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LINEAR ALGORITHMS FOR COMPUTING CLOSED LOOP POLYNOMIALS OF 2D SYSTEMS

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ABSTRACT

In this paper some problems connected with the construction of 2D compensators and observers are analyzed. In particular we are concerned with algebraic criteria and linear algorithms for selecting 2D stable polynomials which can be realized as closed loop characteristic polynomials of a 2D system.

I. INTRODUCTION

Given a 2D system in state space form [1]

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) + \\ &+ B_1 u(h,k+1) + B_2 u(h+1,k) \qquad (1) \\ y(h,k) &= C\, x(h,k) \end{aligned}$$

with m inputs, p outputs and n state variables, one of the fundamental control problems is to construct an asymptotic state observer and synthesize a state feedback law that provides a suitable dynamical behaviour.

The general structure of the solution to this problem comes out from the analysis of a Bézout 2D polynomial matrix equation and the state space realization of a matrix fraction description obtained from the solution of the Bézout equation.

More precisely, introducing the following matrices

$$\mathcal{O}(z_1,z_2) = \begin{bmatrix} I-A_1z_1-A_2z_2 \\ \hline C \end{bmatrix} \qquad (2)$$

and

$$\mathcal{R}(z_1,z_2) = \left[I-A_1z_1-A_2z_2 \,\middle|\, B_1z_1+B_2z_2 \right]$$

which are the PBH test matrices for reconstructibility and controllability of (1), it has been proved [2,3] that

i) it is possible to compute a state observer whose estimation error e(h,k) converges to zero as h+k goes to infinity if and only if $\mathcal{O}(z_1,z_2)$ is full rank for all (z_1,z_2) in the closed polidisc

$$\mathscr{P}_1 = \{(z_1,z_2) : |z_1| \le 1, \ |z_2| \le 1\}$$

ii) given any polynomial $q(z_1,z_2)$ in the ideal generated by the maximal order minors of $\mathcal{O}(z_1,z_2)$ the equation

$$P(z_1,z_2)C+Q(z_1,z_2)(I-A_1z_1-A_2z_2)=q(z_1,z_2)I \qquad (4)$$

is solvable. Moreover, if $\mathcal{O}(z_1,z_2)$ is full rank in $\mathscr{P}_1$, we can select $q(z_1,z_2)$ so that the intersection $\mathscr{V}(q)\cap\mathscr{P}_1$ is empty. In this case, any realization of

$$\hat{W}(z_1,z_2)=\left[Q(z_1,z_2)(B_1z_1+B_2z_2) \,\middle|\, P(z_1,z_2)\right]q(z_1,z_2)^{-1}$$

that satisfies the PBH controllability and reconstructibility criteria, provides an asymptotic observer.

iii) stabilizability by means of a dynamic state feedback compensator is equivalent to the full rank condition of $\mathcal{R}(z_1,z_2)$ for all (z_1,z_2) in $\mathscr{P}_1$.

iv) the equation

$$\begin{aligned} &(B_1z_1+B_2z_2)N(z_1,z_2)+ \qquad (5) \\ &\qquad +(I-A_1z_1-A_2'z_2)M(z_1,z_2) = I\,p(z_1,z_2) \end{aligned}$$

is solvable for any polynomial $p(z_1,z_2)$ in the ideal $\mathscr{I}$ generated by the maximal order minors of $\mathcal{R}(z_1,z_2)$. If $\mathcal{R}(z_1,z_2)$ is full rank in $\mathscr{P}_1$, we can choose $p(z_1,z_2)$ that satisfies $\mathscr{V}(p)\cap\mathscr{P}_1=\emptyset$. Given any state feedback compensator that realizes the matrix function

$$N\,M^{-1} \qquad (6)$$

and satisfies the PBH controllability and reconstructibility tests, the closed loop polynomial is $p(z_1,z_2)^n$. So if $p(z_1,z_2)$ has no zeros in $\mathscr{P}_1$, then the above compensator makes the whole system stable.

Clearly, i), ii) and iii), iv) are relative to dual situations and the solutions are essentially the same. Hence in the sequel we only refer to the

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synthesis of a stabilizing compensator with this objective in mind, the following problems have to be successively tackled:

a) check if the maximal order minors of $\mathcal{R}(z_1,z_2)$ are devoid of common zeros in $\mathcal{P}_1$.

b) if one has a positive answer for a), compute a polynomial $p(z_1,z_2)$ belonging to $\bar{\mathcal{I}}$ and having no zeros in $\mathcal{P}_1$

c) solve the Bézout equation (5)

d) realize NM^{-1} by a 2D state space model that satisfies the controllability and reconstructibility PBH tests.

In the sequel, we shall introduce an algorithm which enables to solve a) and b). The reader is referred to [2] and [3] for a complete solution of c) and d).

II. STABILIZABILITY CRITERION

Consider the ideal generated by the minors of maximal order $\bar{m}_1(z_1,z_2), \bar{m}_2(z_1,z_2),\dots,\bar{m}_s(z_1,z_2)$ of the matrix (3) and compute their GCD $c(z_1,z_2)$.

Denote by $\mathcal{I}$ the ideal generated by the coprime polynomials $m_1(z_1,z_2),\dots,m_s(z_1,z_2)$, where $m_i(z_1,z_2) = \bar{m}_i(z_1,z_2)/c(z_1,z_2)$, $i=1,2,\dots,s$.

Since $\mathcal{V}(\bar{\mathcal{I}})\cap\mathcal{P}_1 = [\mathcal{V}(c)\cap\mathcal{P}_1]\cup[\mathcal{V}(\mathcal{I})\cap\mathcal{P}_1]$, for testing stabilizability it is enough to check separately

$$\mathcal{V}(c)\cap\mathcal{P}_1 = \emptyset \tag{7}$$

and

$$\mathcal{V}(\mathcal{I})\cap\mathcal{P}_1 = \emptyset \tag{8}$$

As far as (1) is concerned, we can use standard tests for 2D polynomial stability [4]. In order to check if (8) is satisfied, we shall introduce a linear algorithm that does not require an explicit computation of $\mathcal{V}(\mathcal{I})$.

Let $G=(g_1,g_2,\dots,g_r)$ be a Gröbner basis in $\mathcal{I}$. Since $\mathcal{V}(\mathcal{I})$ is a finite set, the quotient ring $\mathbb{R}(z_1,z_2)/\mathcal{I}$ is a finite dimensional $\mathbb{R}$-vector space and its dimension is equal to the number of monic monomials $d_1,d_2,\dots,d_k$ that are not multiples of the leading power products of any of the polynomials $g_1,g_2,\dots,g_r$ [5]. Note that this set is empty if and only if the Gröbner basis G contains a non zero constant. In this case $\mathcal{V}(\mathcal{I})=\emptyset$ and (8) is obviously true.

Assume now $k>0$. Thus

$$d_1+\mathcal{I} := \bar{d}_1, d_2+\mathcal{I} := \bar{d}_2,\dots d_k+\mathcal{I} := \bar{d}_k$$

can be assumed as a basis in $\mathbb{R}[z_1,z_2]/\mathcal{I}$.

Consider the following maps

$$\mathcal{Z}_1 : \mathbb{R}[z_1,z_2]/\mathcal{I}\to\mathbb{R}[z_1,z_2]/\mathcal{I} : q+\mathcal{I}\to z_1q+\mathcal{I} \tag{9}$$

$$\mathcal{Z}_2 : \mathbb{R}[z_1,z_2]/\mathcal{I}\to\mathbb{R}[z_1,z_2]/\mathcal{I} : q+\mathcal{I}\to z_2q+\mathcal{I} \tag{10}$$

They are both well defined, commutative linear transformations on $\mathbb{R}[z_1,z_2]/\mathcal{I}$ and are represented by a pair of commutative matrices M_1,M_2 in $\mathbb{R}^{k\times k}$, once a basis $v_1,v_2,\dots,v_k$ in $\mathbb{R}^k$ has been associated with $\bar{d}_1,\bar{d}_2,\dots,\bar{d}_k$. Note that the smallest $\mathcal{Z}_1$ and $\mathcal{Z}_2$-invariant subspace generated by $\bar{d}_1=\bar{1}$ is the whole space $\mathbb{R}[z_1,z_2]/\mathcal{I}$. Thus $M_1^iM_2^jv_1$, $i,j\in\mathbb{N}$, generate $\mathbb{R}^k$.

The construction of M_1 and M_2 essentially requires to express $\overline{z_1d_i}$ and $\overline{z_2d_i}$, $i=1,2,\dots,k$, as linear combinations of $\bar{d}_1,\bar{d}_2,\dots,\bar{d}_k$. This can be accomplished by applying the normal form algorithm with respect to G [5].

The properties of $\mathcal{I}$, as well as those of its variety $\mathcal{V}(\mathcal{I})$, directly reflect into the structure of the pair M_1,M_2. Note first that the mapping

$$\mathbb{R}\to\mathbb{R}^{k\times k} : \alpha\to\alpha I_k$$

is a monomorphism of $\mathbb{R}$ into $\mathbb{R}^{k\times k}$, so that the image set $\mathbb{R}I_k$ is a subfield of $\mathbb{R}^{k\times k}$ isomorphic to $\mathbb{R}$. Since the matrices M_1 and M_2 commute each other and with every element αI_k, it follows that the mapping

$$p(z_1,z_2) = \Sigma_{ij}a_{ij}z_1^iz_2^j \to \Sigma_{ij}a_{ij}M_1^iM_2^j := p(M_1,M_2)$$

is a homomorphism of $\mathbb{R}[z_1,z_2]$ into $\mathbb{R}^{k\times k}$. It is easy to see that the kernel of the homomorphism is the ideal $\mathcal{I}$ that is

$$p(z_1,z_2)\in\mathcal{I} \Leftrightarrow p(M_1,M_2) = 0 \tag{11}$$

As a corollary of the theorem on common eigenvectors for commutative matrices [6], we have that $(\alpha,\beta)\in\mathcal{V}(\mathcal{I})$ if and only if M_1 and M_2 have a common eigenvector v and

$$M_1v = \alpha v \qquad M_2v = \beta v$$

On the other hand, basing on the Frobenius theorem on simultaneous triangularization of commutative matrices, the variety $\mathcal{V}(\mathcal{I})$ can be characterized in the following way. Let $T_1=[t_{ij}^{(1)}]$ and $T_2=[t_{ij}^{(2)}]$ be triangular matrices such that $M_1 = P^{-1}T_1P$ and $M_2=P^{-1}T_2P$ for some invertible matrix P in $\mathbb{C}^{k\times k}$.

Then (α,β) belongs to $(\mathcal{I})$ if and only if

By Lemma 2 the measure of the interval

$$\mathcal{L}_r \stackrel{\Delta}{=} \{\lambda \in [0,1] : |t_{rr}^{(2)\bar{j}} + \lambda(t_{rr}^{(2)\bar{j}} - t_{rr}^{(1)\bar{j}})| < 1\}$$

satisfies the inequalities

$$\text{meas}(\mathcal{L}_r) < \frac{2}{\rho^{\bar{j}}-1} < \frac{1}{K}$$

Since the minimum distance between λ_i and λ_j, $i \neq j$, is $1/K$, each interval contains at most one of the λ_i.

Hence at least one λ_i, say $\lambda_{\bar{i}}$, falls out $\bigcup_{r=1}^{R} \mathcal{L}_r$ and the spectrum of $P_{\bar{i}}$

$$\mu_r = t_{rr}^{(2)\bar{j}} - \lambda_{\bar{i}}[t_{rr}^{(2)\bar{j}} - t_{rr}^{(1)\bar{j}}], \quad r = 1,2,\ldots,k;$$

Theorem 1 and the above observations make possible to set a procedure for checking (8) using the following sequence of steps:

STEP 1 Compute the commutative matrices M_1 and M_2

STEP 2 Compute a positive integer $\bar{q}$ such that $\bar{q}$ does not intersect the spectra of M_1 and M_2

STEP 3 Compute ρ and the integer $\bar{j}$

STEP 4 Solve the Lyapunov equations:

$$P_i^T X P_i - X = I \qquad i = 0,1,\ldots,k$$

The system is stabilizing if and only if at least one of the above equations admits a positive definite solution.

3. COMPUTATION OF A STABLE 2D POLYNOMIAL

Assume that the procedure of section 2 has been successful, which means that the system is stabilizable.

The aim of this section is to solve the problem mentioned at point b) in the Introduction, that is the computation of a stable 2D polynomial in $\mathcal{J}$. As we shall see, the techniques of the previous section constitute the basic tools for obtaining this goal.

Let P_i be one of the matrices considered at step 4 in section 2, whose spectrum lies outside the unit disc

$$P_i = \lambda_i M_1^{\bar{j}} + (1-\lambda_i) M_2^{\bar{j}}$$

Denote by $\Delta_i(z) \in R[z]$ the characteristic polynomial of P_i and introduce the polynomial

$$h(z_1,z_2) = \Delta_i(\lambda_i z_1^{\bar{j}} + (1-\lambda_i) z_2^{\bar{j}}) \tag{16}$$

Recalling (11), it is easy to see that $h(z_1,z_2) \in \mathcal{J}$, since

$$h(M_1,M_2) = \Delta_i(P_i) = 0$$

We have now to prove that $h(z_1,z_2)$ defined in (16) is 2D-stable. For, factorize $\Delta_i(z)$ as a product of prime factors:

$$\Delta_i(z) = \prod_{r=1}^{R} (z-\gamma_r)$$

and notice that $|\gamma_r| > 1$, $r = 1,2,\ldots,k$. Each factor in the corresponding factorization of $h(z_1,z_2)$

$$h(z_1,z_2) = \prod_{r=1}^{R} (\lambda_i z_1^{\bar{j}} + (1-\lambda_i) z_2^{\bar{j}} - \gamma_r) \tag{14}$$

turns out a stable 2D polynomial. In fact assume that (z_1,z_2) is a zero of the r-th factor in (14)

$$\lambda_i z_1^{\bar{j}} + (1-\lambda_i) z_2^{\bar{j}} = \gamma_r$$

Then

$$\lambda_i |z_1|^{\bar{j}} + (1-\lambda_i)|z_2|^{\bar{j}} \geq |\lambda_i z_1^{\bar{j}} + (1-\lambda_i) z_2^{\bar{j}}| = |\gamma_r| > 1$$

proves that $(z_1,z_2) \notin \mathcal{P}_1$, since for any (z_1,z_2) in $\mathcal{P}_1$ we have

$$\lambda_i |z_1|^{\bar{j}} + (1-\lambda_i)|z_2|^{\bar{j}} \leq 1$$

Once $h(z_1,z_2)$ has been obtained, a stable 2D polynomial in $\mathcal{J}$, is given by

$$p(z_1,z_2) = c(z_1,z_2) h(z_1,z_2)$$

Note that by assumption $c(z_1,z_2)$ which is the g.c.d. of the maximal order minors in (3), $c(z_1,z_2)$ satisfies (7) and hence is a stable 2D polynomial.

REMARK: If the entries of A_1, A_2, B_1 and B_2 belong to a subfield K of the real field, the algorithm given above provides a stable polynomial $h(z_1,z_2)$ in $\mathcal{J} \cap K[z_1,z_2]$ i.e. a polynomial having coefficients in K.

4. REFERENCES

[1] Math. Sys. Theory, 12, pp. 59-72, 1978.

[2] IEEE Trans., CAS-32, 12, pp. 1246-54, 1985.

[3] in C.I. Byrnes, pp. 99-113 "Frequency domain .. .. for Linear Systems" Elsevier, 1986.

[4] IEEE Proc., 66, 9, pp. 1018-47, 1978.

[5] in N.K. Bose "Multidimensional System Theory", pp. 184-232, Dekker, 1985.

[6] D.A. Suprunenko, R.I. Tyshkevich "Commutative Matrices" Acad. Press., 1968.

[7] M. Marden: "Geometry of Polynomials", AMS, 1966.