{-1} N_L$ be two coprime MFD's of $W(z_1,z_2)$. Then $\mathcal{I}(N_R,D_R) = \mathcal{I}(D_L,N_L)$.

The proof depends on two technical Lemmas.

Lemma 1 [7] Under the same hypotheses of theorem 1, det D_L = det D_R. Moreover, if C, A and B are 2D polynomial matrices such that $C A^{-1} B = W$, then

i) det $D_L \mid$ det A

ii) det D_L = det A if and only if $C A^{-1}$ and $A^{-1}B$ are factor coprime MFD's.

Lemma 2 Consider the polynomial matrix

$$U = \begin{bmatrix} X & -YC \\ B & A \end{bmatrix} \tag{13}$$

where X and A are square matrices and det A is a nonzero polynomial. Then any r.MFD $N_R D_R^{-1}$ of the rational matrix $C A^{-1}$ satisfies the following equation

$$\det U = \frac{\det A}{\det D_R} \det (X D_R + Y N_R) \tag{14}$$

The proof of Lemma 2 is an immediate consequence of the determinantal formula for block matrices.

proof of Theorem 1. Putting $A=D_L$, $B=N_L$, $C=I$ in lemma 2 and recalling lemma 1, one gets

$$\det \begin{bmatrix} X & -Y \\ N_L & D_L \end{bmatrix} = \det(X D_R + Y N_R) \tag{15}$$

Assume that $[X\ -Y]$ is any permutation of the columns of $[I\ O]$. Then, except for the sign, the right and left hand sides of (15) are maximal order minors of $[N_L\ D_L]$ and $[D_R^T\ N_R^T]$ respectively. Moreover, as $[X\ -Y]$ varies over the set of all permutations, we get a bijective correspondence between the maximal order minors of $[N_L\ D_L]$ and $[D_R^T\ N_R^T]$. So $\mathcal{I}(D_L,N_L) = \mathcal{I}(N_R,D_R)$.

Consequently, there is no ambiguity in defining the transfer matrix ideal $\mathcal{I}(W)$ as the ideal of the maximal order minors associated with an arbitrary right or left coprime MFD of W. The corresponding transfer matrix variety $\mathcal{V}(W) := \mathcal{V}(\mathcal{I}(W))$ is a (possibly empty) finite set, whose points are called the rank singularities of W.

REMARK $\mathcal{V}(W)$ is empty if and only if factor coprime MFD's of $W(z_1,z_2)$ are zero coprime. This makes a substantial difference with respect to 1D transfer matrices, where zero coprimeness and factor coprimeness are equivalent concepts, and $\mathcal{V}(W)$ is always empty. As we shall see, the existence of rank singularities plays an essential role in the closed loop polynomial assignability problem.

Theorem 2 Assume that C, A, B, N_R, D_R and 2D polynomial matrices of suitable sizes with

$$W(z_1,z_2) = C A^{-1} B = N_R D_R^{-1}$$

and $N_R D_R^{-1}$ is right factor coprime

Then

$$\mathcal{V}(A,B) \cup \mathcal{V}(C,A) = \mathcal{V}(W) \cup \mathcal{V}(h)$$

where

$$h = \det A / \det D_R$$

is a 2D polynomial (by Theorem 1).

In proving Theorem 2, we need the following Lemma 3, that provides same additional properties of the matrix U introduced in Lemma 2.

Lemma 3 [9] Let $(z_1^\circ, z_2^\circ) \in \mathbb{C} \times \mathbb{C}$. The matrix $U(z_1^\circ, z_2^\circ)$ is singular for any X and Y if and only if at least one of the matrices $[A \; B]$ and $\begin{bmatrix} A \\ C \end{bmatrix}$ is singular when evaluated at (z_1°, z_2°).

proof of Theorem 2. By Lemma 3, we have

$$(z_1^\circ, z_2^\circ) \in \mathcal{V}(A,B) \cup \mathcal{V}(C\;A) \Leftrightarrow \det U(z_1^\circ, z_2^\circ)=0, \; \forall X, Y \quad (16)$$

and, applying Lemma 2,

$$\det U(z_1^\circ, z_2^\circ) = 0 \Leftrightarrow h \det(XD_R+YN_R)(z_1^\circ, z_2^\circ) = 0 \quad (17)$$

Next observe that

$$(z_1^\circ, z_2^\circ) \in \mathcal{V}(W) \Leftrightarrow \det(XD_R+YN_R)(z_1^\circ, z_2^\circ) = 0, \quad \forall X, Y \quad (18)$$

This follows directly from the equivalence of the statements below:

i) $(z_1^\circ, z_2^\circ) \notin \mathcal{V}(W)$

ii) $\begin{bmatrix} D_R(z_1^\circ, z_2^\circ) \\ N_R(z_1^\circ, z_2^\circ) \end{bmatrix}$ is full rank at (z_1°, z_2°)

iii) there exist constant matrices X°, Y° such that

$$X^\circ D_R(z_1^\circ, z_2^\circ) + Y^\circ N_R(z_1^\circ, z_2^\circ) = I \quad (19)$$

iv) there exist polynomial matrices X and Y such that $X(z_1^\circ, z_2^\circ) = X^\circ$, $Y(z_1^\circ, z_2^\circ) = Y^\circ$ and (19) holds.

Finally, using (16), (17) and (18), one gets that (z_1°, z_2°) is in $\mathcal{V}(A,B) \cup \mathcal{V}(C,A)$ if and only if it belongs to $\mathcal{V}(h) \cup \mathcal{V}(W)$.

COPRIME REALIZATIONS

As we shall see in greater detail in the next section, the compensator synthesis is performed in two steps. The first one consists in solving a 2D Bézout equation, whose coefficients are determined by the plant transfer matrix and by some requirements on the structure of the characteristic polynomial of the closed loop system. The solution provides us with an input/output representation of the compensator and the second step calls for a state space realization of it.

A problem which naturally arises in connection with the realization procedure is how to avoid the inclusion of unwanted "hidden modes" in the closed loop polynomial.

In order to introduce a concrete definition of the concept of "hidden modes" in 2D state space models, we aim to consider two complex varieties, associated with the polynomial matrices of the PBH controllability and reconstructibility criteria, and to establish some connections between these varieties and the rank singularities of the transfer matrix. Interestingly, a 2D realization of $W(z_1,z_2)$ is free of hidden modes if and only if the join of the above varieties coincide with $\mathcal{V}(W)$. So the natural question arises as to whether such realization does exist and how may it be computed.

The realization algorithm presented at the end of this section, gives a positive answer to this question and provides a constructive realization procedure.

In designing state feedback laws and observers of a 2D system $\Sigma = (A_1,A_2,B_1,B_2,C)$, the following two matrices proved to be of paramount importance [4]:

$$\mathcal{R} = [\, I-A_1z_1-A_2z_2 \quad B_1z_1+B_2z_2 \,] \quad (20)$$

$$\mathcal{O} = \begin{bmatrix} I-A_1z_1-A_2z_2 \\ C \end{bmatrix} \quad (21)$$

In fact, the controllability and reconstructibility properties of Σ can be translated in terms of rank conditions on $\mathcal{R}$ and $\mathcal{O}$, that will be therefore called PBH controllability and PBH reconstructibility matrices.

Denote for short by $\mathcal{V}(\mathcal{R})$ and $\mathcal{V}(\mathcal{O})$ the complex varieties $\mathcal{V}(I-A_1z_1-A_2z_2,\; B_1z_1+B_2z_2)$ and $\mathcal{V}(C,\; I-A_1z_1-A_2z_2)$, and assume that $N_RD_R^{-1}$ is any r.c.MFD of the system matrix. Then Theorem 2 can be easily rephrased in terms of $\mathcal{R}$, $\mathcal{O}$ and

$$h = \det(I-A_1z_1-A_2z_2)/\det D_R \,, \quad (22)$$

giving

$$\mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O}) = \mathcal{V}(h) \cup \mathcal{V}(W) \quad (23)$$

Of course, if we assume that h is a nonzero constant, the finite cardinality of the right hand side in (23) implies the factor coprimeness both of $C(I-A_1z_1-A_2z_2)^{-1}$ and $(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)$. Viceversa, if h is a nonconstant polynomial, $\mathcal{R}$ and/or $\mathcal{O}$ are not full rank along the algebraic curves associated with the irreducible factors of h. In this case, the uncontrollable and the unreconstructible modes (collectively, hidden modes) refer to the irreducible factors of h, that appear as common factors of the maximal order minors of $\mathcal{R}$ and $\mathcal{O}$ respectively.

By definition, a realization Σ of $W(z_1,z_2)$ is coprime if Σ is free of hidden modes.

As a matter of fact, there are many equivalent definitions of coprime realizations. These are summarized in the following corollary, whose proof is a stright forward consequence of (23).

Corollary Let $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ be a realization of $W(z_1,z_2)$ and assume that $N_RD_R^{-1}$ is a r.c.MFD of $W(z_1,z_2)$. Then the following facts are equivalent

i) $\det D_R = \det(I-A_1z_1-A_2z_2)$

ii) $C(I-A_1z_1-A_2z_2)^{-1}$ and $(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)$ are right and left coprime MFD's respectively

iii) $\mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O}) = \mathcal{V}(W)$

iv) Σ is a coprime realization.

REMARK Coprime realizations are not necessarily minimal realizations, since their local state space needs not have minimal dimension.

For instance, the coprime realization

$$A_1 = A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D = 0$$

of the transfer function $(z_1+z_2)/(1-z_1-z_2)$ is non minimal. Even more, it is easy to show that whenever a transfer matrix admits a coprime realization, then it admits infinitely many.

The question of the existence of coprime realizations for any proper transfer matrix is positively answered by Theorem 3, that provides also an explicit realization procedure. It should be noticed that the Theorem proves something more, since the statement refers to any (not necessarily coprime) r.MFD of $W(z_1,z_2)$. satisfying $D_R(0,0) = I$.

Theorem 3 Let $N_R D_R^{-1}$ be a r.MFD of the transfer matrix $W(z_1,z_2)$ satisfying $D_R(0,0) = I$. Then there exists a 2D system $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ that realizes W and satisfies the following conditions

i) $\mathcal{R}(z_1,z_2)$ is full rank in $\mathbb{C} \times \mathbb{C}$

ii) $\det(I - A_1 z_1, A_2 z_2) = \det D_R$

In particular, if $N_R D_R^{-1}$ is r.coprime, Σ turns out to be a coprime realization of $W(z_1,z_2)$.

sketch of the proof There is no restriction in assuming $W(z_1,z_2)$ strictly proper, so that $N(0,0) = 0$. Denote by k_i, $i = 1,2,\dots,m$ the column degree of the i-th column of

$$\begin{bmatrix} N_R \\ D_R \end{bmatrix},$$

that is the degree of the polynomial of maximal degree in the i-th column. We can write

$$D_R = I_m - D_{HT}\Psi, \qquad N_R = N_{HT}\Psi$$

where

$$\Psi' = \left[\begin{array}{cccccc|c|cccccc} z_2^{k_1} & z_1 z_2^{k_1-1} & \dots & z_1^{k_1} \dots z_2 & z_1 & & 0\dots\dots & 0 & \dots & \dots & \dots & \dots & \dots \\ \hline & & & & & & \ddots & & & & & & \\ 0 & \dots & \dots & \dots & \dots & & \dots\dots 0 & z_2^{k_m} & z_1 z_2^{k_m-1} & \dots & z_1^{k_m} \dots z_2 & z_1 & \end{array}\right]$$

$$D_{HT} = \begin{bmatrix} D_{11} & \dots\dots & D_{1m} \\ & & \\ D_{m1} & \dots\dots & D_{mm} \end{bmatrix}, \qquad N_{HT} = \begin{bmatrix} N_{11} & \dots\dots & N_{1m} \\ & & \\ N_{p1} & \dots\dots & N_{pm} \end{bmatrix}$$

and D_{ij} and N_{ij} are row-vectors whose elements are the coefficients of the (i,j)-indexed polynomial in $-D_R+I_m$ and N_R.

Introduce now the following matrices

$$A_{10}^{(h)} = \begin{bmatrix} M_h & & & \\ & M_{h-1} & & \\ & & \ddots & \\ & & & M_2 \\ \hline 0 & \dots\dots\dots\dots & & 0 \\ 0 & \dots\dots\dots\dots & & 0 \end{bmatrix}, \qquad B_1^{(h)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 0 \\ 1 \end{bmatrix}$$

$$A_{20}^{(h)} = \begin{bmatrix} N_h & & & \\ & N_{h-1} & & \\ & & \ddots & \\ & & & N_2 \\ \hline 0 & \dots\dots\dots\dots & & 0 \\ 0 & \dots\dots\dots\dots & & 0 \end{bmatrix}, \qquad B_2^{(h)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 0 \\ 1 \end{bmatrix}$$

with

$$M_j = \begin{bmatrix} 0 & 0 & \dots & 0 \\ & & & \\ & I_j & & \end{bmatrix} \qquad N_j = \begin{bmatrix} 1 & 0 & \dots & 0 \\ & & & \\ & & 0_j & \end{bmatrix}$$

and define

$$A_{10} = \mathrm{diag}\left[A_{10}^{(k_1)} \; A_{10}^{(k_2)} \dots A_{10}^{(k_m)}\right]$$

$$A_{20} = \mathrm{diag}\left[A_{20}^{(k_1)} \; A_{20}^{(k_2)} \dots A_{20}^{(k_m)}\right]$$

$$B_1 = \mathrm{diag}\left[B_1^{(k_1)} \; B_1^{(k_2)} \dots B_1^{(k_m)}\right]$$

$$B_2 = \mathrm{diag}\left[B_2^{(k_1)} \; B_2^{(k_2)} \dots B_2^{(k_m)}\right]$$

Then the matrices $A_1 = A_{10}+B_1 D_{HT}$, $A_2 = A_{20}+B_2 D_{HT}$, $B_1, B_2, C = N_{HT}$ furnish a realization of $N_R D_R^{-1}$ satisfying i) and ii). The details of the proof are given in [5].

ASSIGNABILITY OF THE CLOSED LOOP CHARACTERISTIC POLYNOMIAL

At the end of section 2 we posed problems i)-iii) relative to the system of fig. 1, obtained by interconnecting a strictly proper plant $\Sigma = (A_1, A_2, B_1, B_2, C)$ and a compensator $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$.

Our aim now is to give a solution to such problems. Let $W(z_1,z_2)$ and $W_c(z_1,z_2)$ be the transfer matrices of Σ and Σ_c respectively, and consider two MFD's PQ^{-1} and $X^{-1}Y$ satisfying

$$W(z_1,z_2) = PQ^{-1}, \quad \det Q = \det(I - A_1 z_1 - A_2 z_2) \tag{26}$$

$$W_c(z_1,z_2) = X^{-1}Y, \quad \det X = \det(I - F_1 z_1 - F_2 z_2) \tag{27}$$

Then the closed loop characteristic polynomial (8) is given by

$$\hat{\Delta}(z_1,z_2) = \det(X\,Q - Y\,P) \tag{28}$$

On the other hand, by Theorem 3 any left MFD $X^{-1}Y$ with $X(0,0) = I$ admits a realization $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$ that satisfies the condition

$$\det X(z_1,z_2) = \det(I - F_1 z_1 - F_2 z_2)$$

So, as (X,Y) varies over the set of polynomial matrix pairs with $X(0,0) = I$, (28) produces all assignable closed loop polynomials for the given plant Σ.

Let E be a GCRD of P and Q. Then

$$P = N_R E \qquad Q = D_R E \tag{29}$$

and $N_R D_R^{-1}$ is a r.c.MFD of W. As a consequence of (22) and (26), we have

$$h(z_1,z_2) = \det(I - A_1 z_1 - A_2 z_2) / \det D_R = \det E \tag{30}$$

and (28) becomes

$$\hat{\Delta}(z_1,z_2) = h \det(X\,D_R + Y\,N_R) \tag{31}$$

The above formula clearly shows that $h(z_1,z_2)$, which represents the hidden modes of Σ, is an invariant factor of $\hat{\Delta}(z_1,z_2)$ w.r. to feedback compensation. In other words, as far as fixed modes are concerned, 2D systems behave exactly in the same way as 1D systems do. However a deep difference between 2D and 1D systems comes out when we consider the factor $\det(X\,D_R + Y\,N_R)$. In fact, as we established in the proof of Theorem 2, this factor must vanish for any choice of X and Y on the set $\mathcal{V}(W)$ of rank singularities. Such restriction does not exist in the 1D case, where the solvability of the Bézout equation $X\,D_R + Y\,N_R = I$ and hence the complete assignability of the polynomial $\det(X\,D_R + Y\,N_R)$ are consequences of the coprimeness of N_R and D_R.

The next theorem shows how the condition that $\hat{\Delta}$ vanishes on $\mathcal{V}(h)$ and $\mathcal{V}(W)$ and that $\hat{\Delta}(0,0) \neq 0$, represent the only constraints, imposed by the structure of the plant on the closed loop polynomial variety.

<u>Theorem 4</u> Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of the transfer matrix W. For any compensator Σ_c, the closed loop polynomial variety $\mathcal{V}(\hat{\Delta})$ satisfies the inclusion

$$\mathcal{V}(\hat{\Delta}) \supseteq \mathcal{V}(h) \cup \mathcal{V}(W)$$

where h is given by (30) and $\mathcal{V}(W)$ is the set of the rank singularities of W.

Viceversa, given any algebraic curve $\mathcal{C}$ that includes $\mathcal{V}(h) \cup \mathcal{V}(W)$ and excludes the origin, a compensator Σ_c exists such that $\mathcal{V}(\hat{\Delta}) = \mathcal{C}$.

<u>proof</u> The first part of the Theorem has already been proved. For the second, let M_i be the submatrices of maximal order in $[P^T\ Q^T]^T$ that correspond to the minors m_i, $i = 1,2,\ldots,\nu$. Then there exist constant matrices L_i and K_i that satisfy

$$M_i = L_i Q - K_i P \qquad i = 1,2,\ldots,$$

and we have

$$m_i I = (\operatorname{adj} M_i) M_i = (\operatorname{Adj} M_i) L_i Q - (\operatorname{adj} M_i) K_i P \tag{32}$$

Consider a 2D polynomial c such that

$$\mathcal{V}(c) = \mathcal{C}, \quad c(0,0) = 1$$

The inclusion

$$\mathcal{V}(c) \supseteq \mathcal{V}(W) \cup \mathcal{V}(h) = \mathcal{V}(N_R, D_R) \cup \mathcal{V}(E) = \mathcal{V}(P,Q)$$

and Hilbert's Nullstellensatz imply

$$c^r = \sum_{i=1}^{\nu} m_i g_i \in \mathcal{I}(P,Q) \tag{33}$$

for a suitable integer r and suitable 2D polynomials g_i. Tying (32) and (33) together yields

$$c^r I = \left(\sum_{i=1} m_i g_i\right) I = X\,Q - Y\,P, \tag{34}$$

with

$$X = \sum_i g_i (\operatorname{adj} M_i) L_i, \qquad X(0,0) = I$$

$$Y = \sum_i g_i (\operatorname{adj} M_i) K_i$$

By theorem 4, we are able to construct a compensator $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$ that realizes $X^{-1}Y$ under the constraint $\det(I - F_1 z_1 - F_2 z_2) = \det X$. Thus the corresponding closed loop polynomial is given by

$$\hat{\Delta}(z_1,z_2) = \det(XQ - YP) = c^{rm}$$

and $\mathcal{C}$ is the variety of $\hat{\Delta}$.

Let us pause to make some observations.

1. Assignable polynomials of a strictly proper MISO system Σ are easily characterized as the elements with unit constant term in the ideal $h\,\mathcal{I}(W)$. In fact, let q be the characteristic polynomial of Σ and $[p_1, p_2, \ldots, p_p]q^{-1}$ its transfer matrix and consider any poly-

nomial c in $(q,p_1,p_2,\dots,p_p) = h\,\mathcal{I}(W)$ and satisfying $c(0,0) = 1$.
Then there exist 2D polynomials $x, y_1, y_2, \dots, y_p$ such that

$$c = q\,x + \sum_i p_i y_i$$

and $x(0,0) = 1$. Clearly any 2D realization $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$ of

$$x^{-1}[y_1, y_2, \dots, y_p]^T$$

that satisfies $x = \det(I - F_1 z_1 - F_2 z_2)$ gives c as closed loop characteristic polynomial.
So, when dealing with MISO (and, using dual resoning, with SIMO) systems, an alternative characterization of the feedback action is available in terms of polynomial ideals instead of polynomial varieties.

2. As a corollary of Theorem 4, we easily solve the question ii) mentioned at the end of section ii), thus obtaining the following equivalent conditions for the complete assignability of closed loop polynomials
   - i) $h=1$ and $\mathcal{V}(W) = \emptyset$
   - ii) $\mathcal{V}(\mathcal{R}) = \mathcal{V}(\mathcal{O}) = \emptyset$ (i.e. the plant is controllable and reconstructible in the sense of [4,5])
   - iii) the plant admits a dead-beat controller, i.e. a compensator Σ_c such that $\Delta = 1$.

3. Rank singularities are not invariant under output feedback. This can be easily seen by taking, for instance

$$W(z_1,z_2) = z_1(1-2z_2-z_1^2)/(1+2z_2) \tag{34}$$

$$W_c(z_1,z_2) = z_1/(1+2z_2+z_1^2) \tag{35}$$

The closed loop transfer function

$$\hat{W}(z_1,z_2) = W/(1+W\,W_c) = z_1 \tag{36}$$

is devoid of rank singularities, while $\mathcal{V}(W) = \mathcal{V}(W_c) = \{(0,-\frac{1}{2})\}$. Note that, whatever the realizations of $W(z_1,z_2)$ and $W_c(z_1,z_2)$ may be, the resulting closed loop system is internally unstable. In fact, independently of the internal descriptions Σ and Σ_c of W and W_c, the variety of the closed loop characteristic polynomial $\Delta(z_1,z_2)$ must include $\mathcal{V}(W)$ (and $\mathcal{V}(W_c)$) and $\Delta(z_1,z_2)$ is an hidden mode of the closed loop system.

A key problem which naturally arises at this point is to decide whether a given algebraic curve $\mathcal{C}$, described by a polynomial equation $c(z_1,z_2) = 0$, is assignable (i.e. can be viewed as the closed loop polynomial variety of the system depicted in fig. 1).

By Theorem 4, the procedure will break up into three checks:

$$\text{a)}\quad (0,0) \notin \mathcal{C} \tag{37}$$

$$\text{b)}\quad \mathcal{V}(h) \subseteq \mathcal{C} \tag{38}$$

$$\text{c)}\quad \mathcal{V}(W) \subseteq \mathcal{C} \tag{39}$$

Checking a) is trivial and checking b) reduces to verify if h divides a suitable power of c (for instance $c^{\deg h}$). The last check can be algorithmically performed [10] once a set of generators of $\mathcal{I}(W)$ has been found.

The computation of such a set and of h is immediate if a coprime MFD $N_R D_R^{-1}$ of the transfer matrix W is available. In fact the maximal order minors of $[N_R^T\ D_R^T]$ generate $\mathcal{I}(W)$ and equation (30) yields h. Using a coprime MFD of $W(z_1,z_2)$ can be avoided if we refer to the state space model and compute the matrices $\bar{N}$ and $\bar{D}$ in

$$W(z_1,z_2) = [C\ \mathrm{adj}(I-A_1z_1-A_2z_2)(B_1z_1+B_2z_2)]\,[I_m \det(I-A_1z_1-A_2z_2)]^{-1} = \bar{N}\,\bar{D}^{-1}$$

The generators set can be obtained by evaluating the maximal order minors $\{m_1, m_2, \dots, m_\nu\}$ in $[\bar{N}^T\ \bar{D}^T]$ and then by eliminating their g.c.d. $d(z_1,z_2)$. Thus h is given by

$$h = \det(I-A_1z_1-A_2z_2)/\det D_R = \det(I-A_1z_1-A_2z_2)\frac{d}{\det \bar{D}}$$

Assume now that a variety $\mathcal{C}$ that fulfills conditions (37) through (39) has been given. To conclude this section we shall briefly mention a compensator synthesis procedure that produces a closed loop polynomial Δ whose variety is $\mathcal{C}$. First, we must evaluate a r. coprime MFD $N_R D_R^{-1}$ of W. One way to do this is to use the primitive factorization algorithm [7], or other algorithms that do not require primitive factorizations [8]. Next since the maximal order minors $m_1, m_2, \dots, m_\nu$ of $[N_R^T\ D_R^T]$ are known, we can obtain [10] an integer r and polynomials $g_1, g_2, \dots, g_\nu$ such that $c^r = \sum_{i=1}^{\nu} m_i g_i$ and solve the Bézout equation $c^r I_m = X D_R - Y N_R$ by the same technique developed in the proof of Theorem 4. The final step exploits the realization algorithm of Theorem 3 for computing a coprime realization Σ_c of $X^{-1}Y$.

The correctness of the procedure is easily seen, from the following chain of equalities:

$$\begin{aligned}\mathcal{V}(\Delta) &= \mathcal{V}(h) \cup \mathcal{V}(\det(XD_R - YN_R)) && \text{by (31)}\\ &= \mathcal{V}(h) \cup \mathcal{V}(c^r I_m) && \text{by (37)}\\ &= \mathcal{V}(c) = \mathcal{C}\end{aligned}$$

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