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FEEDBACK INVARIANTS IN STABILIZATION OF TWO-DIMENSIONAL FILTERS

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ABSTRACT The recursive structure of 2D systems, which is naturally related to the quarter plane causality introduced in the discrete plane Z x Z, plays an important role in the definition of state feedback laws and gives more possibilities than in 1D case.

In this paper we shall consider the problem of stabilizing a 2D system by a feedback control law generated by another 2D system fed by the output or, as a particular case, by the state of the given system.

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1. INTRODUCTION

The recursive structure of 2D systems is naturally related to the quarter plane causality introduced in the discrete plane Z x Z. The fact that in the discrete plane the future and the past sets of a point do not cover the whole plane, plays an important role in the definition of state feedback laws and gives more possibilities than in 1D case.

As a consequence, in 2D systems theory we have at our disposal more flexible techniques for solving the stabilization problem. Of course the problems are more involved, essentially because the stability criterion of 2D systems relies on the shape of an algebraic curve instead of the position of isolated singularities.

In this paper we shall introduce a stabilization technique based on a dynamic feedback law which preserves the quarter-plane causality. More precisely, we shall consider the problem of stabilizing a 2D system by a feedback control law generated by another 2D system fed by the output or, as a particular case, by the state of the given system.

The stabilization problem exhibits different aspects according to whether we deal with input-output descriptions, in terms of transfer matrices in two variables, or with internal descriptions, in terms of state space models. In both cases, however, a peculiar aspect of the synthesis of stabilizing compensators is that it is not possible to freely assign the variety of the closed loop polynomial. In fact, denoting by $N_R(z_1,z_2)D_R^{-1}(z_1,z_2)$ a coprime matrix fraction description of the transfer matrix, the closed loop variety is constrained to include the set of points $\mathcal{S}$ where the minors of maximal order of the matrix $[N_R \quad D_R]$ simultaneously vanish. So, the set $\mathcal{S}$ is constituted by all points of the closed loop polynomial variety which are invariant under feedback compensation.

The above constraint can be satisfied in a direct way when such points are explicitly computed and do not belong to the unit closed polydisc

In this case it is straightforward to determine a stable separable polynomial vanishing on $\mathcal{S}$, so that a suitable power of this polynomial can be assumed as the closed loop characteristic polynomial.

A different approach is based on the implementation of finite tests for checking feedback stabilizability without any explicit computation of $\mathcal{S}$. In this case a stable closed loop polynomial can be assigned in such a way that the above constraint is automatically satisfied. Keeping with the spirit of the second approach, the content of this paper is based on some constructive methods of the polynomial ideal theory and in particular on a matrix version of the Grobner basis algorithm.

2. PRELIMINARY NOTATIONS AND STATEMENT OF THE PROBLEM.

A 2D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ is a dynamical model [1]

$$x(h+1,k+1) = A_1x(h,k+1) + A_2x(h+1,k) + B_1u(h,k+1) + B_2u(h+1,k) \qquad (1)$$
$$y(h,k) = Cx(h,k) + Du(h,k)$$

where the *local state* x is an n–dimensional vector over the real field **R**, input and output functions take values in $\mathbf{R}^m$ and $\mathbf{R}^p$, A_1, A_2, B_1, B_2, C and D are matrices of suitable dimensions with entries in **R**.

Assuming zero initial conditions $x(i,-i) = 0$, $\forall i \varepsilon \mathbf{Z}$ and denoting by

$$X(z_1,z_2) = \Sigma_{i+j\geq 0}x(i,j)z_1^i z_2^j$$
$$U(z_1,z_2) = \Sigma_{i+j\geq 0}u(i,j)z_1^i z_2^j$$
$$Y(z_1,z_2) = \Sigma_{i+j\geq 0}y(i,j)z_1^i z_2^j$$

the state, input and output functions, respectively, the rational *transfer matrix*

$$W(z_1,z_2) = C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2) + D \qquad (2)$$

gives the input–output map

$$Y(z_1,z_2) = W(z_1,z_2)U(z_1,z_2)$$

Denoting by

$$\Delta(z_1,z_2) = \det(I-A_1z_1-A_2z_2) \qquad (3)$$

the characteristic polynomial of Σ, it has been shown [2] that Σ is internally stable if and only if the variety $V(\Delta)$ does not intersect the unit closed polydisc

$$P_1 = \{(z_1,z_2) \varepsilon \mathbf{C}x\mathbf{C} : |z_1| \leq 1, |z_2| \leq 1\}$$

Suppose now that a 2D *strictly proper* (i.e. D = 0) *plant* $\Sigma = (A_1, A_2, B_1, B_2, C)$ has been given and consider the feedback connection (see fig. 1) with a *compensator* $\Sigma_c = (F_1, F_2, G_1, G_2, H, J)$

$$x'(h+1,k+1) = F_1x'(h,k+1) + F_2x'(h+1,k) + G_1y(h,k+1) + G_2y(h+1,k)$$

$$y'(h,k) = Hx'(h,k) + Jy(h,k)$$
$$u(h,k) = y'(h,k) + v(h,k) \qquad (4)$$

where $v(h,k)$ is the external input at (h,k)

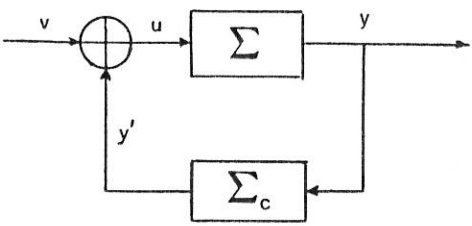

fig 1

The local state $\hat{x} = x \oplus x'$ of the resulting closed loop system $\hat{\Sigma}$ updates according to the following transition matrices

$$\hat{A}_1 = \begin{bmatrix} A_1+B_1JC & B_1H \\ G_1C & F_1 \end{bmatrix} \qquad \hat{A}_2 = \begin{bmatrix} A_2+B_2JC & B_2H \\ G_2C & F_2 \end{bmatrix} \qquad (5)$$

and the corresponding closed loop characteristic polynomial of $\hat{\Sigma}$

$$\hat{\Delta}(z_1,z_2) := \det(I-\hat{A}_1z_1-\hat{A}_2z_2) \qquad (6)$$

depends on the matrices of the compensator. We say that a polynomial $c(z_1,z_2)$ is *assignable* if it can be assumed as the closed loop characteristic polynomial of the feedback connection of Σ and Σ_c , for a suitable compensator Σ_c.

Given Σ, the set of assignable polynomials is a proper subset of $R[z_1,z_2]$. A first obvious constraint on assignable polynomials is that the constant term must be one. Depending on the structure of Σ, further constraints can arise, relative either to the plant transfer matrix or to the particular state space model that realizes it.

Referring to that, our objectives are the following:

i) for a given plant, characterize the subset of assignable polynomials

ii) derive the conditions to be fulfilled in order that the subset above includes some 2D stable polynomials. In this case we say that Σ is output feedback *stabilizable*

iii) in case Σ is stabilizable, explicitely compute an assignable stable polynomial

iv) given any specific $c(z_1, z_2)$ in $\mathbb{R}[z_1, z_2]$, decide about its assignability (or at least the assignability of its variety $\mathcal{V}(c)$) and then realize Σ_c.

3. STRUCTURAL PROPERTIES OF 2D SYSTEMS : PBH CONTROLLABILITY, RECONSTRUCTIBILITY AND HIDDEN MODES

Before embarking in the solution of these problems, some preliminary notions are needed concerning systems structural properties that play a fundamental role in the feedback analysis.

Our exposition will be as concise as possible and we refer to [3,4] for a more accurate discussion and for the proofs.

Let

$$W(z_1, z_2) = N_R D_R^{-1} = D_L^{-1} N_L$$

be a right coprime and a left coprime MFD of the transfer matrix of the plant and denote by $\mathfrak{I}(N_R, D_R)$ and by $\mathfrak{I}(D_L, N_L)$ the ideals generated by the maximal order minors of the matrices $[D_L \;\; N_L]$ and $[N_R^T \;\; D_R^T]$ respectively.

Fact 1. $\mathfrak{I}(N_R, D_R) = \mathfrak{I}(D_L, N_L)$

So, there is no ambiguity on defining the *transfer matrix ideal* $\mathfrak{I}(W)$ as the ideal of the maximal order minors associated with an arbitrary right or left coprime MFD of W. The corresponding *transfer matrix variety* $\mathcal{S} := \mathcal{V}(\mathfrak{I}(W))$ is a (possibly empty) finite set, whose points are called the *rank singularities* of W.

Given a 2D system Σ in state space form, we are allowed to recover the structure of any coprime MFD of the system transfer matrix W. The converse, however, is not true, since the knowledge of the zero state input/output behaviour of Σ does not provide a complete information on the state space structure of Σ. In particular, $W(z_1, z_2)$ uniquely determines $\det D_R$ (modulo some nonzero constant factor) but does not determine the characteristic polynomial of Σ. This is easily seen [4] from the equation

$$\Delta(z_1, z_2) = h(z_1, z_2) \det D_R \tag{7}$$

where the factor h in the right hand cannot be recovered from the transfer matrix.

Introduce the so called *PBH controllability* and *PBH reconstructibility* matrices

$$\mathfrak{R} = \quad [\, I-A_1z_1-A_2z_2 \quad B_1z_1-B_2z_2 \,] \tag{8}$$

$$\mathfrak{O} = \begin{bmatrix} I-A_1z_1-A_2z_2 \\ C \end{bmatrix} \tag{9}$$

and denote by $\mathcal{V}(\mathfrak{R})$ and $\mathcal{V}(\mathfrak{O})$ the complex varieties of the maximal order minors in (8) and (9) respectively.

Fact 2 $\mathcal{V}(\mathfrak{R}) \cup \mathcal{V}(\mathfrak{O}) = \mathcal{V}(h) \cup \mathcal{S}$ (10)

This shows that h is a non constant polynomial if and only if $\mathfrak{R}$ and $\mathfrak{O}$ are not full rank along some algebraic curves, which are associated with the irreducible factors of h. The uncontrollable and/or unreconstructible modes (*hidden modes*) refer to the irreducible factors of the maximal order minors of $\mathfrak{R}$ and $\mathfrak{O}$ respectively.

By definition, a realization Σ of $W(z_1,z_2)$ is *coprime* if Σ is free of hidden modes (i.e. if h is a nonzero constant). As a matter of fact, there are many equivalent definitions of coprime realizations. These are summarized in the following Proposition

Proposition 1 *Let* $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ *be a realization of* $W(z_1,z_2)$ *and assume that* $N_R D_R^{-1}$ *is a r.c.MFD of* $W(z_1,z_2)$. *Then the following facts are equivalent*

i) $\det D_R = \det(I-A_1z_1-A_2z_2)$

ii) $C(I-A_1z_1-A_2z_2)^{-1}$ *and* $(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)$ *are right and left coprime MFD's respectively*

iii) $\mathcal{V}(\mathfrak{R}) \cup \mathcal{V}(\mathfrak{O}) = \mathcal{S}$

iv) Σ *is a coprime realization of* $W(z_1,z_2)$

The question of the existence of coprime realizations for any proper transfer matrix has been positively answered in [5]. Actually it can be proved something more, i.e. that given a coprime MFD $N_R D_R^{-1}$ of $W(z_1,z_2)$ and a polynomial $h \in \mathbb{R}[z_1,z_2]$, a realization $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ of $W(z_1,z_2)$ exists such that $\Delta(z_1,z_2) = h \det D_R$

4. OUTPUT FEEDBACK STABILIZATION

Let $W(z_1,z_2)$ and $W_c(z_1,z_2)$ be the transfer matrices of Σ and Σ_c respectively, and consider two MFD's PQ^{-1} and $X^{-1}Y$ satisfying

$$W(z_1,z_2) = PQ^{-1} \quad , \quad \det Q = \det(I-A_1z_1-A_2z_2) \tag{11}$$

$$W_c(z_1,z_2) = X^{-1}Y \quad , \quad \det X = \det(I-F_1z_1-F_2z_2) \tag{12}$$

Then the closed loop characteristic polynomial (6) is given by

$$\hat{\Delta}(z_1,z_2) = \det(XQ-YP) \tag{13}$$

On the other hand, any left MFD $X^{-1}Y$ with $X(0,0) = I$ admits a realization $\Sigma_c = (F_1,F_2,G_1,G_2,H,J)$ that satisfies the condition

$$\det X = \det(I-F_1z_1-F_2z_2)$$

So, as (X,Y) varies over the set of polynomial matrix pairs with $X(0,0) = I$, (13) produces all assignable closed loop polynomials for the given plant Σ.

Let E be the GCRD of P and Q. Then

$$P = N_RE \qquad Q = D_RE \tag{14}$$

and $N_RD_R^{-1}$ is a right coprime MFD of $W(z_1,z_2)$. As a consequence of (7) and (13), we have

$$\det E = h(z_1,z_2) \tag{15}$$

$$\hat{\Delta}(z_1,z_2) = h\det(XD_R-YN_R) \tag{16}$$

The above formula clearly shows that $h(z_1,z_2)$, which represents the hidden modes of Σ, is an *invariant factor* of $\hat{\Delta}(z_1,z_2)$ w.r. to feedback compensation. In other words, as far as fixed modes are concerned, 2D systems behave exactly in the same way as 1D sustems do. However a deep difference between 2D and 1D systems comes out when we consider the factor $\det(XD_R-YN_R)$. In fact, this factor must vanish for any choice of X and Y on the set $\mathcal{S}$ of rank singularities. Such resticlion does not exist in the 1D case, where the solvability of the Bèzout equation $XD_R-YN_R = I$, and hence the complete assignability of the polynomial $\det(XD_R-YN_R)$, are consequence of the coprimeness of N_R and D_R.

The conditions that $\hat{\Delta}$ vanishes on $\mathcal{S}$ and $\mathcal{V}(h)$ and that $\hat{\Delta}(0,0) \neq 0$ represent the only constraints imposed by the structure of the plant on the closed loop polynomial variety. In fact, given any algebraic curve $\mathcal{C}$ that includes $\mathcal{V}(h) \cup \mathcal{S}$ and excludes the origin, a compensator Σ_c exists such that $\mathcal{V}(\Delta) = \mathcal{C}$ [5].

The most obvious conclusion is that $\mathcal{V}(h) \cap P_1 = \emptyset$ and $\mathcal{S} \cap P_1 = \emptyset$ are both necessary stabilizability conditions for Σ.

These conditions are also sufficient. In fact, assume that

$$\mathcal{S} = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_t, \beta_t)\}$$

satisfies $|\alpha_i| > 1$, $i=1, 2, \ldots s$ $|\beta_i| > 1$, $i=s+1, s+2, \ldots t$ and let $\mathcal{V}(h) \cap P_1 = \emptyset$

Then the variety of the polynomial

$$c(z_1, z_2) = h(z_1, z_2) \prod_{i=1, 2, \ldots s}(z-\alpha_i) \prod_{i=s+1, s+2, \ldots t}(z-\beta_i)$$

is an algebraic curve $\mathcal{C}$ that

(1) does not intersect P_1

(2) includes $\mathcal{V}(h) \cup \mathcal{S}$

We therefore have that a compensator Σ_c exists such that $\mathcal{C}$ is the closed loop polynomial variety. Clearly this compensator stabilizes the plant, since $\mathcal{C} \cap P_1 = \emptyset$

5. STABILIZABILITY CRITERIA

Stabilizability criteria are available, based on separate checks for the stabilizability conditions

$$\mathcal{V}(h) \cap P_1 = \emptyset \qquad (17)$$

$$\mathcal{V}(W) \cap P_1 = \emptyset \qquad (18)$$

As far as (17) is concerned, we can use standard tests for 2D polynomial stability [6]. In order to check if (18) is satisfied, we shall introduce a linear algorithm that does not require an explicit computation of $\mathcal{S}$ [7].

Let $G = (g_1, g_2, \ldots g_r)$ be a Grobner basis in $\mathcal{I}(W)$. Since $\mathcal{S}$ is a finite set, the quotient ring $R[z_1, z_2]/\mathcal{I}(W)$ is a finite dimensional R–vector space and its dimension is equal to the number of monic monomials $d_1, d_2, \ldots d_k$ that are not multiples of the leading power products of any of the polynomials $g_1, g_2, \ldots g_r$ [8]. Note that this set is empty if and only if the Grobner basis G contains a non zero constant. In this case $\mathcal{S} = \emptyset$ and (18) is obviously true.

Assume now $k > 0$. Thus

$$d_1+\mathfrak{I} := \bar{d}_1\ ,\ d_2+\mathfrak{I} := \bar{d}_2\ ,\ \ldots\ \ d_k+\mathfrak{I} := \bar{d}_k$$

can be assumed as a basis in $R[z_1, z_2]/\mathfrak{I}$.

Consider the following maps

$$\mathfrak{z}_1 : R[z_1, z_2]/\mathfrak{I} \rightarrow R[z_1, z_2]/\mathfrak{I} : q+\mathfrak{I} \rightarrow z_1 q+\mathfrak{I} \tag{19}$$

$$\mathfrak{z}_2 : R[z_1, z_2]/\mathfrak{I} \rightarrow R[z_1, z_2]/\mathfrak{I} : q+\mathfrak{I} \rightarrow z_2 q+\mathfrak{I} \tag{20}$$

They are both well defined, commutative, linear transformations on $R[z_1, z_2]/\mathfrak{I}$ and are represented by a pair of commutative matrices M_1, M_2 in $R^{k\times k}$, once a basis $v_1, v_2, \ldots$ v_k in R^k has been associated with $\bar{d}_1, \bar{d}_2, \ \ldots\ \bar{d}_k$. Note that the smallest $\mathfrak{z}_1$ and $\mathfrak{z}_2$-invariant subspace generated by $\bar{d}_1 = \bar{1}$ is the whole space $R[z_1, z_2]/\mathfrak{I}$. Thus $M_1{}^i M_2{}^j v_1$, $i, j \in N$, generate R^k.

The construction of M_1 and M_2 essentially requires to express $\overline{z_1 d_i}$ and $\overline{z_2 d_i}$, $i = 1, 2, \ldots k$ as linear combinations of $\bar{d}_1, \bar{d}_2, \ \ldots\ \bar{d}_k$. This can be accomplished by applying the normal form algorithm with respect to $\mathcal{G}$.

Since the matrices M_1 and M_2 commute each other and with every element αI_k, it follows that the mapping

$$p(z_1, z_2) = \Sigma_{ij} a_{ij} z_1{}^i z_2{}^j \rightarrow \Sigma_{ij} a_{ij} M_1{}^i M_2{}^j := p(M_1, M_2)$$

is a homomorphism of $R[z_1, z_2]$ into $R^{k\times k}$. It is easy to see that the kernel of the homomorphism is the ideal $\mathfrak{I}(W)$, that is

$$p(z_1, z_2) \in \mathfrak{I}(W) \Leftrightarrow p(M_1, M_2) = 0 \tag{21}$$

As a corollary of the Frobenius theorem [9] on simultaneous triangularization of commutative matrices,

$$\mathcal{S} \cap \mathcal{P}_1 = \emptyset \tag{22}$$

if and only if the pairs of eigenvalues appearing along the diagonal of the triangular form do not belong to the unit polydisc $\mathcal{P}_1$.

We can check (22) and construct a 2D stable assignable polynomial using the following procedure:

Step 1 Compute a real number $\rho > 1$ such that M_1 and M_2 are devoid of eigenvalues in the open set $\{z \in C : 1 < |z| < \rho\}$. An algorithm for obtaining such a ρ has been presented in [6].

Step 2 Let j be the smallest integer greater than $\log(2k+1)/\log\rho$. Let $\lambda_i := i/k$, $i = 0, 1, \ldots k$ and define the matrices

$$P_i := \lambda_i M_1^j + (1-\lambda_i)M_2^j \quad , i = 0, 1, \ldots k$$

Solve the Lyapunov equations

$$P_i^T X P_i - X = I \qquad i = 0, 1, \ldots k \tag{23}$$

$\mathcal{S}$ does not intersect $\mathcal{P}_1$ if and only if at least one of the above equations admits a positive definite solution.

Step 3 Assume that P_i is positive definite for some i and denote by $\Delta_i(z) \in R[z]$ its characteristic polynomial. Then the polynomial

$$g(z_1, z_2) = \Delta_i(\lambda_i z_1^j + (1-\lambda_i)z_2^j) \tag{24}$$

is a 2D stable polynomial in $\mathfrak{I}(W)$ and $\mathcal{V}(hg)$ is an assignable closed loop variety that does not intersect $\mathcal{P}_1$.

It can be shown that hg^r is a closed loop assignable polynomial for some positive integer r.

To conclude this section, we briefly sketch the solution of problem iv) in section ii). The procedure breaks up into three checks:

$$\text{a)} \quad (0,0) \notin \mathcal{V}(c) \tag{25}$$
$$\text{b)} \quad \mathcal{V}(h) \subset \mathcal{V}(c) \tag{26}$$
$$\text{c)} \quad \mathcal{V}(W) \subset \mathcal{V}(c) \tag{27}$$

Checking a) is trivial and checking b) reduces to verify if h divides a suitable power of c (for instance $c^{\deg h}$). The last check can be algorithmically performed [4] once a set of generators of $\mathfrak{I}(W)$ has been found. The computation of such a set is immediate if a coprime MFD $N_R D_R^{-1}$ of the transfer matrix is available. In fact the maximal order minors of $[N_R^T \;\; D_R^T]$ generate $\mathfrak{I}(W)$ and equation (17) yiels h.

Using a coprime MFD of $W(z_1, z_2)$ can be avoided if we refer to the state space model and compute the matrices N and D in

$$W(z_1,z_2) = [C\,adj(I-A_1z_1-A_2z_2)(B_1z_1+B_2z_2)][I_m \det(I-A_1z_1-A_2z_2)]^{-1} = \bar{N}\bar{D}^{-1}$$

The generators can be obtained by evaluating the maximal order minors $\{m_1, m_2 \ldots m_\nu\}$ in $[\bar{N}^T \;\; \bar{D}^T]$ and then by eliminating their GCD $d(z_1,z_2)$. Thus h is given by

$$h = \det(I-A_1z_1-A_2z_2)/\det D_R = \det(I-A_1z_1-A_2z_2)d(z_1,z_2)/\det\bar{D}$$

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