On some algebraic aspects of 2D dynamic feedback control

M.Bisiacco, E.Fornasini, G.Marchesini

Abstract The paper investigates some algebraic problems related with the assignability of the closed loop polynomial of 2D systems and the connections between the compensator synthesis and the solution of a polynomial matrix row bordering problem. The possibility of extending the algorithms to 3D systems is also discussed.

1 Matrix Fraction Description of 2D Systems

A 2D system in state space form $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ is a dynamical model [2]

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) \\ &\quad + B_1 u(h,k+1) + B_2 u(h+1,k) \\ y(h,k) &= Cx(h,k) + Du(h,k) \end{aligned} \tag{1.1}$$

where the local state x is an n-dimensional vector over the real field $\mathbf{R}$, input and output functions take value in $\mathbf{R}^m$ and $\mathbf{R}^p$, A_1, A_2, B_1, B_2, C and D are matrices of suitable dimensions, with entries in $\mathbf{R}$. The proper rational transfer matrix that provides the input/output map can be represented using left or right matrix fraction descriptions (MFD's) [3], namely

$$\begin{aligned} W(z_1,z_2) &= C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D \\ &= N_R(z_1,z_2) D_R(z_1,z_2)^{-1} \\ &= D_L(z_1,z_2)^{-1} N_L(z_1,z_2) \end{aligned} \tag{1.2}$$

Among the infinitely many different MFD's of $W(z_1,z_2)$, we draw particular attention to irreducible left MFD's, that are characterized by the following equivalent properties:

1. $D_L = E\bar{D}_L$ and $N_L = E\bar{N}_L$ imply that $\det E$ is a nonzero constant;

2. the maximal order minors $q_1, q_2, \ldots, q_\nu$ of $[D_L \; N_L]$ are coprime polynomials;

3. the variety of the ideal $\mathcal{I}(q_1, q_2, \ldots, q_\nu) := \mathcal{I}(D_L, N_L)$ is a finite, possibly nonempty set.

Irreducible right MFD's can be defined with the obvious changes. The connections between irreducible right and left MFD's are summarized in the following

PROPOSITION 1.1 [1] *Let $N_R D_R^{-1} = D_L^{-1} N_L$ be two irreducible MFD's of $W(z_1, z_2)$. Then, except for a nonzero constant, the maximal order minors obtained by selecting the $i_1, i_2, \ldots, i_p$-th columns in $[D_L \; N_L]$ and the rows with complementary indices in $\begin{bmatrix} N_R \\ D_R \end{bmatrix}$ coincide and, in particular,*

$$\det D_R(z_1, z_2) = \det D_L(z_1, z_2) \tag{1.3}$$

As a straightforward consequence of proposition 1.1, the ideals $\mathcal{I}(D_L, N_L)$ and $\mathcal{I}(D_R, N_R)$ coincide. The corresponding variety is invariant w.r.to left and right irreducible representations of the transfer matrix $W(z_1, z_2)$ and will be denoted by $\mathcal{V}(W)$. The points of $\mathcal{V}(W)$ are called rank singularities and correspond to the values of (z_1, z_2) where the matrices $[D_L \; N_L]$ and $\begin{bmatrix} N_R \\ D_R \end{bmatrix}$ are not full rank.

REMARK In case of transfer matrices in three indeterminates, the irreducibility of $D_L^{-1} N_L$, as stated by property 1., is not equivalent to the coprimality of the maximal order minors in $[D_L \; N_L]$ (see [6]). Also, proposition 1.1 does not hold any longer, as shown by the following example. The 3D transfer matrix

$$W(z_1, z_2, z_3) = \begin{bmatrix} \dfrac{z_2}{1+z_1} & \dfrac{z_3}{1+z_1} \end{bmatrix}$$

admits an irreducible left MFD $D_L^{-1} N_L = [1 + z_1]^{-1} [z_2 \; z_3]$. However, no right MFD $N_R D_R^{-1}$ with elements in $\mathbf{R}[z_1, z_2, z_3]$ can be found such that $\det D_R = 1 + z_1$. In fact, assume by contradiction that there exist polynomials $a, b, c, d, e, f \in \mathbf{C}[z_1, z_2, z_3]$ such that

$$W(z_1, z_2, z_3) = [e \quad f] \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

with $ad - bc = 1 + z_1$. Then we have

(i) $ad - bc = 1 + z_1$ (ii) $ed - fc = z_2$ (iii) $af - be = z_3$

which imply the following column bordering problem:

$$\det \begin{bmatrix} z_2 & a & b \\ z_3 & c & d \\ 1+z_1 & -e & -f \end{bmatrix} = (1+z_1)^2 + z_2^2 + z_3^2 \tag{1.4}$$

As it will be discussed in sec.3, this problem cannot be solved in $\mathbf{R}[z_1, z_2, z_3]$.

The characteristic polynomial of Σ given by

$$\Delta(z_1, z_2) := \det(I - A_1 z_1 - A_2 z_2) \tag{1.5}$$

enables one to deduce explicit results concerning the internal stability of the system. This follows from the property that Σ is internally stable if and only if $\Delta(z_1, z_2)$ is devoid of zeros in the closed unit bidisk $\mathcal{P}_1$.

Given an irreducible MFD $N_R D_R^{-1}$ of a transfer matrix $W(z_1, z_2)$, the characteristic polynomial $\Delta(z_1, z_2)$ of any state space realization of W is a multiple of $\det D_R$ and the polynomial

$$h(z_1, z_2) := \Delta(z_1, z_2) / \det D_R(z_1, z_2) \tag{1.6}$$

provides the so called "hidden modes" of the realization. It can be shown [1] that any proper transfer matrix W admits a state space realizations free of hidden modes ("coprime" realization), so that h is a nonzero constant. More generally, any (not necessarily coprime) MFD $Q^{-1}P$ of W admits a realization whose characteristic polynomial coincides with $\det Q$. Rank singularities and hidden modes are connected with the matrices of the so called PBH criteria of controllability and reconstructibility

$$\mathcal{R} = [\, I - A_1 z_1 - A_2 z_2 \quad B_1 z_1 + B_2 z_2 \,] \qquad \mathcal{O} = \begin{bmatrix} I - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix} \tag{1.7}$$

Actually, denoting by $\mathcal{V}(\mathcal{R})$ and $\mathcal{V}(\mathcal{O})$ the varieties associated with the maximal order minors of $\mathcal{R}$ and $\mathcal{O}$, we have

$$\mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O}) = \mathcal{V}(h) \cup \mathcal{V}(W) \tag{1.8}$$

2 Closed loop characteristic polynomial assignment

Suppose now $D = 0$ in (1.1) (strictly proper system) and consider the connection of Σ with an output feedback compensator $\Sigma_C = (F_1, F_2, G_1, G_2, H, J)$. The local state of the resulting closed loop system $\hat{\Sigma}$ updates according to the following transition matrices

$$\hat{A}_1 = \begin{bmatrix} A_1 - B_1JC & -B_1H \\ G_1C & F_1 \end{bmatrix} \qquad \hat{A}_2 = \begin{bmatrix} A_2 - B_2JC & -B_2H \\ G_2C & F_2 \end{bmatrix} \tag{2.1}$$

and the corresponding characteristic polynomial is

$$\hat{\Delta}(z_1, z_2) = \det(I - \hat{A}_1 z_1 - \hat{A}_2 z_2) \tag{2.2}$$

We say that a polynomial $c(z_1, z_2)$ is assignable if it can be assumed as the closed loop characteristic polynomial of the output feedback connection of Σ and Σ_C for a suitable compensator Σ_C.

Given Σ, some questions arise at this point in a natural way:

1. what is the class of closed loop characteristic polynomials that can be achieved by varying Σ_C ?

2. to what extent can we modify the closed loop polynomial variety $\mathcal{V}(\hat{\Delta})$?

3. how can we decide if a given polynomial or an algebraic curve can be viewed as the characteristic polynomial or the variety of an output feedback connection of Σ and Σ_C ?

The MFD approach provides the natural setting for studying these problems. Let $Q^{-1}P$ and YX^{-1} be two MFD's of the transfer matrices of Σ and Σ_C respectively, and assume that

$$\begin{aligned} \det Q &= \det(I - A_1 z_1 - A_2 z_2) \\ \det X &= \det(I - F_1 z_1 - F_2 z_2) \end{aligned} \tag{2.3}$$

Then, recalling (1.6) and (2.1), the characteristic polynomial of $\hat{\Sigma}$ is given by

$$\hat{\Delta}(z_1, z_2) = \det(QX+PY) = h\det(D_LX+N_LY) = h\det\left([D_L \;\; N_L]\begin{bmatrix} X \\ Y \end{bmatrix}\right) \tag{2.4}$$

Equation (2.4) provides three necessary conditions for the assignability of a polynomial $q(z_1, z_2) \in \mathbf{R}[z_1, z_2]$. The first condition, $q(0,0) = 1$, is obvious and descends from the definition of characteristic polynomial. The second condition is that $q(z_1, z_2)$ is a multiple of $h(z_1, z_2)$. In fact, by (2.4), $h(z_1, z_2)$ is an invariant factor of the characteristic polynomial w.r. to feedback compensation. In other words, as far as the fixed modes are concerned, 2D systems behave exactly in the same way as 1D systems do. A deep difference between 1D and 2D systems comes out, however, when we consider the factor $\det(D_L X + N_L Y)$. Applying the Binet-Cauchy formula, we easily obtain, as a third constraint, that $q(z_1, z_2)/h(z_1, z_2)$ must belong to the ideal generated by the maximal order minors of $[D_L \ N_L]$ and, consequently, must vanish on the set $\mathcal{V}(W)$ of the rank singularities. Such a restriction does not exist in the 1D case, where the solvability of the Bézout equation $D_L X + N_L Y = I$, and hence the complete assignability (except for the zero degree coefficient) of the polynomial $\det(D_L X + N_L Y)$, are consequences of the coprimeness of D_L and N_L.

The above conditions can be interpreted as constraints on the closed loop polynomial variety, that is

$$\begin{aligned} (0,0) &\notin \mathcal{V}(\hat{\Delta}) \\ \mathcal{V}(h) \cup \mathcal{V}(W) &\subseteq \mathcal{V}(\hat{\Delta}) \end{aligned} \tag{2.5}$$

Next theorem shows that (2.5) constitute the only constraints imposed on the closed loop polynomial variety by the structure of Σ

THEOREM 2.1 [1] *Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of the strictly proper transfer matrix $W(z_1, z_2)$ and let $\mathcal{V}(h)$ and $\mathcal{V}(W)$ denote the variety of hidden modes and the set of rank singularities. Given any algebraic curve $\mathcal{C}$ that includes $\mathcal{V}(h) \cup \mathcal{V}(W)$ and excludes the origin, there exists a compensator Σ_C such that the closed loop polynomial variety $\mathcal{V}(\hat{\Delta})$ of $\hat{\Sigma}$ is $\mathcal{C}$.*

Theorem 2.1 has an important corollary concerning feedback stabilization. Indeed the existence of a stabilizing compensator is equivalent to the possibility of obtaining a closed loop polynomial variety that does not intersect the unit closed bidisk. In view of the above theorem, we need only to check that both $\mathcal{V}(h) \cap \mathcal{P}_1$ and $\mathcal{V}(W) \cap \mathcal{P}_1$ are empty sets. In a sense, this provides a complete answer to the stabilizability problem and, more generally, to the assignability of the closed loop polynomial variety. From the computational point of view, however, it would be more useful if a complete characterization of assignable polynomials were available. As far as we know, there are only partial answers to such problem.

According to (2.4), the problem above requires to consider all proper $p \times m$ right MFD's YX^{-1} and to evaluate the corresponding polynomials $\det(D_L X + N_L Y)$ in $\mathcal{I}(D_L, N_L)$. We distinguish two cases.

First, assume $\mathcal{V}(W) = \emptyset$. Thus the matrices D_L and N_L are zero coprime and the Bézout equation

$$D_L X + N_L Y = I \tag{2.6}$$

is solvable in $\mathbf{R}[z_1, z_2]$ using linear algorithms [3]. In order to obtain $h(z_1, z_2)$ as closed loop characteristic polynomial, all what has to be done is to compute a solution of (2.6) and to construct a coprime realization Σ_C of the transfer matrix YX^{-1}. If the closed loop polynomial we need is a multiple of h, say $\hat{\Delta} = qh$, we consider a polynomial matrix M with $\det M = q$. Then $(\tilde{X}, \tilde{Y}) := (XM, YM)$ satisfies $\det(D_L\tilde{X} + N_L\tilde{Y}) = q$ and Σ_C is a realization of $\tilde{Y}\tilde{X}^{-1}$ satisfying $\det \tilde{X} = \det(I - F_1 z_1 - F_2 z_2)$. An alternative procedure leading to a compensator free of hidden modes has been presented in [1].

The above procedure can be viewed as an extension to the 2D case of some known results of 1D theory. The situation is completely different when one assumes $\mathcal{V}(W) \neq \emptyset$. As before, the polynomials $\det(D_L X + N_L Y)$ assume the value 1 at $(0,0)$ and belong to the ideal $\mathcal{I}(N_L, D_L)$. However, in this case $\mathcal{I}(N_L, D_L)$ is a proper ideal of $\mathbf{R}[z_1, z_2]$ and, in general, it is not known if all polynomials in $\mathcal{I}(N_L, D_L)$ have the form $\det(D_L X + N_L Y)$. The problem can be solved for systems having m and/or p equal to 1.

Assume first $p = 1$ and let $N_L = [\, n_1 \quad n_2 \quad \dots \quad n_m \,]$. Since $D_L^{-1} N_L$ is irreducible and strictly proper, we have $n_i(0,0) = 0$, $i = 1, 2, \dots, m$ and it is not restrictive to assume $D_L(0,0) = 1$. Denoting by $y_1, y_2, \dots, y_m$ the elements of Y, the equation

$$q = \det(D_L X + N_L Y) = X D_L + \sum_{i=1}^{m} y_i n_i$$

is solvable for any $q \in \mathcal{I}(N_L, D_L)$ with $q(0,0) = 1$. Moreover $1 = q(0,0) = X(0,0)D(0,0)$ implies that YX^{-1} is a proper transfer matrix.

The case $m = 1$ can be solved along the same lines, using a right MFD for the transfer matrix of Σ.

3 Column/row bordering techniques

In this section we show how the closed loop polynomial assignment can be reformulated in terms of column (or row) bordering of a suitable polynomial matrix in two variables. The key point consists in the observation that, if $D_L^{-1} N_L = N_R D_R^{-1}$ are irreducible MFD's, then $\det D_L = \det D_R$ and, consequently, for any pair of polynomial matrices X and Y, we have

$$\det(D_L X + N_L Y) = \det \begin{bmatrix} X & N_R \\ -Y & D_R \end{bmatrix} \tag{3.1}$$

So, given a polynomial $q(z_1, z_2)$, the search for a pair X, Y such that $\det(D_L X + N_L Y) = q$ is equivalent to column bordering the matrix $\begin{bmatrix} N_R \\ D_R \end{bmatrix}$ up into a square $(p+m) \times (p+m)$ matrix with determinant q.

If $\mathcal{V}(W) = \emptyset$ and $q(z_1, z_2) = 1$, the above problem corresponds to column bordering $\begin{bmatrix} N_R \\ D_R \end{bmatrix}$ up into a unimodular 2D matrix. The feasibility of such a bordering depends on some results of [1], that provide an alternative proof of the Quillen-Suslin theorem for polynomial matrices in two indeterminates.

If $\mathcal{V}(W) \neq \emptyset$, the possibility of determining X and Y such that

$$\det \begin{bmatrix} X & N_R \\ -Y & D_R \end{bmatrix} = q \tag{3.2}$$

for all $q \in \mathcal{I}(N_L, D_L) = \mathcal{I}(N_R, D_R)$ can be viewed as a partial extension of the Serre conjecture. Actually, the Serre conjecture, proved by Quillen and Suslin, is true for an arbitrary number of indeterminates, provided that D_R and N_R are zero coprime [7]. In (3.2), N_R and D_R are factor coprime, and this constitutes an extension of the Serre conjecture. However, it should be pointed out that the extension is false when considering polynomials in three indeterminates, as shown by the following counterexample,where the equation

$$\det \begin{bmatrix} X & z_1 \\ -Y & z_2 \\ & z_3 \end{bmatrix} = z_1^2 + z_2^2 + z_3^2 \in \mathcal{I}(z_1, z_2, z_3)$$

cannot be solved in $\mathbf{R}[z_1, z_2, z_3]$ [4].

There are two cases where the above conjecture can be easily proved. The first of these corresponds to the column bordering up of a $k \times (k-1)$ matrix R into a $k \times k$ matrix, whose determinant is a preassigned polynomial q in the ideal generated by the maximal order minors $r_1, r_2, \ldots, r_k$ of R. In this case, the solution consists in finding k polynomials $q_1, q_2, \ldots, q_k$ such that $q = \sum_{i=1}^{k} q_i r_i$, and these are obtained using the Gröbner basis algorithm. Except for the sign, the polynomials q_i are the elements of the column we were looking for.
In the second case we want to border up a column $R = [\, r_1 \quad r_2 \quad \ldots \quad r_k \,]^T$ into a $k \times k$ matrix whose determinant is a preassigned polynomial $q \neq 0$ in the ideal $\mathcal{I}(r_1, r_2, \ldots, r_k)$. As before, using the Gröbner basis algorithm we write q as $\sum_{i=1}^{k} r_i q_i$, where, without loss of generality, $r_1 q_1$ is assumed to be nonzero. Next, applying a linear algorithm by Lai and Chen [5], we compute a right MFD UV^{-1} such that

$$[q_1]^{-1} [\, q_2 \quad q_3 \quad \ldots \quad q_k \,] = UV^{-1}$$

and $\det V = q_1$. Since we have

$$\det \begin{bmatrix} r_1 & -U \\ r_2 & \\ \cdots & V \\ r_k & \end{bmatrix} = \det V \left(r_1 + UV^{-1} \begin{bmatrix} r_2 \\ \vdots \\ r_k \end{bmatrix} \right)$$

$$= q_1 \left(r_1 + [q_1]^{-1} [\, q_2 \quad \cdots \quad q_k \,] \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix} \right)$$

$$= \sum_{i=1}^{k} q_i r_i = q$$

the matrix $\begin{bmatrix} -U \\ V \end{bmatrix}$ provides the solution of the column bordering problem.

References

1. M.Bisiacco, E.Fornasini, G.Marchesini *Dynamic regulation of 2D systems: a state space approach* to appear in Lin. Alg. and Appl.

2. E.Fornasini and G.Marchesini *Doubly indexed dynamical systems: state space models and structural properties* Math. Sys. Th. , vol. 12, pp. 59-72, 1978

3. S.Kung, B.Lévy, M.Morf, T.Kailath *New results in 2D systems theory: part I, polynomial matrices, factorization and coprimeness* Proc IEEE , vol.65, pp.861-72, 1977

4. R. Gattazzo *Row bordering of polynomial matrices with given determinant* submitted

5. Y.S.Lai, C.T.Chen *Coprime fraction computation of 2D rational matrices* IEEE Trans. on Aut. Contr., vol AC-32, pp.333-36, 1987

6. D.C.Youla, G.Gnavi *Notes on n-dimensional system theory* IEEE Trans. Circ. Sys., vol. CAS-26, pp 105-11, 1979

7. D.C.Youla, P.F.Pickel *The Quillen-Suslin theorem and the structure of n-dimensional elementary polynomial matrices* IEEE Trans Circ. Sys., vol CAS-31, pp.513-18, 1984