CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol.4, No.4, pp.381-393, December, 1988

C88069

Contributed Paper

FAILURE DETECTION IN 2D SYSTEMS*

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Abstract. Dynamical redundancy relations of 2D systems allow for an implementation of parity checks by means of 2D dynamical models. This paper presents a complete characterization of the admissible parity checks, both in time and in formal power series domains, and provides a dynamic implementation which does not increase the intrinsic delays of the failure detection process.

Key Words—2D systems, failure detection, matrix fraction description, parity check, realization.

1. Introduction

Over the past decade several contributions to the problem of failure detection in dynamical systems have been presented in the literature (Chow and Willsky, 1984; Lan et al., 1986; Massoumnia, 1986; Isermann, 1984).

Any failure detection method essentially involves some processing technique of the measured variables and is based on the use of redundancy among them. Redundancy relations fall in two classes: *direct redundancy* exploits the relationships among instantaneous outputs of sensors, while *temporal redundancy* takes advantage of the relationships among the histories of sensor outputs and actuator inputs. In both cases, the signal generated by the detection process—the *residual*—depends on the difference between the measured and expected values of some function of the plant output. In the absence of a failure, a zero residual should testify the normal behaviour of the plant.

In a linear environment, the residual generation process based on temporal redundancy is easily described by a moving average (MA) model. Thus it seems quite natural to look for a *residual generator* in state space form. The task it performs conceptually splits in two different steps, though very often inextricably mixed in the operation of the state model. The first step consists in reconstructing the free output of the plant, by eliminating the forced evolution, the second one relies on checking if the space of the admissible outputs contains the signal generated in the first step. If the signal is not contained in it, the corresponding residual will be nonzero, which constitutes an alarm evidentiating that a failure occurred.

The above procedure requires to perform a *parity check* on the outputs. The set of parity checks can be explicitly described in terms of linear functionals associated to the observation matrix of the plant (Lan et al., 1986).

* Received by the editors August 2, 1988.

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This paper deals with the solution of the failure detection problem for 2D systems. As well known, 2D systems provide input/output and state-space models representing physical processes which depend on two independent variables. Typically, they apply to two-dimensional data processing in several fields, as seismology, X-ray image enhancement, image deblurring, digital pictures processing, etc. Also, 2D systems constitute a natural framework for modelling multivariable networks, large scale systems obtained by interconnecting many subsystems and, in general, physical processes where both space and time have to be taken into account. Finally, they are needed when synthesizing 2D control laws (Bisiacco, 1985).

2D systems constitute a relatively recent area of research and up to now there are no results in the literature concerning the 2D failure detection problem. The input and output signals that are needed in 2D failure detection are defined on the discrete plane $Z \times Z$ or, more frequently, on a suitable half-plane of $Z \times Z$. Moreover, since the quarter plane causality is assumed, the output value at (i, j) only depends on the input values and initial conditions of the system on the set $\{(h, k): h \leq i,\ k \leq j\}$.

Clearly the failure detection based on direct redundancy only keeps into account the outputs of the sensors at the single point (i, j). Here the causal structure of the system is not relevant and the detection problem can be tackled along the same lines as the 1D case.

Vice versa, when considering temporal redundancy, the difference between the causality structures calls for a specific treatment of the 2D case. As it is well known, a 1D parity check processes a data set,

$$y(t),\ y(t+1),\ \cdots\ y(t+T),$$

whose support is a finite time interval and the parity check cannot be performed before $t=T$ if the data are available starting from $t=0$.

On the other hand, the 2D data entering the residual generation process constitute a finite set whose support is contained in a triangular window of the discrete plane $Z \times Z$,

$$\left.\begin{array}{ccccc} & & & & y(i-\nu, j) \\ & & & y(i-\nu+1, j-1) & y(i-\nu+1, j) \\ & & \cdots\cdots & & \\ & y(i-1, j-\nu+1) & \cdots & y(i-1, j-1) & y(i-1, j) \\ y(i, j-\nu) & y(i, j-\nu+1) & \cdots & y(i, j-1) & y(i, j) \end{array}\right\}. \quad (1)$$

It comes out that if we have to perform the parity check and to make available the corresponding residual at (i, j), we need data in the past of (i, j) that belong to a "band" constituted by $\nu+1$ diagonals. In particular, if the data are known in the half plane $\{(h, k): h+k \geq 0\}$, the residual can be computed at (i, j) if and only if $i+j \geq \nu$. In this case, if no failure occurred in the system, the parity check leads to the same result if the data set (1) is substituted by any other set we obtain from (1) once (i, j) has been substituted by (h, k) with $h+k \geq \nu$.

As one can expect, because of the above invariance property, the residual generation process is naturally represented by MA models in two variables. Consequently a 2D residual generator in state space form can be synthesized as a 2D system that realizes the MA model.

The paper is organized as follows. In Sec. 2 we give a fairly complete analysis of the redundancy relations that underlie 2D parity checks. The first characterization of the parity checks expresses temporal redundancy in terms of orthogonality conditions in suitable linear spaces. These are the images of matrices which depend on the dimension of the triangular data window and exhibit a structure that reminds of the local observability matrix introduced in Fornasini and Marchesini (1978).

A further characterization of the parity checks is introduced in that section. This leads to a representation of parity checks as elements of a free module over the ring of polynomials in two variables, whose structure is completely specified by a finite set of generators computed from the matrix fraction description of the system.

In the last section of the paper we assume that a specific parity check has been given and we present an explicit realization procedure of the polynomial matrix in two variables that constitutes the transfer matrix of the corresponding residual generator.

As well known, there are several different state space realization techniques for a polynomial transfer matrix in two variables. In this context we are interested in obtaining a finite memory realization whose unforced dynamics goes to zero in a minimum number of steps. The reason for this requirement is that the response of the residual generator provides exactly the parity check only when the unforced response has vanished.

The realization given in this paper proves that the time interval required for a "on line" implementation of the parity check by a minimum memory 2D system is the same as for an "off line" implementation. Therefore the minimum memory 2D systems (which need not have minimal dimension) exhibit the same properties as minimal dimension 1D systems in the 1D theory of residual generation.

2. 2D parity relations

Consider a 2D system (*plant*), represented by the state model (Fornasini and Marchesini, 1978),

$$\left.\begin{aligned} x(h+1,\ k+1) &= A_1x(h,\ k+1) + A_2x(h+1,\ k) + B_1u(h,\ k+1) \\ &\qquad + B_2u(h+1,\ k) \\ y(h,\ k) &= Cx(h,\ k) + Du(h,\ k) \end{aligned}\right\}, \tag{2}$$

where x is an n-dimensional *local state* vector, u is an m-dimensional vector of known inputs, y is a p-dimensional vector of measured outputs and A_1, A_2, B_1, B_2, C and D are known matrices of appropriate dimensions. Assume further that C is full rank, which rules out direct redundancy among the instantaneous values of the sensors.

The *transfer matrix* of (2) is given by

$$W(z_1, z_2) = C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2) + D. \tag{3}$$

A *parity relation* is a linear combination of a finite *window* of present and lagged values of u and y, that is identically zero if no failures occur in (2). This should be verified for any location of the data window in the discrete plane, which implies that the parity criterion is invariant with respect to two-dimensional shifts and hence is associated with a 2D moving average model.

Let us first assume that the plant undergoes a free state evolution starting from some initial *global state*,

$$\mathscr{X}_0 = \{x(i, -i) : i \in Z\}.$$

Denote by

$$Y(z_1, z_2) = \sum_{i+j\geq 0} y(i, j)z_1{}^i z_2{}^i = C(I-A_1z_1-A_2z_2)^{-1}\mathscr{X}_0, \tag{4}$$

the formal power series associated with the output values in the half plane $\{(i, j): i+j\geq 0\}$ and for any (i, j) and $\nu\geq 0$ introduce the $(1+2+\cdots+(\nu+1))p$-dimensional vector,

$$y_\nu(i, j) \triangleq [y^T(i-\nu, j)y^T(i-\nu+1, j-1)\cdots y^T(i, j-\nu)|$$
$$\cdots |y^T(i-1, j)y^T(i, j-1)|y^T(i, j)]^T.$$

Note that $y_\nu(i, j)$ represents the output data contained in a νth *order triangular window* with vertices (i, j), $(i-\nu, j)$, $(i, j-\nu)$.

Let now decompose $Y(z_1, z_2)$, $\det(I-A_1z_1-A_2z_2)$ and $\mathrm{adj}(I-A_1z_1-A_2z_2)$ as follows:

$$Y(z_1, z_2) = Y_0+Y_1+Y_2+\cdots,$$

$$\det(I-A_1z_1-A_2z_2) = 1+H_1+H_2+\cdots+H_n,$$

$$\mathrm{adj}(I-A_1z_1-A_2z_2) = I+M_1+M_2+\cdots+M_{n-1}.$$

Here H_i, $i=0, 1, \cdots, n$ are homogeneous forms in two variables of degree i, Y_i and M_i, $i=0, 1, \cdots$ are vectors and matrices whose elements are homogeneous forms in two variables of degree i.

So, recalling (3), we have

$$(1+H_1+\cdots+H_n)(Y_0+Y_1+Y_2+\cdots) = I+M_1+\cdots+M_{n-1}, \tag{5}$$

which implies

$$H_nY_k + H_{n-1}Y_{k+1} + \cdots + H_1Y_{k+n-1} + Y_{k+n} = 0$$

for any $k\geq 0$. Denoting by h_{ij} the coefficient of $z_1{}^i z_2{}^j$ in H_{i+j} and introducing the real $(1+2+\cdots+(\nu+1))p$-dimensional row vector,

$$w^T \triangleq [h_{n0}h_{n-1,1} \cdots h_{0n} | \cdots | h_{20}h_{11}h_{02} | h_{10}h_{01} | 1] \otimes [\underbrace{1 \quad 1 \cdots 1}_{p \text{ times}}], \qquad (6)$$

one gets the parity relation,

$$w^T y_n(i, n+k-i) = 0. \qquad (7)$$

Clearly, w^T provides a parity check on the free outputs space, in the sense that, if the product (7) is different from zero, a failure has occurred in the plant.

As one can easily intuit, the convolution of the output series with the characteristic polynomial $\det(I - A_1 z_1 - A_2 z_2)$, given by (5), does not constitute the unique way for obtaining a parity check. Actually, let

$$q^T(z_1, z_2) = [q_1(z_1, z_2) \quad q_2(z_1, z_2) \cdots q_p(z_1, z_2)]$$

be a polynomial row vector and denote by $q^T(z_1, z_2) = Q_0 + Q_1 + \cdots + Q_s$ its representation as sum of homogeneous terms. Assuming that

$$q^T(z_1, z_2) C (I - A_1 z_1 - A_2 z_2)^{-1} \qquad (8)$$

is a polynomial matrix of degree $\nu - 1$, the degree of $q^T(z_1, z_2)C$ and, by the rank assumption on C, the degree s of $q^T(z_1, z_2)$ cannot exceed ν.

Since the degree of the nonzero homogeneous terms of $q^T(z_1, z_2) Y(z_1, z_2)$ is less than or equal to $\nu - 1$, we have

$$Q_s Y_{-s+k} + \cdots + Q_1 Y_{-1+k} + Q_0 Y_{+k} = 0$$

for any $k \geq 0$. Denoting by q_{ij}^T the coefficient of $z_1^i z_2^j$ in Q_{i+j}, the $(1+2+\cdots+(\nu+1))p$-dimensional real vector

$$v \triangleq [0 \cdots 0 | q_{s0}^T q_{s-1,1}^T \cdots q_{0s}^T | q_{20}^T q_{11}^T q_{02}^T | q_{10}^T q_{01}^T | q_{00}^T]^T \qquad (9)$$

is orthogonal to the free output vector $y_\nu(i, \nu+k-i)$ for any $k \geq 0$ and for any i.

Note that as $q^T(z_1, z_2)$ varies, the corresponding row vectors v^T given by (9) constitute a class of parity checks containing the vector w^T defined by (6).

The converse is also true, that is, given a $(1+2+\cdots+(\nu+1))p$-dimensional vector v, orthogonal to the free outputs $y_\nu(i, \nu+k-i)$ for any $k \geq 0$ and any i, its entries can be viewed as the coefficients of a polynomial vector $q^T(z_1, z_2) \in R[z_1, z_2]^p$ and $q^T(z_1, z_2) C (I - A_1 z_1 - A_2 z_2)^{-1}$ is a polynomial matrix.

So doing, we have obtained a complete characterization of the parity checks which can be performed on the output data contained in a νth order triangular window. As ν varies, the above polynomial vectors $q^T(z_1, z_2)$ constitute a free submodule S of $R[z_1, z_2]^p$ which can be characterized starting from a left coprime matrix fraction description (MFD) (Kung et al., 1977) $M^{-1}(z_1, z_2) N(z_1, z_2)$ of $C(I - A_1 z_1 - A_2 z_2)^{-1}$. The parity checks consist of the polynomial row vectors $q^T(z_1, z_2)$ which make

$$q^T(z_1, z_2) M^{-1}(z_1, z_2) N(z_1, z_2)$$

to be polynomial. Since $M^{-1}N$ is coprime, by Lemma 5.3 in Kung et al. (1977), this is equivalent to the requirement that $q^T(z_1, z_2)M^{-1}(z_1, z_2)$ is polynomial, i.e., that $q^T(z_1, z_2)$ belongs to the free module generated by the rows of $M(z_1, z_2)$.

Example 1. Consider the 2D system given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = 0.$$

It is easy to check that

$$\begin{bmatrix} 1-z_1-z_2 & 2z_2 \\ -z_1/2 & 1-z_1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

provides a left coprime MFD of $C(I-A_1z_1-A_2z_2)^{-1}$.

Hence the parity relations are represented by the row polynomial vectors given by

$$h(z_1, z_2)[1-z_1-z_2 \,|\, 2z_2] + k(z_1, z_2)[-z_1/2 \,|\, 1-z_1], \qquad h, k \in R[z_1, z_2].$$

An alternative complete characterization of the parity checks can be given in the time domain by exploiting a generalization of the observability matrix. Define inductively the following matrices (Fornasini and Marchesini, 1980):

$$A_1{}^r \sqcup\!\sqcup\, {}^0A_2 = A_1{}^r$$

$$A_1{}^0 \sqcup\!\sqcup\, {}^sA_2 = A_2{}^s$$

$$A_1{}^{r+1} \sqcup\!\sqcup\, {}^{s+1}A_2 = A_1(A_1{}^r \sqcup\!\sqcup\, {}^{s+1}A_2) + A_2(A_1{}^{r+1} \sqcup\!\sqcup\, {}^sA_2)$$

and note that the contribution of the state $x(0, 0)$ to the output value at (r, s) is given by $C(A_1{}^r \sqcup\!\sqcup\, {}^sA_2)x(0, 0)$.

The output values belonging to the triangular window with vertices $(i, v-i+k)$, $(i-v, -i+k+v)$ and $(i, -i+k)$ only depend on the local states on the segment with vertices $(i, -i+k)$ and $(i-v, -i+k+v)$ and are expressed by

$$y(i, v-i+k) = \mathcal{O} \begin{bmatrix} x(i, -i+k) \\ x(i-1, -i+1+k) \\ \cdots \\ \cdots \\ x(i-v, -i+v+k) \end{bmatrix},$$

where $\mathcal{O}$ is the $p((v+1)+v+\cdots+1)\times n(v+1)$ matrix given by

$$\mathscr{O} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & C \\ 0 & 0 & 0 & \cdots & C & 0 \\ & & \cdots & & & \\ C & 0 & 0 & & 0 & 0 \\ & & \cdots & & & \\ & & \cdots & & & \\ 0 & CA_1{}^{\nu-1} & CA_1{}^{\nu-2} \sqcup\!\sqcup^1 A_2 & \cdots & CA_1{}^1 \sqcup\!\sqcup^{\nu-2} A_2 & CA_2{}^{\nu-1} \\ CA_1{}^{\nu-1} & CA_1{}^{\nu-2} \sqcup\!\sqcup^1 A_2 & CA_1{}^{\nu-3} \sqcup\!\sqcup^2 A_2 & \cdots & CA_2{}^{\nu-1} & 0 \\ CA_1{}^{\nu} & CA_1{}^{\nu-1} \sqcup\!\sqcup^1 A_2 & CA_1{}^{\nu-2} \sqcup\!\sqcup^2 A_2 & \cdots & CA_1{}^1 \sqcup\!\sqcup^{\nu-1} A_2 & CA_2{}^{\nu} \end{bmatrix}.$$

A parity check $\boldsymbol{v}$ relative to the output data contained in a νth order triangular window must satisfy the condition

$$\boldsymbol{v}^T \boldsymbol{y}(i,\ \nu-i+k) = 0, \qquad k \geq 0, \qquad i = 0,\ \pm 1,\ \pm 2,\ \cdots \tag{10}$$

for any initial global state $\mathscr{X}_0$. Hence $\boldsymbol{v}$ belongs to the orthogonal complement of $\mathrm{Im}(\mathscr{O})$ and vice versa any vector in $\mathrm{Im}(\mathscr{O})^{\perp}$ is a parity check.

Clearly, the dimension of $\mathrm{Im}(\mathscr{O})$ linearly increases with ν, while the dimension of $\mathrm{Im}(\mathscr{O})^{\perp}$ is proportional to ν^2 or, equivalently, to the number of output values contained in the window we take into account.

The parity checks previously introduced apply also when the input of the plant is different from zero. This of course requires that the free output evolution should have been previously reconstructed from the actual input and output functions.

Denote by $\boldsymbol{u}_\nu(i,\ j)$ the $(1+2+\ \cdots\ +(\nu+1))m$-dimensional input vector

$$\begin{aligned} \boldsymbol{u}_\nu(i,\ j) \triangleq\ & [u^T(i-\nu,\ j)u^T(i-\nu+1,\ j-1) \cdots u^T(i,\ j-\nu) | \\ & \cdots\ | u^T(i-1,\ j)u^T(i,\ j-1) | u^T(i,\ j)]^T, \end{aligned}$$

and let $\boldsymbol{y}_\nu(i,\ j)$ be the corresponding output vector.

Starting from Eq. (1) and using lengthy but simple computations, it is easy to show that the free output vector can be written in the form

$$\boldsymbol{y}_\nu(i,\ j) - \mathscr{Y} \boldsymbol{u}_\nu(i,\ j), \tag{11}$$

where $\mathscr{Y}$ is the following block triangular matrix

$$\mathscr{Y} = \begin{bmatrix} F_{11} & & & & \\ F_{21} & & F_{22} & & \\ F_{31} & & F_{32} & F_{33} & \\ & \cdots & & & \\ F_{\nu+1,1} & & F_{\nu+1,2} & F_{\nu+1,3} & \cdots\ F_{\nu+1,\nu+1} \end{bmatrix}.$$

The diagonal blocks of $\mathscr{Y}$ are given by

$$F_{ii} = \mathrm{diag}\{\underbrace{D \quad D \cdots D}_{i \text{ times}}\}$$

and $F_{h,h-k}$ are block matrices of the following form

$$F_{h,h-k} = \begin{bmatrix} 0 & CA_1{}^{k-1}B_1 & CA_1{}^{k-2} ⧢ {}^1A_2B_1 + CA_1{}^{k-1}B_2 \cdots & CA_2{}^{k-1}B_2 \\ & \vdots & \vdots & \vdots \\ & CA_1{}^{k-1}B_1 & CA_1{}^{k-2} ⧢ {}^1A_2B_1 + CA_1{}^{k-1}B_2 \cdots & CA_2{}^{k-1}B_2 \\ CA_1{}^{k-1}B_1 & CA_1{}^{k-2} ⧢ {}^1A_2B_1 + CA_1{}^{k-1}B_2 \cdots & CA_2{}^{k-1}B_2 & 0 \end{bmatrix}.$$

In this case the parity check condition (9) is replaced by

$$\boldsymbol{v}^T[\boldsymbol{y}(i, \ -i+\nu+k) \ - \ \mathscr{Y}\boldsymbol{u}(i, \ -i+\nu+k)] = 0$$

for any $k \geq 0$ and any i.

An analogous procedure applies when a formal power series approach is used. In this case the formal power series that represents the free output is given by

$$[I \mid -W(z_1, z_2)]\begin{bmatrix} Y(z_1, z_2) \\ U(z_1, z_2) \end{bmatrix} \tag{12}$$

with

$$U(z_1, z_2) + \sum_{i+j \geq 0} u(i, j) z_1{}^i z_2{}^j.$$

If $q^T(z_1, z_2)$ is any row polynomial vector in the submodule S, the coefficients $r(i, j)$ of the series resulting from the discrete convolution

$$q^T(z_1, z_2)[I \mid -W(z_1, z_2)]\begin{bmatrix} Y(z_1, z_2) \\ U(z_1, z_2) \end{bmatrix} \tag{13}$$

are zero whenever $i+j \geq \nu$, for some positive integer ν. So, the above convolution represents a residual generation process, in the sense that $r(i, j) \neq 0$ for $i+j \geq \nu$ indicates that some failure occurred in the system.

3. Realization of a 2D residual generator

The aim of this section is to realize the residual generation process introduced at the end of Sec. 2 by means of a 2D dynamical system driven by the inputs and the outputs of the plant.

Let $q^T(z_1, z_2)$ be a parity check for (2), so that the matrix (8) and, consequently $q^T(z_1, z_2)W(z_1, z_2)$ are polynomial. The application of the parity check to the formal power series (12), representing the free output evolution, reduces to apply the row vector,

$$g^T(z_1, z_2) \triangleq q^T(z_1, z_2)[I \mid -W(z_1, z_2)], \tag{14}$$

to the output and input data vector,

$$\begin{bmatrix} Y(z_1, z_2) \\ U(z_1, z_2) \end{bmatrix}.$$

So, the residual $r(h, k)$ can be viewed as the output of a 2D system $\Sigma_g = (F_1, F_2, G_1, G_2, H, J)$,

$$\left.\begin{aligned} x'(h+1, k+1) &= F_1 x'(h, k+1) + F_2 x'(h+1, k) \\ &\quad + G_1\begin{bmatrix} y(h, k+1) \\ u(h, k+1) \end{bmatrix} + G_2\begin{bmatrix} y(h+1, k) \\ u(h+1, k) \end{bmatrix} \\ r(h, k) &= Hx'(h, k) + J\begin{bmatrix} y(h, k) \\ u(h, k) \end{bmatrix} \end{aligned}\right\}, \tag{15}$$

which is driven by $y(h, k)$ and $u(h, k)$ and realizes the polynomial vector $g^T(z_1, z_2)$.

Actually the residual $r(h, k)$ generated by Σ_g is the sum of a forced term, that provides the expected parity check on the pair $y(h, k)$ and $u(h, k)$, and a second term that depends on the initial conditions of Σ_g, which are in general unknown. However, since $g^T(z_1, z_2)$ is a polynomial vector, we can assume that in Σ_g the matrices F_1 and F_2 satisfy the condition $\det(I-F_1z_1-F_2z_2)=1$. In this way Σ_g is a finite memory 2D dynamical system (Bisiacco, 1985), in the sense that there exists a positive integer μ such that, for any initial global state $\mathscr{X}_0$, the free state evolution,

$$X(z_1, z_2) = \sum_{h+k\geq 0} x(h, k) z_1{}^h z_2{}^k$$

is zero for $h+k\geq\mu$.

So, after a finite number of steps, the (undesired) second term vanishes and the output of Σ_g provides the correct parity check.

Equation (8) shows that the parity check $q^T(z_1, z_2)$ requires that processing of output data determined by $\mathscr{X}_0$ should be extended at least up to the terms appearing in the νth diagonal $\{(i, j)\colon i+j=\nu\}$. Hence the parity check is reliable from the νth diagonal onwards.

We shall prove now that Σ_g can be realized in such a way that the transient of $r(h, k)$ due to nonzero initial conditions in Σ_g vanishes on the νth diagonal. This shows that the existence of a nonzero initial global state $\mathscr{X}_0$ for Σ_g does not impair the performance of the residual generator.

Consider preliminarily a polynomial transfer matrix $Q(z_1, z_2)$ of degree $\nu>0$. Whatever realization we refer to, a pulse in $(0, 0)$ gives rise to a nonzero output and hence to nonzero local states on the νth diagonal. So, bearing in mind that the state updating equation introduces a single step delay between inputs and states, there exist values of $x(0, 0)$ leading to nonzero local states on the $(\nu-1)$th diagonal.

What will be proved in Lemmas 1 and 2 below, is that there exist realizations of $Q(z_1, z_2)$ whose free state evolution is zero on the diagonals with indices greater than $\nu-1$. The above arguments clearly show that these realizations exhibit a minimum length dynamical memory.

Lemma 1. The transfer matrix $Qz_1{}^iz_2{}^j$, $Q\in R^{p\times m}$, $i+j>0$, can be realized by a 2D system $\Sigma=(F_1, F_2, G_1, G_2, H, J)$ whose free state evolution, given by

$$X(z_1, z_2) = \sum_{h+k\geq 0} x(h, k) z_1{}^h z_2{}^k = (I-F_1z_1-F_2z_2)^{-1}\mathscr{X}_0$$

satisfies the condition $x(h, k)=0$ when $h+k\geq i+j$.

Proof. There is no restriction in assuming $i>0$.

Let first consider the scalar case $m=p=1$. The following realization

$$\left.\begin{array}{l}
F_1 = \left[\begin{array}{cccc:c}
\multicolumn{4}{l:}{\overbrace{}^{i\times i}} & \\
0 & 1 & & & \\
0 & 0 & 1 & & \\
 & & & & \mathbf{0} \\
 & & & 1 & \\
0 & 0 & \cdots & 0 & \\
\hdashline
 & & \mathbf{0} & & \mathbf{0}
\end{array}\right], \quad
F_2 = \left[\begin{array}{cc:ccc}
\mathbf{0} & & & & \\
 & 1 & & & \\
\hdashline
 & & 1 & & \\
\mathbf{0} & & & & 1 \\
 & 0 & 0 & \cdots & 0 \\
\multicolumn{2}{c}{} & \multicolumn{3}{c}{\underbrace{}_{j\times j}}
\end{array}\right] \\[2em]
G_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Q \end{bmatrix}, \qquad G_2 = 0, \qquad H = [0 \quad 0 \cdots 1], \qquad J = 0
\end{array}\right\} \tag{16}$$

satisfies

$$(F_1 z_1 + F_2 z_2)^{i+j} = 0.$$

Let now assume $p=1$ and $m>1$, so that

$$Q z_1{}^i z_2{}^j = [q_1 \quad q_2 \cdots q_m] z_1{}^i z_2{}^j. \tag{17}$$

Then a realization of (17) can be obtained assuming F_1, F_2 and H as in (16), G_2 and $J=0$ and

$$G_1 = \begin{bmatrix} & & \mathbf{0} & \\ & & & \\ q_1 & q_2 & \cdots & q_m \end{bmatrix}.$$

Finally, when p and q are both greater than 1, so that

$$Q z_1{}^i z_2{}^j = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ q_{21} & q_{22} & \cdots & q_{2m} \\ & & \cdots & \\ q_{p1} & q_{p2} & \cdots & q_{pm} \end{bmatrix}, \tag{18}$$

the realization $(\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{H}, \boldsymbol{J})$ given by

$$\boldsymbol{F}_1 = I_p \otimes F_1, \qquad \boldsymbol{F}_2 = I_p \otimes F_2, \qquad \boldsymbol{H} = I_p \otimes H, \qquad \boldsymbol{J} = 0,$$

$$\boldsymbol{G}_1 = Q \otimes \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \qquad \boldsymbol{G}_2 = 0$$

satisfies

$$(\boldsymbol{F}_1 z_1 + \boldsymbol{F}_2 z_2)^{i+j} = 0.$$

Lemma 2. The polynomial transfer matrix

$$Q(z_1, z_2) = \sum_{i+j\geq 0} Q_{ij} z_1{}^i z_2{}^j, \qquad Q_{ij} \in R^{p\times m}, \qquad \nu > 0$$

can be realized by a 2D system $\Sigma=(F_1, F_2, G_1, G_2, H, J)$ whose free state evolution satisfies the condition

$$x(h, k) = 0, \qquad h+k \geq \nu.$$

Proof. Clearly $J=Q_{00}$. Moreover, for any $(i, j)\neq(0, 0)$, consider the realization $\Sigma^{(i,j)}=(F_1{}^{(i,j)}, F_2{}^{(i,j)}, G_1{}^{(i,j)}, G_2{}^{(i,j)}, H^{(i,j)}, 0)$ of $Q_{ij} z_1{}^i z_2{}^j$, obtained as in Lemma 1.

The matrices F_1, F_2, G_1, G_2, H of a realization of $Q(z_1, z_2)$ are given by the parallel connection of $\Sigma^{(i,j)}$, $i+j\leq\nu$. In particular

$$F_1 = \oplus_{i,j} F_1{}^{(i,j)}, \qquad F_2 = \oplus_{i,j} F_2{}^{(i,j)}$$

satisfy $(F_1 z_1 + F_2 z_2)^{\nu}=0$.

We are now in a position to prove the main result of this section:

Theorem 1. Let $q^T(z_1, z_2)\in S$ and assume that

$$p^T(z_1, z_2) \triangleq q^T(z_1, z_2) C (I - A_1 z_1 - A_2 z_2)^{-1}$$

is a polynomial row vector with degree $\nu-1$.

Then the parity check associated with $q^T(z_1, z_2)$ can be implemented by a residual generator Σ_g whose unforced motion $x'(h, k)$ vanishes for $h+k\geq\nu$.

Proof. By Lemma 2, there exists a realization Σ_g of $g^T(z_1, z_2)$ having a free state evolution which satisfies $x'(h, k)=0$ for $h+k\geq \deg g^T(z_1, z_2)$. So we are reduced to prove that

$$\deg g^T(z_1, z_2)\leq\nu.$$

By (14), the degree of the polynomial matrix $g^T(z_1, z_2)$ is the maximum between $\deg q^T(z_1, z_2)$ and $\deg q^T(z_1, z_2) W(z_1, z_2) = \deg q^T(z_1, z_2)[C(I-A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D]$.

Now the assumption $\deg p^T(z_1, z_2)=\nu-1$ obviously implies

$$\deg q^T(z_1, z_2) C (I - A_1 z_1 - A_2 z_2)^{-1} (B_1 z_1 + B_2 z_2) \leq \nu.$$

Furthermore, by the full rank assumption on C, we have

$$q^T(z_1, z_2) = p^T(z_1, z_2)(I - A_1 z_1 - A_2 z_2) C^T (C C^T)^{-1}$$

which gives $\deg q^T(z_1, z_2)\leq\nu$.

So $\deg g^T(z_1, z_2)$ is less than or equal to ν.

As a consequence of the theorem above, the dynamical system Σ_g constitutes the best residual generator we can expect when implementing the parity check associated with $q^T(z_1, z_2)$. In fact the free evolution of Σ_g vanishes on the *separation sets*

$$\mathscr{C}_i = \{(h, k): h+k = i\}$$

for all $i \geq \nu$.

On the other hand, we process output and input values of the plant Σ that are located on the separation sets $\mathscr{C}_i$ for all $i \geq 0$. Since the parity check utilizes a data set that belongs to $\nu+1$ consecutive separation sets, the output values of Σ_g on $\mathscr{C}_\nu$ constitute the first set of residuals which are significant for the parity check.

Example 2. Consider the parity relation,

$$q^T(z_1, z_2) = [1-z_1-z_2 \,|\, 2z_2] \tag{19}$$

of the 2D system introduced in Example 1, whose transfer matrix is

$$W(z_1, z_2) = \begin{bmatrix} 2z_2(1-2z_1) \\ z_1(1-z_1) \end{bmatrix} (1-2z_1-z_2+z_1{}^2+2z_1z_2)^{-1}.$$

We therefore have that the transfer matrix of the residual generator Σ_g associated with the parity relation (19) is the polynomial matrix

$$g^T(z_1, z_2) = q^T(z_1, z_2)[I-W(z_1, z_2)] = [1-z_1-z_2 \,|\, 2z_2 \,|\, -2z_2]. \tag{20}$$

Note that the residual generator Σ_g has

-1 output, that is the residual of the fault detection

-3 inputs, i.e., the 2 outputs and the single input of the plant Σ.

The realization procedure described in Lemmas 1 and 2 consists in three steps:

Step 1: Decompose $g^T(z_1, z_2)$ as follows:

$$\begin{aligned} g^T(z_1, z_2) &= [1 \quad 0 \quad 0] + [-1 \quad 0 \quad 0]z_1 + [-1 \quad 2 \quad -2]z_2 \\ &= Q^{(0,0)} + Q^{(1,0)}z_1 + Q^{(0,1)}z_2. \end{aligned}$$

Step 2: Assume $J = Q^{(0,0)}$ and realize separately $Q^{(1,0)}z_1$ and $Q^{(0,1)}z_2$,

$$Q^{(1,0)}z_1 \to \Sigma^{(1,0)}: \quad \begin{aligned} F_1{}^{(1,0)} &= [0], & F_2{}^{(1,0)} &= [0], \\ G_1{}^{(1,0)} &= [-1 \quad 0 \quad 0], & G_2{}^{(1,0)} &= [0 \quad 0 \quad 0], \\ H^{(1,0)} &= [1], \end{aligned}$$

$$Q^{(0,1)}z_2 \to \Sigma^{(0,1)}: \quad \begin{aligned} F_1{}^{(0,1)} &= [0], & F_2{}^{(0,1)} &= [0], \\ G_1{}^{(0,1)} &= [0 \quad 0 \quad 0], & G_2{}^{(0,1)} &= [-1 \quad 2 \quad -2], \\ H^{(0,1)} &= [1]. \end{aligned}$$

Step 3: The matrices of the parallel connection of $\Sigma^{(1,0)}$ and $\Sigma^{(0,1)}$,

$$F_i = F_i^{(1,0)} \oplus F_i^{(0,1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad i = 1,\ 2,$$

$$G_1 = \begin{bmatrix} G_1^{(1,0)} \\ G_2^{(0,1)} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ G_2 = \begin{bmatrix} G_2^{(1,0)} \\ G_2^{(0,1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix},$$

$$H = [H^{(1,0)} \quad H^{(0,1)}] = [1 \quad 1],$$

and $J = Q^{(0,0)} = [1\ \ 0\ \ 0]$ provide a minimum memory state space realization of the residual generator.

Final remark: So far, we assumed C to be full row rank, which excludes direct redundancy among the sensor outputs of the plant.

If rank $C = p' < p$, there are $p - p'$ rows of C that linearly depend on the others and $p - p'$ linearly independent vectors in the orthogonal complement of $\mathrm{Im}(C)$.

Denote by $U \in R^{(p-p')\times p}$ a constant matrix whose rows generate $\mathrm{Im}(C)$. Then, if the sensors operate correctly,

$$U[y(h,\ k) - Du(h,\ k)] = UCx(h,\ k) = 0$$

must hold for any $(h,\ k)$. Consequently $p - p'$ instantaneous parity checks can be implemented on the behaviour of the sensors by exploiting direct redundancy among the outputs.

The procedures for obtaining the dynamic parity checks considered in Sec. 2 still apply, provided that a suitable selection of p' independent outputs has been performed. This can be formally accomplished by using a matrix $T \in R^{p'\times p}$ that selects p' independent outputs or, more generally, any matrix $T \in R^{p'\times p}$ such that TC has rank p'. The fault detection procedure previously described applies then to the p'-output plant $\Sigma' = (A_1,\ A_2,\ B_1,\ B_2,\ TC,\ TD)$.

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