SIAM J. CONTROL AND OPTIMIZATION
Vol. 28, No. 3, pp. 582-601, May 1990

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OPTIMAL CONTROL OF TWO-DIMENSIONAL SYSTEMS*

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Abstract. Necessary and sufficient conditions for the existence and the uniqueness of the solution of the optimal control problem of discrete-time linear time invariant two-dimensional systems are determined. Given a system that satisfies these conditions, the optimal control law is obtained using an algebraic Riccati equation with coefficients in the polynomial ring $R[z]$. Since the feedback implementation of this law does not preserve the causal structure of the system, suboptimal control laws are also discussed that lead to a weakly causal two-dimensional system.

Finally, some important connections between optimal control in an l_2 setting and l_∞ stabilization are investigated.

Key words. two-dimensional systems, linear quadratic optimal control, Riccati equation, l_2 stabilizability, l_∞ stability

AMS(MOS) subject classifications. 93C55, 93D15, 49E20

1. Introduction. Underlying a study of a two-dimensional system is often a motivation to improve its dynamical behaviour.

Recently, some authors [1]-[3] concentrated their efforts in two-dimensional state space models stabilization by means of feedback compensators. Their synthesis objective being to obtain a prescribed stable closed-loop characteristic polynomial, the approach they followed is reminiscent of the classical one-dimensional pole placement method. However, the pole placement design usually exhibits a poor control on the short-term system response, since it essentially affects the asymptotic evolution.

In this paper we take a different approach and concentrate our study on design procedures that are based on the minimization of a quadratic cost functional J. The two-dimensional system, which is the end result of this optimal design, is not merely internally stable, but satisfies additional requirements on the state and input evolutions that are summarized by the structure of J.

The class of discrete time two-dimensional systems we will consider has as a prototype the linear model described by the state updating equation [4]

$$x(h+1, k+1) = A_1x(h, k+1) + A_2x(h+1, k) + B_1u(h, k+1) + B_2u(h+1, k) \tag{1.1}$$

where $x(h, k) \in R^n$ and $u(h, k) \in R^m$ are the local state and the input value at (h, k) and A_1, A_2, B_1, B_2 are real matrices of suitable dimensions.

Assuming that the initial local states $x(i, -i)$ have been assigned, the state dynamics is completely determined by the input function $u(\cdot, \cdot)$. Our prime concern in the next section will be to pose in precise terms the optimum LQ problem for the system above. In fact, if no constraint is imposed on the structure of the infinite set of initial local states we might expect that initial conditions contribute *per se* an infinite value to the corresponding cost functional. It turns out that a satisfactory theory requires us to assume that both the space of allowed initial conditions and the space of input functions are l_2.

Within this framework, a fundamental property is the one that reduces the existence and the uniqueness of two-dimensional optimal control to a pair of rank conditions on two-dimensional polynomial matrices. This result constitutes a nontrivial extension

* Received by the editors August 3, 1988; accepted for publication (in revised form) August 5, 1989.
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of the corresponding one-dimensional polynomial criteria based on PBH controllability and reconstructibility matrices.

One further point of contact with the one-dimensional theory is that the optimal control can be expressed in a linear feedback form, so that the resulting closed-loop system is also a linear dynamic system. In this respect, however, a deep difference arises, since one-dimensional optimal control law preserves the original causality structure of the system, while two-dimensional causality is completely lost. This is essentially due to the fact that the optimal control value at (h, k) depends on an infinite number of local states that are not located in the past of (h, k).

The main tool for solving the problem is a special extension of the algebraic Riccati equation, whose coefficients are polynomial matrices in one variable. When the solvability conditions are satisfied, this equation admits a unique stabilizing solution, which induces a feedback matrix analytic in an open neighbourhood of the unit circle.

The analyticity property is extremely important in two respects. First, it allows us to obtain the optimal feedback law using the coefficients of a Laurent series expansion. Second, it naturally calls for an approximation procedure that provides a suboptimal control through the truncation of the above series.

A noteworthy advantage is that this suboptimal control stabilizes the closed-loop system while preserving a weakly causal two-dimensional structure, that recursively generates the feedback input values.

The $\mathfrak{l}_2$ approach followed in the paper is mainly motivated by the necessity of guaranteeing that the optimal control problem is a meaningful one. Usually, when dealing with the internal stability of two-dimensional systems, a more general approach is taken into account [5], since an $\mathfrak{l}_\infty$ constraint is the only requirement imposed on the set of initial conditions. The control law we obtained by solving the $\mathfrak{l}_2$ optimal control problem still holds in an $\mathfrak{l}_\infty$ setting and, interestingly enough, the $\mathfrak{l}_2$ stabilizability criterion, based on the rank of a two-dimensional polynomial matrix, provides a necessary and sufficient condition for $\mathfrak{l}_\infty$ stabilizability also.

2. Problem formulation. Denote by

$$\mathscr{C}_k = \{(i, j): i + j = k\}, \qquad k = 0, 1, 2, \cdots$$

the kth *separation set* in $Z \times Z$. For purposes of future manipulation, it is useful to single out the restrictions of the input function $u(\cdot, \cdot)$ and the state evolution $x(\cdot, \cdot)$ to the separation sets and to consider such restrictions either as bilateral sequences or as Laurent formal power series in one variable. So, the restrictions of $u(\cdot, \cdot)$ and $x(\cdot, \cdot)$ to $\mathscr{C}_k$ will be denoted by the sequences

$$\mathfrak{U}_k := \{u(-i, k+i);\ i \in Z\}, \tag{2.1}$$

$$\mathfrak{X}_k := \{x(-i, k+i);\ i \in Z\}, \tag{2.2}$$

or by the series

$$\mathfrak{U}_k(\xi) := \sum_{i=\infty}^{+\infty} u(-i, k+i)\xi^i, \tag{2.3}$$

$$\mathfrak{X}_i(\xi) := \sum_{i=-\infty}^{+\infty} x(-i, k+i)\xi^i. \tag{2.4}$$

In the following, sequences (2.2) will be called *global states*.

For each initial global state $\mathcal{X}_0$ on the separation set $\mathcal{C}_0$ and each input function $u(\cdot,\cdot)$, we introduce the quadratic *cost functional*[1]

$$J(u,\mathcal{X}_0):=\sum_{h+k\geqq 0}[x^T(h,k)Qx(h,k)+u^T(h,k)Ru(h,k)] \tag{2.5}$$

with $R>0$ and $Q\geqq 0$. The optimal control problems we aim to solve are the following:

(i) Given $\mathcal{X}_0$, derive conditions for the existence and the uniqueness of an input $u(\cdot,\cdot)$ that minimizes the cost J.

(ii) Whenever these conditions are satisfied, explicitly compute the optimal input and the corresponding value of J.

It is apparent from the structure of J that admissible input functions must belong to the space $l_2^{2D}(R^m)$ of R^m-valued functions $u(\cdot,\cdot)$ defined on

$$Z_+^2:=\{(h,k)\in Z\times Z\colon h+k\geqq 0\}=\bigcup_{k\geqq 0}\mathcal{C}_k$$

and satisfying the finite norm condition

$$\|u(\cdot,\cdot)\|_2^2:=\sum_{h+k\geqq 0}u^T(h,k)u(h,k)<\infty.$$

Furthermore, we are only interested in state dynamics $x(\cdot,\cdot)$ that belong to $l_2^{2D}(R^n)$. Although this condition is not necessary for guaranteeing the finiteness of J in case Q is singular, it fulfills the natural requirement of imposing a stable pattern on the admissible state evolutions. In fact, $x(\cdot,\cdot)\in l_2^{2D}$ implies that the associated global states $\mathcal{X}_t$ satisfy

$$\|\mathcal{X}_t\|_2^2:=\sum_{i=-\infty}^{+\infty}x^T(-i,t+i)x(-i,t+i)<\infty, \tag{2.6}$$

$$\sum_{t=0}^{\infty}\|\mathcal{X}_t\|_2^2=\|x(\cdot,\cdot)\|_2^2 \tag{2.7}$$

showing that $\|\mathcal{X}_t\|\to 0$ as $t\to\infty$.

Just putting $t=0$ in (2.6), we argue that the allowable bilateral sequences of initial conditions must belong to $l_2(R^n)$. In this way, the optimization problem we aim to solve is completely couched in an l_2 setting.

It is well known that the infinite time least squares problem for stationary linear one-dimensional dynamical systems may be treated analytically via the algebraic Riccati equation. The questions of the existence and uniqueness of a stabilizing optimal solution, however, can be settled a priori, without explicitly solving the equation. In fact, a necessary and sufficient condition for both properties is that the polynomial matrix

$$[I-Aw \quad Bw] \tag{2.8}$$

has full rank for any w in the closed unit disk and the polynomial matrix

$$\begin{bmatrix} Q \\ I-Aw \end{bmatrix} \tag{2.9}$$

has full rank for any complex w in the unit cirle [6], [7]:

$$\gamma_1:=\{w\in C\colon |w|=1\}.$$

[1] Throughout the paper, ^{T} means transpose, $^\vee$ conjugate, and $*$ conjugate transpose.

In two-dimensional optimal control problems, the existence and uniqueness properties of a two-dimensional stabilizing control still reduce to rank conditions on polynomial matrices in two variables and the optimal control law is obtained via an algebraic Riccati equation whose coefficients are polynomial matrices in one variable. A precise statement of the main results is given by Theorem 1.

THEOREM 1. *The following facts are equivalent*:

(1) OS (*optimal solution*). *For each global state* $\mathfrak{X}_0$ *in* $\mathfrak{l}_2(R^n)$ *there exists an* $\mathfrak{l}_2^{2D}$ *solution of the optimal control problem, i.e., there exists an input* $u(\cdot,\cdot)$ *in* $\mathfrak{l}_2^{2D}(R^m)$ *such that* $x(\cdot,\cdot)$ *belongs to* $\mathfrak{l}_2^{2D}(R^n)$ *and the corresponding value of* J *is minimized.*

(2) RC (*rank conditions*). *The two-dimensional polynomial matrix*

$$[I-A_1z_1-A_2z_2 \quad B_1z_1+B_2z_2] \tag{2.10}$$

has full rank on the set

$$\mathcal{M}=\{(z_1,z_2)\in C\times C: |z_1|=|z_2|\leqq 1\}$$

and the two-dimensional polynomial matrix

$$\begin{bmatrix} Q \\ I-A_1z_1-A_2z_2 \end{bmatrix} \tag{2.11}$$

has full rank on the unit torus

$$\mathcal{T}=\{(z_1,z_2)\in C\times C: |z_1|=|z_2|=1\}.$$

(3) AREz (*algebraic Riccati equation*). *The following polynomial algebraic Riccati equation*:

$$\begin{aligned} P(z)=Q&+(A_1^T+A_2^Tz^{-1})P(z)(A_1+A_2z)\\ &-(A_1^T+A_2^Tz^{-1})P(z)(B_1+B_2z)[R+(B_1^T+B_2^Tz^{-1})P(z)(B_1+B_2z)]^{-1}\\ &\times(B_1^T+B_2^Tz^{-1})P(z)(A_1+A_2z)\end{aligned} \tag{2.12}$$

in the unknown matrix $P(z)$ *admits a unique solution in an open annulus that includes* γ_1, *with the following properties*:

(i) $P(e^{j\omega})=P^*(e^{j\omega})\geqq 0$, *for all* $\omega\in[0,2\pi]$.

(ii) *The matrix*

$$K(z):=-[R+(B_1^T+B_2^Tz^{-1})P(z)(B_1+B_2z)]^{-1}(B_1^T+B_2^Tz^{-1})P(z)(A_1+A_2z) \tag{2.13}$$

is analytic in an open annulus that includes γ_1.

(iii) *The matrix*

$$\hat{\Gamma}(\omega):=(A_1+A_2e^{j\omega})+(B_1+B_2e^{j\omega})K(e^{j\omega}) \tag{2.14}$$

is asymptotically stable for all ω *in* $[0,2\pi]$.

Moreover, whenever the above conditions are fulfilled, the input $u(\cdot,\cdot)$ considered in OS is uniquely determined by the stabilizing feedback law

$$u(h,k)=\sum_{i=-\infty}^{+\infty}K_ix(h+i,k-i) \tag{2.15}$$

where the matrices K_i are the coefficients of the Laurent series expansion

$$K(z)=\sum_{i=-\infty}^{+\infty}K_iz^i. \tag{2.16}$$

The implications OS $\to$ RC of the above theorem will be proved in § 3 and the implications RC $\to$ AREz $\to$ OS in § 4. In § 5 an $\mathfrak{l}_\infty$ extension of some results will be given, while in § 6 a suboptimal solution is discussed, which seems quite attractive from the implementation point of view.

3. Necessary conditions for $\mathfrak{l}_2$ solvability. An obvious necessary solvability condition of the $\mathfrak{l}_2$ optimal control problem is that for any initial global state $\mathcal{X}_0$ in $\mathfrak{l}_2$, there exists some input $u(\cdot,\cdot)$ in $\mathfrak{l}_2^{2D}$ that provides an $\mathfrak{l}_2^{2D}$ state evolution.

Denoting by $\mathcal{V}(\mathcal{X}_0)$ the affine variety of all inputs in $\mathfrak{l}_2^{2D}$ with this property, the above requirement is formally restated as

$$\mathcal{V}(\mathcal{X}_0)\neq\varnothing \qquad \forall \mathcal{X}_0\in\mathfrak{l}_2. \tag{3.1}$$

It is intuitively clear, however, that the existence of input functions inducing finite values in the cost functional does not necessarily imply that the infimum of $J(\mathcal{X}_0,\cdot)$ is effectively attained for some input function in $\mathcal{V}(\mathcal{X}_0)$. So we expect that additional conditions, besides (3.1), must be fulfilled to guarantee the existence of an optimal control in $\mathcal{V}(\mathcal{X}_0)$.

The following theorem shows that in some sense it is meaningful to discuss separately the existence of $\mathfrak{l}_2^{2D}$ state evolutions and that of optimal controls. Actually, the first problem is connected with the rank of the matrix (2.10), whereas the second depends on the rank of both (2.10) and (2.11).

The results of the theorem that provide necessary conditions for solving these problems will be supplemented by those of § 4, showing that the same conditions are also sufficient. Thus an elegant check is available for the feasibility of two-dimensional optimal control which constitutes a nontrivial extension of the results already available in the one-dimensional case.

THEOREM 2. *If $\mathcal{V}(\mathcal{X}_0)\neq\varnothing$ for all initial global states $\mathcal{X}_0\in\mathfrak{l}_2$, the polynomial matrix* (2.10) *has full rank for all (z_1,z_2) in $\mathcal{M}$. If moreover, for each $\mathcal{X}_0\in\mathfrak{l}_2$, there exists an input $u_{\mathrm{opt}}\in\mathcal{V}(\mathcal{X}_0)$ such that*

$$J(\mathcal{X}_0,u)\geqq J(\mathcal{X}_0,u_{\mathrm{opt}}) \quad \forall u\in\mathcal{V}(\mathcal{X}_0), \tag{3.2}$$

then the polynomial matrix (2.11) *has full rank for all (z_1,z_2) in $\mathcal{T}$.*

Proof. Suppose that (2.10) is not full rank at $(z_1^0,z_2^0)=(b\,e^{j\vartheta_1^0},b\,e^{j\vartheta_2^0})\in\mathcal{M}$.

Then there exists a nonzero vector $v\in C^n$ such that

$$v^T(I-A_1z_1^0-A_2z_2^0)=0,\qquad v^T(B_1z_1^0+B_2z_2^0)=0. \tag{3.3}$$

We now introduce the following initial global state $\mathcal{X}_0\in\mathfrak{l}_2$:

$$\mathcal{X}_0:=\{x(i,-i)=0 \quad \text{for } i\neq 0;\ x(0,0)=v\}$$

and suppose that there exists an input $u(\cdot,\cdot)\in\mathfrak{l}_2^{2D}$ whose corresponding state evolution $x(\cdot,\cdot)$ is in $\mathfrak{l}_2^{2D}$. Since $0<b\leqq 1$, the functions u_b and x_b defined by

$$u_b(h,k)=u(h,k)b^{h+k},\qquad h+k\geqq 0, \tag{3.4}$$

$$x_b(h,k)=x(h,k)b^{h+k},\qquad h+k\geqq 0, \tag{3.5}$$

are in $\mathfrak{l}_2^{2D}$ and x_b represents the state evolution determined by $\mathcal{X}_0$ and u when A_1, A_2, B_1, B_2 in (1.1) are replaced by bA_1, bA_2, bB_1, bB_2. Therefore, taking the double Fourier transforms of u_b and x_b

$$\hat{u}_b(\omega_1,\omega_2):=\sum_{h+k\geqq 0}u_b(h,k)\,e^{-ih\omega_1}\,e^{-ik\omega_2}, \tag{3.6}$$

$$\hat{x}_b(\omega_1,\omega_2):=\sum_{h+k\geqq 0}x_b(h,k)\,e^{-ih\omega_1}\,e^{-ik\omega_2}, \tag{3.7}$$

it is easy to prove that

$$v = [I - bA_1\, e^{-j\omega_1} - bA_2\, e^{-j\omega_2}]\hat{x}_b(\omega_1, \omega_2) - [bB_1\, e^{-j\omega_1} + bB_2\, e^{-j\omega_2}]\hat{u}_b(\omega_1, \omega_2) \tag{3.8}$$

holds almost everywhere in $[0, 2\pi) \times [0, 2\pi)$. Letting

$$a(\omega_1, \omega_2) := \begin{bmatrix} (I - bA_1^T\, e^{j\omega_1} - bA_2^T\, e^{j\omega_2})v \\ (-bB_1^T\, e^{j\omega_1} - bB_2^T\, e^{j\omega_2})v \end{bmatrix}, \tag{3.9}$$

premultiplication of (3.8) by v^* gives

$$\|v\|_2^2 = \left\langle a(\omega_1, \omega_2), \begin{bmatrix} \hat{u}_b(\omega_1, \omega_2) \\ \hat{x}_b(\omega_1, \omega_2) \end{bmatrix} \right\rangle \leqq \|a(\omega_1, \omega_2)\|_2 \cdot \left\| \begin{bmatrix} \hat{u}(\omega_1, \omega_2) \\ \hat{x}(\omega_1, \omega_2) \end{bmatrix} \right\|_2$$

which in turn implies that

$$\|\hat{u}_b(\omega_1, \omega_2)\|_2^2 + \|\hat{x}_b(\omega_1, \omega_2)\|_2^2 \geqq \|v\|_2^4 / \|a(\omega_1, \omega_2)\|_2^2. \tag{3.10}$$

Since $a(\vartheta_1^0, \vartheta_2^0) = 0$, it is easy to obtain a quadratic upper bound of the following form:

$$\|a(\omega_1, \omega_2)\|_2^2 \leqq M[(\omega_1 - \vartheta_1^0)^2 + (\omega_2 - \vartheta_2^0)^2]$$

and inequality (3.10) can be replaced by

$$\|\hat{u}_b(\omega_1, \omega_2)\|_2^2 + \|\hat{x}_b(\omega_1, \omega_2)\|_2^2 \geqq \|v\|_2^4 / M[(\omega_1 - \vartheta_1^0)^2 + (\omega_2 - \vartheta_2^0)^2]. \tag{3.11}$$

The right-hand side of the above inequality being not summable on $[0, 2\pi) \times [0, 2\pi)$, we have that the same is a fortiori true for the left-hand side. This contradicts the original assumption that $u_b(\cdot, \cdot)$ and $x_b(\cdot, \cdot)$ were l_2^{2D} functions. In fact that assumption implied, by Parseval's Theorem, the summability of both $\hat{u}_b$ and $\hat{x}_b$ on $[0, 2\pi) \times [0, 2\pi)$.

Note that in the above proof a complex valued global state $\mathfrak{X}_0$ was considered. However the conclusion of the theorem is correct even when only real global states are allowed: we have just to analyze the dynamics that correspond to the real or imaginary part of $\mathfrak{X}_0$.

The proof of the second part of the theorem is quite long and will be omitted for sake of brevity. It may be found in [8]. □

4. A Riccati equation for two-dimensional systems. It is very well known that the algebraic Riccati equation plays a crucial role in the solution of one-dimensional optimal control problems. In this section we will derive a Riccati equation for two-dimensional systems that provides a closed-loop optimal solution to the problem of minimizing the quadratic cost functional J defined in (2.5).

As a matter of fact, the actual evaluation of the minimum cost and the explicit computation of the optimal input function lead us to study the existence of a particular solution of the Riccati equation, which we characterize in terms of positive definiteness and analyticity.

Let us first start with a preliminary analysis of the open-loop system dynamics in terms of Fourier transforms. We assume that $\mathfrak{X}_0$ and $\mathfrak{U}_t$ belong to l_2. Then, by equation (1.1) all global states $\mathfrak{X}_t$, $t = 1, 2, \cdots$ are in l_2 and the Fourier transforms

$$\begin{aligned} \hat{\mathfrak{U}}_t(\omega) &= \sum_{h=-\infty}^{+\infty} u(t+h, -h)\, e^{-ih\omega}, \\ \hat{\mathfrak{X}}_t(\omega) &= \sum_{h=-\infty}^{+\infty} x(t+h, -h)\, e^{-ih\omega} \end{aligned} \tag{4.1}$$

have components in $L_2[0, 2\pi]$. Letting

$$\hat{A}(\omega) = A_1 + e^{j\omega}A_2, \qquad \hat{B}(\omega) = B_t + e^{j\omega}B_2,$$

equation (1.1) can be rewritten as a first-order recursive equation

$$\hat{\mathfrak{x}}_{t+1}(\omega) = \hat{A}(\omega)\hat{\mathfrak{x}}_t(\omega) + \hat{B}(\omega)\hat{\mathfrak{u}}_t(\omega), \tag{4.2}$$

whereas Parseval's and Beppo Levi's Theorems allow us to express the cost functional in the following form:

$$\begin{aligned} J &= \sum_{t=0}^{\infty} (2\pi)^{-1} \int_0^{2\pi} \mathfrak{x}_t^*(\omega) Q \hat{\mathfrak{x}}_t(\omega) + \hat{\mathfrak{u}}_t^*(\omega) R \hat{\mathfrak{u}}_t(\omega)\, d\omega \\ &= (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} [\hat{\mathfrak{u}}_t^*(\omega) \quad \hat{\mathfrak{x}}_t^*(\omega)] \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \hat{\mathfrak{u}}_t(\omega) \\ \hat{\mathfrak{x}}_t(\omega) \end{bmatrix} d\omega. \end{aligned} \tag{4.3}$$

Suppose that the $\mathfrak{l}_2^{2D}$ norms of the input function $u(\cdot,\cdot)$ and of the state dynamics $x(\cdot,\cdot)$ are both finite, i.e.,

$$\begin{aligned} \|u(\cdot,\cdot)\|_2^2 &= (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} \hat{\mathfrak{u}}_t^*(\omega)\hat{\mathfrak{u}}_t(\omega)\, d\omega < \infty, \\ \|x(\cdot,\cdot)\|_2^2 &= (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} \hat{\mathfrak{x}}_t^*(\omega)\hat{\mathfrak{x}}_t(\omega)\, d\omega < \infty. \end{aligned} \tag{4.4}$$

Then, for every Hermitian matrix $\hat{P}(\omega) = \hat{P}^*(\omega)$ with elements in $L_\infty[0, 2\pi]$, we obtain the following identity:

(4.5)

$$\begin{aligned} 0 &= \hat{\mathfrak{x}}_0^*(\omega)\hat{P}(\omega)\hat{\mathfrak{x}}_0(\omega) - \sum_{t=0}^{\infty} \hat{\mathfrak{x}}_t^*(\omega)\hat{P}(\omega)\hat{\mathfrak{x}}_t(\omega) \\ &\quad + \sum_{t=0}^{\infty} [\hat{\mathfrak{u}}_t^*(\omega)\hat{B}^*(\omega) + \hat{\mathfrak{x}}_t^*(\omega)\hat{A}^*(\omega)]\hat{P}(\omega)[\hat{A}(\omega)\hat{\mathfrak{x}}_t(\omega) + \hat{B}(\omega)\hat{\mathfrak{u}}_t(\omega)] \\ &= \hat{\mathfrak{x}}_0^*(\omega)\hat{P}(\omega)\hat{\mathfrak{x}}_0(\omega) \\ &\quad + \sum_{t=0}^{\infty} [\hat{\mathfrak{u}}_t^*(\omega)\hat{\mathfrak{x}}_t^*(\omega)] \begin{bmatrix} \hat{B}^*(\omega)\hat{P}(\omega)\hat{B}(\omega) & \hat{B}^*(\omega)\hat{P}(\omega)\hat{A}(\omega) \\ \hat{A}^*(\omega)\hat{P}(\omega)\hat{B}(\omega) & \hat{A}^*(\omega)\hat{P}(\omega)\hat{A}(\omega) - \hat{P}(\omega) \end{bmatrix} \begin{bmatrix} \hat{\mathfrak{u}}_t(\omega) \\ \hat{\mathfrak{x}}_t(\omega) \end{bmatrix}. \end{aligned}$$

Integrating (4.5) between zero and 2π and adding the resulting identity to J, we obtain

$$\begin{aligned} J &= (2\pi)^{-1} \int_0^{2\pi} \hat{\mathfrak{x}}_0^*(\omega)\hat{P}(\omega)\hat{\mathfrak{x}}_0(\omega)\, d\omega \\ &\quad + (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} [\hat{\mathfrak{s}}_t^*(\omega) \quad \hat{\mathfrak{x}}_t^*(\omega)] \\ &\quad \times \begin{bmatrix} R + \hat{B}^*(\omega)\hat{P}(\omega)\hat{B}(\omega) & 0 \\ 0 & \hat{E}(\omega) \end{bmatrix} \begin{bmatrix} \hat{\mathfrak{s}}_t(\omega) \\ \hat{\mathfrak{x}}_t(\omega) \end{bmatrix} \end{aligned} \tag{4.6}$$

where

$$\hat{K}(\omega) := -[R + \hat{B}^*(\omega)\hat{P}(\omega)\hat{B}(\omega)]^{-1}\hat{B}^*(\omega)\hat{P}(\omega)\hat{A}(\omega), \tag{4.7}$$

$$\hat{\mathfrak{s}}_t(\omega) := \hat{\mathfrak{u}}_t(\omega) - \hat{K}(\omega)\hat{\mathfrak{x}}_t(\omega), \tag{4.8}$$

$$\begin{aligned} \hat{E}(\omega) &= -\hat{P}(\omega) + Q + \hat{A}^*(\omega)\hat{P}(\omega)\hat{A}(\omega) - \hat{A}^*(\omega)\hat{P}(\omega)\hat{B}(\omega) \\ &\quad \times [R + \hat{B}^*(\omega)\hat{P}(\omega)\hat{B}(\omega)]^{-1}\hat{B}^*(\omega)\hat{P}(\omega)\hat{A}(\omega). \end{aligned} \tag{4.9}$$

Clearly, if we are able to choose $\hat{P}(\omega)$ in such a way that $\hat{E}(\omega)$ is zero almost everywhere in $[0, 2\pi]$, then (4.6) reduces to

$$J = (2\pi)^{-1} \int_0^{2\pi} \hat{\mathfrak{x}}_0^*(\omega)\hat{P}(\omega)\hat{\mathfrak{x}}_0(\omega)\, d\omega + (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} \hat{\mathfrak{s}}_t^*(\omega) \times [R + \hat{B}^*(\omega)\hat{P}(\omega)\hat{B}(\omega)]\hat{\mathfrak{s}}_t(\omega)\, d\omega$$

and the minimum value of J

$$J_{\min} = (2\pi)^{-1} \int_0^{2\pi} \hat{\mathfrak{x}}_0^*(\omega)\hat{P}(\omega)\hat{\mathfrak{x}}_0(\omega)\, d\omega \tag{4.10}$$

is attained using the closed-loop control given by

$$\hat{\mathfrak{u}}_t(\omega) = \hat{K}(\omega)\hat{\mathfrak{x}}_t(\omega). \tag{4.11}$$

The conclusion we have drawn so far depicts the situation in a way that may convince us of the intuitive reasonableness of the result. However, some caveats are in order, since the validity of the procedure depends heavily on the existence of $\hat{P}(\omega)$ and on the fact that both the input and the state dynamics given by (4.11) and (4.2) belong to $\mathfrak{l}_2^{2D}$.

More precisely, the solution of the optimal control problem outlined above makes sense if we are able to give a positive answer to the following questions:

(i) Is there any solution $\hat{P}(\omega) = \hat{P}^*(\omega)$ of the equation $\hat{E}(\omega) = 0$, i.e., of the ω-dependent Riccati equation (AREω)

$$\hat{P}(\omega) = Q + \hat{A}^*(\omega)\hat{P}(\omega)\hat{A}(\omega) - \hat{A}^*(\omega)\hat{P}(\omega)\hat{B}(\omega) \times [R + \hat{B}^*(\omega)\hat{P}(\omega)\hat{B}(\omega)]^{-1}\hat{B}^*(\omega)\hat{P}(\omega)\hat{A}(\omega)?$$

(ii) Among these solutions, is there any solution $\hat{P}(\omega)$ that provides, through (4.7), a feedback matrix $\hat{K}(\omega)$ mapping $L_2[0, 2\pi]$ into $L_2[0, 2\pi]$? This requirement is necessary for guaranteeing that the feedback law (4.11) (reinterpreted in the time domain) always transforms an $\mathfrak{l}_2$ global state $\mathfrak{X}_t$ into an $\mathfrak{l}_2$ input sequence $\mathfrak{U}_t$.

(iii) In particular, does (AREω) possess any (Hermitian) solution that ensures asymptotic stability of the closed-loop system, in the sense that, for any $\mathfrak{X}_0 \in \mathfrak{l}_2$, the resulting global states sequence $\{\mathfrak{X}_t\}$ can be viewed as an element of $\mathfrak{l}_2^{2D}$? Note that this condition is needed in order to have a feedback input (4.11) that belongs to $\mathcal{V}(\mathfrak{X}_0)$.

For every fixed ω in $[0, 2\pi]$, (AREω) is the algebraic Riccati equation of a one-dimensional system over the complex field. So, if the rank conditions of Theorem 1 are fulfilled, for each ω in $[0, 2\pi]$ the equation has a unique positive semidefinite solution $\hat{P}(\omega) = \hat{P}^*(\omega)$, that makes the one-dimensional closed-loop system matrix

$$\hat{\Gamma}(\omega) = \hat{A}(\omega) + \hat{B}(\omega)\hat{K}(\omega) \tag{4.12}$$

asymptotically stable [7].

Clearly $P(\omega)$, viewed as a matrix function of ω, satisfies the first question we raised above. Actually, it provides a solution that also satisfies questions (ii) and (iii). However showing this property deserves an accurate investigation of the analytic structure of $\hat{P}(\omega)$. A first result in this direction is provided by Theorem 3, which shows that the map

$$P : \gamma_1 \to C^{n \times n} : e^{j\omega} \to \hat{P}(\omega) \tag{4.13}$$

admits an analytic extension to a suitable open annulus including γ_1, and the extension $P(z)$ satisfies (2.12).

THEOREM 3. *Assume that the matrices* (2.10) *and* (2.11) *are full rank for any* (z_1, z_2) *in* $\mathcal{M}$ *and in* $\mathcal{T}$, *respectively. Then the equation* (2.12) *admits a (unique) solution* $P(z)$ *that fulfills the following conditions*:

(i) $P(z)$ *is analytic in an open annulus that includes the unit circle* γ_1.

(ii) *For all* ω *in* $[0, 2\pi]$, $P(e^{j\omega})$ *coincides with the unique Hermitian positive-semidefinite stabilizing solution* $\hat{P}(\omega)$ *of* (AREω).

For the proof, see the Appendix.

To completely answer questions (ii) and (iii), we need to discuss certain important properties of the solution $P(z)$ obtained in Theorem 3 that shed some light on the structure of the optimal feedback law and on its stabilizing character.

(1) Because of the analytic nature of $P(z)$, there exists a Laurent series expansion

$$P(z) = \sum_{h=-\infty}^{+\infty} P_h z^h \tag{4.14}$$

that converges in an open annulus including γ_1. The coefficients P_h of (4.14) are real matrices that satisfy the conditions

$$P_h = P_{-h}^T, \qquad h = 0, 1, 2, \cdots. \tag{4.15}$$

The proof of this property depends on the following lemma.

LEMMA 1. *Let* $P(z)$ *be the solution of* (AREz) *considered in Theorem* 3. *Then*

$$P(z) = P^T(z^{-1})$$

in a suitable open annulus that includes γ_1.

Proof. For all $\omega \in [0, 2\pi]$, the matrix $P(e^{j\omega}) = \hat{P}(\omega)$ is a solution of (AREω). On the other hand, taking the transpose of both sides of (AREω) and substituting ω with $-\omega$ we check easily that $P^T(e^{-j\omega})$ is still a solution of (AREω). So $P(e^{j\omega})$ and $P^T(e^{-j\omega})$ are both Hermitian positive-semidefinite solutions of (AREω) for any ω in $[0, 2\pi]$. Because of the uniqueness of the stabilizing solution, proving that these solutions coincide reduces to show that the matrix

$$\hat{A}(\omega) - \hat{B}(\omega)[R + \hat{B}^*(\omega)P^T(^{-j\omega})\hat{B}(\omega)]^{-1}\hat{B}^*(\omega)P^T(e^{-j\omega})\hat{A}(\omega) \tag{4.16}$$

is asymptotically stable for any ω in $[0, 2\pi]$. This is again obvious, since the conjugate of (4.16) is $\hat{\Gamma}(-\omega)$, which is asymptotically stable by hypothesis.

Thus $P(z)$ and $P^T(z^{-1})$ are analytical in an open annulus that includes γ_1 and assume the same values on γ_1. By the identity principle of analytic functions, this implies $P(z) = P^T(z^{-1})$ for any z in the annulus. $\square$

We therefore have (4.15), as an immediate consequence of the lemma. Moreover, in (4.14) the Hermiticity of $P(e^{j\omega})$ gives $P_h = P_{-h}^*$, $h = 0, 1, 2, \cdots$.

The above equalities and (4.15) imply $P_{-h}^* = P_{-h}^T$, which proves the realness of all matrices P_h.

(2) The coefficients P_h in the expansion of $P(z)$ decay exponentially as $|h|$ increases, i.e., there exist $M > 0$ and $\lambda \in (0, 1)$ such that

$$\|P_h\| < M\lambda^{|h|}, \qquad h \in Z. \tag{4.17}$$

(3) Since $P(e^{j\omega})$ is positive semidefinite for any ω in $[0, 2\pi]$,

$$R + (B_1^T + B_2^T z^{-1})P(z)(B_1 + B_2 z)$$

is invertible for every $z \in \gamma_1$ and, by a continuity argument, for every z in an open annulus that includes γ_1. Hence the matrix

$$K(z) = -[R + (B_1^T + B_2^T z^{-1})P(z)(B_1 + B_2 z)]^{-1}(B_1^T + B_2^T z^{-1})P(z)(A_1 + A_2 z)$$

extends analytically $\hat{K}(\omega)$ in the annulus and therefore admits a Laurent power series expansion

$$K(z) = \sum_{h=-\infty}^{\infty} K_h z^h. \tag{4.18}$$

Clearly, the feedback law (4.11) is well defined, since it associates an input $\hat{\mathfrak{U}}_t(\omega) \in L_2[0, 2\pi]$ to every global state $\hat{\mathfrak{X}}_t(\omega) \in L_2[0, 2\pi]$. This provides a positive answer to question (ii).

We conclude at once from these properties that the state dynamics $x(\cdot, \cdot)$ and the corresponding input function $u(\cdot, \cdot)$ are, for any initial global state $\mathfrak{X}_0$ in $\mathfrak{l}_2$, elements of $\mathfrak{l}_2^{2D}$, which is all we need to answer question (iii). Actually, the Lyapunov equation

$$\hat{V}(\omega) = I + \hat{\Gamma}^*(\omega)\hat{V}(\omega)\hat{\Gamma}(\omega) \tag{4.19}$$

admits a unique positive-definite solution, given by the sum of the following pointwise convergent series:

$$\hat{V}(\omega) = \sum_{h=0}^{\infty} \hat{\Gamma}^*(\omega)^h \hat{\Gamma}(\omega)^h.$$

Furthermore, the linearity of (4.19) and the uniqueness of its solution for every ω in $[0, 2\pi]$ imply that the matrix $\hat{V}(\omega)$ is a continuous function of ω and hence its spectral radius $\rho(\omega)$ is uniformly bounded by some positive ρ.

Combining all these properties and applying Beppo Levi's and Parseval's Theorems, we obtain

$$\begin{aligned} \|x(\cdot, \cdot)\|_2^2 &= \sum_{t=0}^{\infty} \|\mathfrak{X}_t\|_2^2 = (2\pi)^{-1} \sum_{t=0}^{\infty} \int_0^{2\pi} \hat{\mathfrak{X}}_0^*(\omega)\hat{\Gamma}^*(\omega)^t \hat{\Gamma}(\omega)^t \hat{\mathfrak{X}}_0(\omega)\, d\omega \\ &= (2\pi)^{-1} \int_0^{2\pi} \mathfrak{X}_0^*(\omega)\hat{V}(\omega)\mathfrak{X}_0(\omega)\, d\omega \leqq \rho \|\mathfrak{X}_0\|_2^2. \end{aligned}$$

So $x(\cdot, \cdot)$ and, obviously, $u(\cdot, \cdot)$ belong to $\mathfrak{l}_2^{2D}$.

Tying together the results of Theorem 3 and its consequences, discussed at points (1)-(3), we have that the implications RC $\rightarrow$ ARE $\rightarrow$ OS in Theorem 1 are completely proved.

To conclude this section, we wish to investigate some important consequences of the time domain structure of the optimal control law

$$u(h, k) = \sum_{i=-\infty}^{+\infty} K_i x(h+i, k-i). \tag{4.20}$$

Clearly the input value at (h, k) linearly depends on the whole sequence of local states on the separation set including (h, k). So, the quarter plane causality is completely lost in the closed-loop system.

As the coefficients K_i decay to zero exponentially, it is reasonable to expect that a suboptimal control law could be achieved by truncating the infinite series (4.18) and hence by using a finite number of local states in the feedback law (4.20). This will be discussed in detail in the sequel. Here we only remark that the input (4.20) actually minimizes the cost functional, whose value is given by

$$
\begin{aligned}
J_{\min}(\mathfrak{X}_0) &= (2\pi)^{-1} \int_0^{2\pi} \hat{\mathfrak{X}}_0^*(\omega) \hat{P}(\omega) \hat{\mathfrak{X}}_0(\omega)\, d\omega \\
&= \sum_{h,k=-\infty}^{+\infty} x^T(h,-h) P_{h-k} x(k,-k) \\
&= [\cdots x^T(1,-1) x^T(0,0) x^T(-1,1) \cdots] \\
&\qquad \begin{bmatrix} \ddots & \ddots & \ddots & \\ \ddots P_0 & \ddots P_1 & P_2 \ddots & \\ \ddots P_{-1} & P_0 & P_1 \ddots & \\ \ddots P_{-2} & P_{-1} & P_0 & \\ & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x(1,-1) \\ x(0,0) \\ x(-1,1) \\ \vdots \end{bmatrix}.
\end{aligned} \tag{4.21}
$$

Remark. In case the system is autonomous (i.e., $B_1 = B_2 = 0$), the stabilizability condition (2.10) reduces to

$$\det(I - A_1 z_1 - A_2 z_2) \neq 0 \tag{4.22}$$

for $|z_1| = |z_2| \leqq 1$ or, equivalently, to the internal stability of the one-dimensional matrices $A_1 + A_2 e^{j\omega}$ for all ω in $[0, 2\pi]$. Note that the stability of $A_1 + A_2 e^{j\omega}$ for all ω is equivalent to two-dimensional internal stability [5] and hence to $\det(I - A_1 z_1 - A_2 z_2) \neq 0$ in the unit closed polydisk $\mathcal{P}_1 = \{(z_1, z_2): |z_1| \leqq 1, |z_2| \leqq 1\}$.

Under the same hypothesis, (AREω) reduces to the ω-independent Lyapunov equation for two-dimensional systems and (4.21) gives the Lyapunov function associated with its free dynamical evolution [5].

5. l_∞ stabilization. The feedback law (4.20) we obtained in the previous section, using an l_2 spaces approach, is well defined even when the initial global state $\mathfrak{X}_0$ belongs to an l_∞ space

$$\|\mathfrak{X}_0\|_\infty := \sup_{h \in Z} \|x(h,-h)\|_2 < \infty. \tag{5.1}$$

Actually, the exponential decay of the matrix sequence $\{K_1\}$ as $i \to \pm\infty$ implies that (4.20) converges for all bounded sequences of local states, which shows that $\mathfrak{U}_t$ and $\mathfrak{X}_t$ are in l_∞ for any $t \geqq 0$. Now it seems quite natural to ask whether the closed-loop asymptotic stability (4.20) realized in the l_2 case is preserved when l_∞ initial global states are allowed.

Here asymptotic stability means that $x(h,k)$ converges to zero uniformly as $h + k \to \infty$.

We first note that there exist positive constants M and λ, $0 < \lambda < 1$, such that

$$\|\hat{\Gamma}^t(\omega)\|_2 < M\lambda^t, \qquad t = 0, 1, 2, \cdots$$

for any ω in $[0, 2\pi]$ (for a proof, see Theorem 5 in the Appendix). Let us assume that all local states of $\mathfrak{X}_0$ are zero except for a single local state x. Then we have

$$\|\mathfrak{X}_t\|_\infty \leqq \|\mathfrak{X}_t\|_2 < M\lambda^t \|x\|_2. \tag{5.2}$$

In case $\mathfrak{X}_0$ is an arbitrary l_∞ sequence, any local $x(h, t-h) \in \mathfrak{X}_t$ is the superposition of $t+1$ contributions determined by the initial local states $x(h,-h)$, $x(h-1,-h+1), \cdots, x(h-t,-h+t)$. Consequently,

$$\|\mathfrak{X}_t\|_\infty = \sup_{h \in Z} \|x(h,-h+t)\|_2 < (t+1) M\lambda^t \|\mathfrak{X}_0\|_\infty \tag{5.3}$$

shows that the global states converge uniformly to zero.

Note that the above discussion implies that an l_∞ stabilizing control does exist as soon as matrix (2.10) is full rank on $\mathcal{M}$. In fact, assuming $Q = I_n$ in the cost functional, an l_2 optimal feedback law is computable through the solution of (2.12) and, as the above discussion shows, the same law provides an l_∞ stabilizing control.

It turns out that the rank condition on (2.10) is not only sufficient but also necessary for the existence of l_∞ input functions that drive any initial global state $\mathfrak{X}_0 \in l_\infty$ uniformly to zero. This is proved in the following theorem.

THEOREM 4. *Assume that the matrix* $[I - A_1 z_1 - A_2 z_2 \;\; B_1 z_1 + B_2 z_2]$ *is not full rank for some* $(z_1^0, z_2^0) \in \mathcal{M}$. *Then there exists an initial global state* $\mathfrak{X}_0 \in l_\infty$, *with* $\|\mathfrak{X}_0\|_\infty = 1$, *having the following property. For any sequence* $\{\mathfrak{U}_t\}$ *with elements in* l_∞, *the corresponding sequence of global states* $\{\mathfrak{X}_t\}$ *satisfies*

$$\|\mathfrak{X}_t\|_\infty \geqq 1, \qquad t = 1, 2, \cdots. \tag{5.4}$$

Proof. Let $(z_1^0, z_2^0) = (\rho\, e^{j\vartheta_1}, \rho\, e^{j\vartheta_2})$, $0 < \rho \leqq 1$ and define

$$\mu := e^{j(\vartheta_2 - \vartheta_1)}, \quad F := A_1 + \mu A_2, \quad G := B_1 + \mu B_2.$$

Since the one-dimensional polynomial matrix $[I - Fz \;\; G]$ is not full rank at $z = \rho\, e^{j\vartheta_1}$, the one-dimensional system (F, G) is not stabilizable. This implies that, modulo a change of basis in the state space, the matrices F and G have the following block structure:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \qquad G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

and the spectrum of F_{22} includes the eigenvalue $\gamma = \rho^{-1}\, e^{-j\vartheta_1}$.

Referring the local state space of the original two-dimensional system to the same basis and partitioning its matrices conformably with the partition of F and G

$$A_i = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_1^{(i)} \\ B_2^{(i)} \end{bmatrix}, \quad i = 1, 2,$$

we have

$$A_{21}^{(1)} + \mu A_{21}^{(2)} = 0, \qquad B_2^{(1)} + \mu B_2^{(2)} = 0. \tag{5.5}$$

An easy inductive argument shows that the polynomial matrices $(A_1 + A_2\xi)^r$ and $(A_1 + A_2\xi)^{r-1}(B_1 + B_2\xi)$, $r = 1, 2, \cdots$ have the following form:

$$(A_i + A_2\xi)^r = \begin{bmatrix} (*) & (*) \\ M_{21}^{(r)}(\xi)(\xi - \mu) & M_{22}^{(r)}(\xi)(\xi - \mu) + F_{22}^r \end{bmatrix}, \tag{5.6}$$

$$(A_1 + A_2\xi)^{r-1}(B_1 + B_2\xi) = \begin{bmatrix} (*) \\ N_{r-1}(\xi)(\xi - \mu) \end{bmatrix} \tag{5.7}$$

where $M_{21}^{(r)}(\xi)$, $M_{22}^{(r)}(\xi)$, $N_{r-1}(\xi)$, and $(*)$ denote polynomial matrices with elements in $C[\xi]$.

We now introduce the r-steps reachability matrix

$$\mathcal{R}_r = [(B_1 + B_2\xi) \quad (A_1 + A_2\xi)(B_1 + B_2\xi) \cdots (A_1 + A_2\xi)^{r-1}(B_1 + B_2\xi)]. \tag{5.8}$$

Then the global state $\mathfrak{X}_r(\xi)$ that corresponds to an initial global state $\mathfrak{X}_0(\xi)$, and to inputs $\mathfrak{U}_0(\xi), \mathfrak{U}_1(\xi) \cdots \mathfrak{U}_{r-1}(\xi)$, is expressed as [9]

$$\mathfrak{X}_r(\xi) = (A_1 + A_2\xi)^r \mathfrak{X}_0(\xi) + \mathcal{R}_r(\xi) \begin{bmatrix} \mathfrak{U}_{r-1}(\xi) \\ \vdots \\ \mathfrak{U}_0(\xi) \end{bmatrix}. \tag{5.9}$$

Using (5.7), we rewrite the reachability matrix as

$$\mathcal{R}_r(\xi)=\begin{bmatrix} I & 0 \\ 0 & (\xi-\mu)I \end{bmatrix}\begin{bmatrix} (*) & (*) & & (*) \\ N_0(\xi) & N_1(\xi) & \cdots & N_{r-1}(\xi) \end{bmatrix},$$

which shows that the forced state evolution in (5.9) has the following structure:

$$\text{(5.10)} \qquad \begin{bmatrix} (*) \\ (\xi-\mu)\sum_{i=0}^{r-1} N_i(\xi)\mathfrak{U}_{r-1-i}(\xi) \end{bmatrix} := \begin{bmatrix} (*) \\ (\xi-\mu)q(\xi) \end{bmatrix}.$$

To satisfy (5.4), we introduce an initial global state

$$\text{(5.11)} \qquad \mathfrak{X}_0(\xi)=\begin{bmatrix} 0 \\ v \end{bmatrix}\sum_{h=-\infty}^{+\infty} \mu^{-h}\xi^h$$

where v is a unitary eigenvector of F_{22} associated to the eigenvalue γ. Since $\mathfrak{X}_0(\xi)(\xi-\mu)=0$, it is easy to see that the corresponding free state evolution in (5.9) is

$$\text{(5.12)} \qquad (A_1+A_2\xi)^r\mathfrak{X}_0(\xi)=\gamma^r\mathfrak{X}_0(\xi)+\begin{bmatrix} (*) \\ 0 \end{bmatrix}.$$

Here $(*)$ denotes some arbitrary bilateral formal power series.

Since $\mathfrak{U}_0(\xi), \mathfrak{U}_1(\xi)\cdots\mathfrak{U}_{r-1}(\xi)$ are bounded (i.e., $\|\mathfrak{U}_i\|_\infty<\infty$, $i=0,1,\cdots,r-1$), $\mathfrak{X}_r(\xi)$ is also bounded. So, we combine (5.10) and (5.12) and get the inequality

$$\text{(5.13)} \qquad \begin{aligned} \|\mathfrak{X}_r(\xi)\|_\infty &= \left\|\begin{bmatrix} [*] \\ \gamma^r v\sum_k \mu^{-k}\xi^k+(\xi-\mu)q(\xi) \end{bmatrix}\right\|_\infty \\ &\geqq \|\gamma^r v\textstyle\sum_k \mu^{-k}\xi^k+(\xi-\mu)q(\xi)\|_\infty. \end{aligned}$$

One additional consequence of the boundedness of $\mathfrak{U}_i(\xi)$ is that the series $q(\xi)=\sum_k q_k\xi^k$ we introduced in (5.10) is $\mathfrak{l}_\infty$ and therefore there exists a positive M such that $\|q_k\|<M$ for all k in Z.

Now consider in (5.13) the coefficients of the series

$$\gamma^r v\sum_{k=-\infty}^{+\infty}\mu^{-k}\xi^k+(\xi-\mu)q(\xi):=\sum_{k=-\infty}^{+\infty} g_k\xi^k.$$

It is immediate that $\mu^k g_k=v\gamma^r+\mu^k q_{k-1}-\mu^{k+1}q_k$, and summing over k yields

$$\text{(5.14)} \qquad \sum_{k=0}^{N-1} g_k\mu^k=Nv\gamma^r+q_{-1}-\mu^N q_{N-1}$$

where N is an arbitrary positive integer. We therefore have

$$\text{(5.15)} \qquad \begin{aligned} \|\mathfrak{X}_r(\xi)\|_\infty &\geqq \sup_{k\in Z}\|g_k\| \geqq \left(\frac{1}{N}\right)[\|g_0\|+\|g_1\|+\cdots\|g_{N-1}\|] \\ &= \left(\frac{1}{N}\right)[\|g_0\|+\|\mu g_1\|+\cdots\|\mu^{N-1}g_{N-1}\|] \\ &\geqq \left(\frac{1}{N}\right)\left\|\sum_{k=0}^{N-1} g_k\mu^k\right\| \geqq \rho^{-r}\|v\|-N^{-1}\|q_{-1}+\mu^N q_{N-1}\| \geqq \rho^{-r}-\frac{2M}{N} \end{aligned}$$

and since $N\geqq 1$ was arbitrary, $\|\mathfrak{X}_r(\xi)\|_\infty\geqq\rho^{-r}\geqq 1$. This shows that (5.4) holds independently of the choice of the inputs $\mathfrak{U}_i$ in $\mathfrak{l}_\infty$.

By the argument used in the conclusion of Theorem 2, the statement of the theorem is correct even when only real global states are allowed. □

Remark 1. When dealing with globally reachable two-dimensional systems, i.e., systems whose reachability matrix $\mathcal{R}_n(\xi)$ is right invertible in $R(\xi)$, it has been proved in [9] that every initial global state $\mathfrak{X}_0$ can be driven to zero in a finite number of steps, irrespective of the rank condition of Theorem 4. Actually this does not involve a contradiction, since the finite-time control considered in [9] was not restricted to use only l_∞ inputs. For instance, assuming in (1.1) $A_1 = A_2 = 1$, $B_1 = -B_2 = 1$ gives a globally reachable two-dimensional system whose PBH matrix (2.10) is zero at $(z_1^0, z_2^0) = (1, 1) \in \mathcal{M}$. The global state

$$\mathfrak{X}_0(\xi) = \sum_{-\infty}^{+\infty}{}_k\, \xi^k$$

is controlled to zero in one step by the unbounded input

$$\mathfrak{U}_0(\xi) = -\sum_{-\infty}^{+\infty}{}_k\, k\xi^k.$$

Remark 2. A sufficient stabilizability condition based on the rank of (2.10) has been proved in [10] using different techniques and referring to dynamical models where the local state at (h, k) linearly depends on all local states and input values of the separation set $\mathscr{C}_{h+k-1}$.

In Kamen's paper, however, stabilizability means by definition the existence of a stabilizing state feedback, whereas the stabilizability definition considered in the present paper is essentially open loop. Actually, no a priori hypothesis has been assumed here on the way the stabilizing input functions could be generated, and the possibility of implementing the stabilizing control by a state feedback law is a theorem rather than an assumption.

The major consequence of this approach is that open-loop stabilizability, closed-loop stabilizability, and the full rank of (2.10) on $\mathcal{M}$ are equivalent properties.

6. Weakly causal suboptimal feedback. The control law (4.20), we obtained through the solution of (AREz), provides a state feedback that stabilizes (1.1) both in the l_2 and in the l_∞ settings.

Although this approach is conceptually appealing, the difficulties when no approximation is used can be very great. We already noted that, in general, the input value at (h, k) depends on the (infinitely many) local states $x(h-i, k+i)$, $i \in Z$. So, implementing (4.20) completely destroys the quarter plane causality of the original system and produces a half plane causal two-dimensional system, whose updating equation is required in principle to cope with an infinite-dimensional state vector. Moreover, to determine the solution of (2.12) is by no means a trivial task. In particular, a difficult problem that has no one-dimensional counterpart is that of obtaining the analytic structure of the feedback matrix $K(z)$ and computing the coefficients K_i that provide, in the time domain, the optimal feedback law.

We will give here two examples. The first shows how the solvability conditions based on the rank of (2.10) and (2.11) reflect into the analytic structure of $\hat{P}(\omega)$. The second gives an idea of some difficulties involved in the computation of $K_i s$ even in dimension one.

Example 1. Assume as in (1.1) $m = n = 1$ and $A_1 = B_1 = B_2 = 1$ and $A_2 = -1$. Furthermore, let $R = Q = 1$ be the weighting matrices of J.

In this case the solution of AREz can be obtained in closed form as

$$P(z) = \frac{1 \pm \sqrt{5 + 2z + 2z^{-1}}}{2(2 + z + z^{-1})}.$$

Letting $z = e^{jw}$, we obtain a negative solution and the solution

$$\hat{P}(\omega) = \frac{1}{-1+\sqrt{5+4\cos\omega}}.$$

The first cannot be taken into account, since we are looking for nonnegative solutions only; the second is positive for all ω in $[0, 2\pi]$, except at $\omega = \pi$, where $\hat{P}(\omega)$ diverges. Actually this is not surprising, because (2.10) is not full rank at $(\frac{1}{2}, -\frac{1}{2}) \in \mathcal{M}$. Hence a stabilizing optimal feedback law does not exist for some initial global state in $\mathfrak{l}_2$.

Example 2. Let us change only the sign of A_2 in the previous example. In this case the unique positive-definite solution of (AREω) is given by

$$\hat{P}(\omega) = \frac{2}{\sqrt{1+16(1+\cos\omega)^2}-(3+4\cos\omega)}$$

and the corresponding feedback matrix is

$$\hat{K}(\omega) = \frac{4(1+\cos\omega)}{1+\sqrt{1+16(1+\cos\omega)^2}}.$$

A plot of $\hat{P}(\omega)$ is given in Fig.1.

Since $\hat{P}(\omega)$ attains its minimum value at $\omega = \pi$, we have

$$J_{\min}(\mathfrak{X}_0) = (2\pi)^{-1}\int_0^{2\pi} \hat{P}(\omega)|\hat{\mathfrak{X}}_0(\omega)|^2\, d\omega \geqq \|\mathfrak{X}_0\|_2^2\hat{P}(\pi)$$

and $J_{\min}$ can be made arbitrarily close to the lower bound if we consider initial global states whose spectral content is concentrated in a narrow neighbourhood of π.

The computation of the K_h's depends on the evaluation of the following integrals:

$$K_h = (2\pi)^{-1}\int_0^{2\pi} \frac{4(1+\cos\omega)\cos(\omega h)}{1+\sqrt{1+16(1+\cos\omega)^2}}\, d\omega.$$

Since infinitely many K_h's are different from zero, the optimal feedback law (2.15) cannot be implemented by a finite-dimensional device, and the resulting closed-loop system is a half-plane causal two-dimensional system.

To overcome the storage and computation problems, it seems natural to investigate whether, in the case (1.1) satisfying the rank condition of Theorem 3, the stabilizing feedback matrix could be constrained to have all elements in the bilateral polynomials ring $R[z, z^{-1}]$. An obvious advantage of this control law is that $u(h, k)$ would only

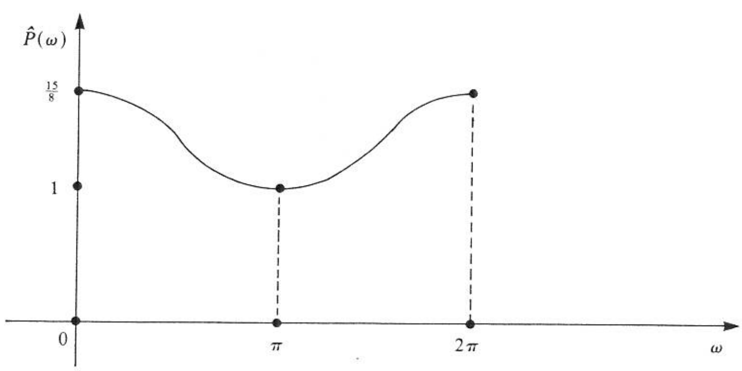

FIG. 1

depend on a finite number of local states, which makes the closed-loop system weakly causal [11].

The question above can be positively answered. Actually, we will prove that the bilateral polynomial matrix

$$K_N(z) := \sum_{i=-N}^{N} K_1 z^1$$

obtained by truncation of the Laurent series (4.18) gives an (l_2 and l_∞) stabilizing state feedback, provided that N is large enough. Furthermore, when N diverges and l_2 initial states are considered, the corresponding cost functional J_N asymptotically converges to the minimum value $J_{\min}$.

To prove the first statement, recall that the coefficients K_i exponentially decay as $|i| \to \infty$. This implies that $K_N(e^{i\omega})$ and the corresponding closed-loop matrix $\Gamma_N(e^{j\omega})$ uniformly converge to $K(e^{j\omega})$ and $\Gamma(e^{j\omega})$, respectively.

Denoting by $\rho < 1$ the maximum spectral radius of $\Gamma(e^{j\omega})$ as ω varies in $[0, 2\pi]$, it will suffice to prove that for every ω the spectral radius of $\Gamma_N(e^{j\omega})$ does not exceed $(1+\rho)/2$ for large values of N. It even suffices to prove that for every ω the distance between the eigenvalues of $\Gamma_N(e^{j\omega})$ and those of $\Gamma(e^{j\omega})$ is less than $(1-\rho)/2$ for large values of N.

By the Ostrowski Theorem [12], there exists a positive real δ such that the distance between the eigenvalues of $\Gamma_N(e^{j\omega})$ and those of $\Gamma(e^{j\omega})$ is less than $(1-\rho)/2$ if

$$\|\Gamma(e^{j\omega}) - \Gamma_N(e^{j\omega})\| < \delta \tag{6.1}$$

for every ω in $[0, 2\pi]$. But the uniform convergence of $\Gamma_N(e^{j\omega})$ to $\Gamma(e^{j\omega})$ guarantees that (6.1) becomes true as N diverges.

To prove that J_N converges to $J_{\min}$ we introduce a frequency-dependent quadratic Lyapunov function that provides a very convenient integral representation of the cost functional J.

LEMMA 2. *Assume that the feedback law* $\hat{\mathfrak{u}}_t(\omega) = \hat{K}(\omega)\hat{\mathfrak{x}}_t(\omega)$ *stabilizes the system* (1.1) *in the usual* l_2 *sense and denote by* $\hat{\Gamma}(\omega)$ *the corresponding closed-loop matrix. The cost functional associated to an initial global state* $\hat{\mathfrak{x}}_0(\omega)$ *is given by*

$$J = (2\pi)^{-1} \int_0^{2\pi} \hat{\mathfrak{x}}_0^*(\omega)\hat{P}(\omega) n \hat{\mathfrak{x}}_0(\omega)\, d\omega \tag{6.2}$$

where $\hat{P}(\omega)$ *is the (unique) solution of the following Lyapunov equation*:

$$\hat{P}(\omega) = \hat{\Gamma}^*(\omega)\hat{P}(\omega)\hat{\Gamma}(\omega) + [Q + \hat{K}^*(\omega)R\hat{K}(\omega)]. \tag{6.3}$$

Proof. By Parseval's and Beppo Levi's Theorems, the cost functional can be represented as

$$\begin{aligned} J &= (2\pi)^{-1} \sum_{h=0}^{\infty} \int_0^{2\pi} [\hat{\mathfrak{x}}_h^*(\omega) Q \hat{\mathfrak{x}}_h(\omega) + \hat{\mathfrak{u}}_h^*(\omega) R \hat{\mathfrak{u}}_h(\omega)]\, d\omega \\ &= (2\pi)^{-1} \int_0^{2\pi} \hat{\mathfrak{x}}_0^*(\omega) \sum_{h=0}^{\infty} \hat{\Gamma}^*(\omega)^h [Q + \hat{K}^*(\omega)R\hat{K}(\omega)]\hat{\Gamma}(\omega)^h \hat{\mathfrak{x}}_0(\omega)\, d\omega. \end{aligned}$$

Since $\hat{\Gamma}(\omega)$ is asymptotically stable for every ω in $[0, 2\pi]$, it is easy to check that the series

$$\sum_{h=0}^{\infty} \hat{\Gamma}^*(\omega)^h [Q + \hat{K}^*(\omega)R\hat{K}(\omega)]\hat{\Gamma}(\omega)^h$$

converges pointwise for every ω to the unique (positive-definite) solution of (6.3).

As a consequence of the above lemma, we have

$$J_N - J_{\min} = (2\pi)^{-1} \int_0^{2\pi} \hat{\mathfrak{x}}_0^*(\omega)[\hat{P}_N(\omega) - \hat{P}(\omega)]\hat{\mathfrak{x}}_0(\omega)\, d\omega. \tag{6.4}$$

Here $\hat{P}(\omega)$ is both the stabilizing solution of (AREω) and the solution of the Lyapunov equation (6.3) that includes the optimal feedback law $\hat{K}(\omega)$ and the corresponding closed-loop system matrix $\hat{\Gamma}(\omega)$. $\hat{P}_N(\omega)$ is the solution of a Lyapunov equation that includes the truncation $\hat{K}_N(\omega)$ of the optimal feedback law and the corresponding closed-loop system matrix $\hat{\Gamma}_N(\omega)$.

The matrix solution $\hat{P}(\omega)$ of (6.3), associated with the optimal feedback law, is unique and its elements $\hat{p}_{ij}(\omega)$ continuously depend on the elements of $\hat{\Gamma}(\omega)$ and $\hat{K}(\omega)$. Therefore the uniform convergence of $\hat{K}_N(\omega)$ and $\hat{\Gamma}_N(\omega)$ to $\hat{K}(\omega)$ and $\hat{\Gamma}(\omega)$ implies that $\hat{P}_N(\omega)$ uniformly converges to $\hat{P}(\omega)$. Using (6.4) we conclude that J_N converges to $J_{\min}$. □

The stabilizability condition we referred to in this paper

$$[I - A_1z_1 - A_2z_2 \quad B_1z_1 + B_2z_2] \quad \text{full rank in } \mathcal{M} \tag{6.5}$$

is weaker than the condition

$$[I - A_1z_1 - A_2z_2 \quad B_1z_1 + B_2z_2] \quad \text{full rank in } P_1, \tag{6.6}$$

which is necessary and sufficient [1] for the existence of a stabilizing state feedback law that preserves the quarter plane causality of the closed-loop system.

Clearly, in the case where (6.5) holds and (6.6) does not, losing quarter plane causality is the price we pay for achieving the closed-loop stabilization.

Although condition (6.5) is in general weaker than (6.6), if we assume $B_1 = B_2 = 0$ both conditions collapse. Actually, in this case neither causal nor noncausal feedback can stabilize the system, unless it is originally stable.

Appendix.

Proof of Theorem 3. Let $P = [p_{ij}]$ belong to $C^{n\times n}$ and introduce the map $f: C \times C^{n\times n} \to C^{n\times n}$ given by

$$\begin{aligned} f(z, P) = P - Q - (A_1^T + A_2^T z^{-1})P(A_1 + A_2 z) \\ + (A_1^T + A_2^T z^{-1})P(B_1 + B_2 z)[R + (B_1^T + B_2^T z^{-1})P(B_1 + B_2 z)]^{-1} \\ \times (B_1^T + B_2^T z^{-1})P(A_1 + A_2 z). \end{aligned} \tag{A1}$$

We therefore have that the problem of obtaining the solutions of (2.12) reduces to that of solving, with respect to the matrix variable P, the implicit equation

$$f(z, P) = 0. \tag{A2}$$

The proof will break up into two parts. The first is devoted to a local solution of the implicit equation on the neighbourhood of an arbitrary point of the unit circle. It will be shown that, given $e^{j\bar{\omega}}$, there exists a unique analytic matrix $P_{\bar{\omega}}(\cdot)$, defined on an open disk centered in $e^{j\bar{\omega}}$, that solves (A2) and satisfies the condition

$$P_{\bar{\omega}}(e^{j\omega}) = \hat{P}(\omega).$$

The second part is concerned with the existence of a global solution of (A2). An analytic continuation $P(z)$ of the local solution will be provided on an open neighbourhood of γ_1 in such a way that the condition

$$P(e^{j\omega}) = \hat{P}(\omega)$$

holds for any $\omega \in [0, 2\pi]$.

As far as the local solution of (A2) is concerned, the definition of $\hat{P}(\omega)$ implies $f(e^{i\bar{\omega}}, \hat{P}(\bar{\omega}))=0$, so that, to apply the Implicit Function Theorem, we must check that the Jacobian matrix of f with respect to the variables p_{ij} is nonsingular at $(e^{j\bar{\omega}}, \hat{P}(\bar{\omega}))$. Assume that the entries of P and the components of $f(z, P)$ have been lexicographically ordered, so that equation (A2) takes the following form;

(A3)
$$\begin{aligned} &f_{11}(z, p_{11}, p_{12}, \cdots, p_{1n}, p_{21}, \cdots, p_{2n}, \cdots, p_{nn})=0,\\ &f_{12}(z, p_{11}, p_{12}, \cdots, p_{1n}, p_{21}, \cdots, p_{2n}, \cdots, p_{nn})=0,\\ &\cdots\\ &f_{nn}(z, p_{11}, p_{12}, \cdots, p_{1n}, p_{21}, \cdots, p_{2n}, \cdots, p_{nn})=0. \end{aligned}$$

Letting

$$\begin{aligned} \Gamma(z, P)=(A_1+A_2z)-(B_1+B_2z)[R+(B_1^T+B_2^Tz^{-1})P(B_1+B_2z)]^{-1}\\ \times(B_1^T+B_2^Tz^{-1})P(A_1+A_2z), \end{aligned}$$

some elementary algebraic manipulations on (A1) yield the (i, j)th indexed columns of the Jacobian matrix

(A4)
$$\frac{\partial f}{\partial p_{ij}}=\frac{\partial P}{\partial p_{ij}}-\Gamma^T(z^{-1}, P^T)\left(\frac{\partial P}{\partial p_{ij}}\right)\Gamma(z, P).$$

In particular, if (A4) is evaluated at $(e^{j\bar{\omega}}, \hat{P}(\bar{\omega}))$, we obtain

(A5)
$$\frac{\partial f}{\partial p_{ij}}=e_ie_j^T-\hat{\Gamma}^*(\bar{\omega})e_ie_j^T\hat{\Gamma}(\bar{\omega})$$

where e_i denotes the ith column of the $n\times n$ identity matrix and $\hat{\Gamma}(\omega)$ has been defined in (4.12).

Thus, for $i, j, r, s=1, 2, \ldots, n$, the entries of the Jacobian matrix are

$$\frac{\partial f_{rs}}{\partial p_{ij}}=\delta_{(r,s)(i,j)}-\hat{\Gamma}^V_{i,r}(\bar{\omega})\hat{\Gamma}_{j,s}(\bar{\omega})$$

and its (r, s)th row can be expressed as

$$e_r^T\otimes e_s^T-(e_r^T\hat{\Gamma}^*(\bar{\omega}))\otimes(e_s^T\hat{\Gamma}^T(\bar{\omega}))=(e_r^T\otimes e_s^T)[I-\hat{\Gamma}^*(\bar{\omega})\otimes\hat{\Gamma}^T(\bar{\omega})].$$

This shows that the Jacobian matrix of f with respect to P is given by

(A6)
$$I-\hat{\Gamma}^*(\bar{\omega})\otimes\hat{\Gamma}^T(\bar{\omega})$$

and is a nonsingular matrix because of the asymptotic stability of $\hat{\Gamma}(\bar{\omega})$.

Before beginning with the "global part" of the proof, we need to investigate some properties of the local solution. Since the closed-loop matrix

$$\hat{\Gamma}(\bar{\omega})=\Gamma(e^{j\bar{\omega}}, P_{\bar{\omega}}(e^{j\bar{\omega}}))$$

is asymptotically stable, by a continuity argument $P_{\bar{\omega}}(e^{j\omega})$ is a stabilizing solution of (AREω) for any $e^{j\omega}$ in a suitable open arc $\alpha(\bar{\omega})$ of γ_1 centered in $e^{j\bar{\omega}}$.

Thus both $P_{\bar{\omega}}(e^{j\omega})$ and $\hat{P}(\omega)$ provide a stabilizing solution of (AREω) in $\alpha(\bar{\omega})$ and, by the uniqueness of the Hermitian stabilizing solution of (AREω), it suffices to prove that $P_{\bar{\omega}}(e^{j\omega})$, $e^{j\omega}$ in $\alpha(\bar{\omega})$, is an Hermitian matrix for concluding that

(A7)
$$\hat{P}(\omega)=P_{\bar{\omega}}(e^{j\omega})\quad\forall\, e^{j\omega}\in\alpha(\bar{\omega}).$$

Actually, taking the conjugate transpose of the identity $f(e^{j\omega}, P_{\bar{\omega}}(e^{j\omega}))=0$, $e^{j\omega}$ in $\alpha(\bar{\omega})$, we obtain $f(e^{j\omega}, P^*_{\bar{\omega}}(e^{j\omega}))=0$, $e^{j\omega}\in\alpha(\bar{\omega})$. On the other hand, we have

$$P^*_{\bar{\omega}}(e^{j\bar{\omega}})=\hat{P}^*(\bar{\omega})=\hat{P}(\bar{\omega})=P_{\bar{\omega}}(e^{j\bar{\omega}})$$

so that the uniqueness of the solution of (A2) in a neighbourhood of $(e^{j\bar{\omega}}, \hat{P}(\bar{\omega}))$ implies

$$P_{\bar{\omega}}(e^{j\omega})=P^*_{\bar{\omega}}(e^{j\omega}), \qquad e^{j\omega}\in\alpha(\bar{\omega})\cap D(\bar{\omega})$$

where $D(\bar{\omega})$ is a suitable open disk centered at $e^{j\bar{\omega}}$.

This last result has really been our main goal. We use it to associate with each point $e^{j\bar{\omega}}\in\gamma_1$ an open disk $D(\bar{\omega})$, centered in $e^{j\bar{\omega}}$ and an analytic function $P_{\bar{\omega}}(z)$, defined in $D(\bar{\omega})$ and satisfying $P_{\bar{\omega}}(e^{j\omega})=\hat{P}(\omega)$ on $\gamma_1\cap D(\bar{\omega})$.

Extracting from the infinite open covering $\{D(\omega)\}_{\omega}\in[0,2\pi]$ a finite subcovering of γ_1 and piecing together all functions that correspond to it, we obtain a function $P(z)$ that is analytical in an open annulus including γ_1 and that fulfills the condition

$$P(e^{j\omega})=\hat{P}(\omega) \quad \forall\omega\in[0,2\pi]. \tag{A8}$$

□

THEOREM 5. *Let $\hat{A}(\cdot):[0,2\pi]\to C^{n\times n}$ be a continuous function and assume that $\hat{A}(\omega)$ is asymptotically stable for any ω in $[0,2\pi]$. Then there exist $M>0$ and $\lambda\in(0,1)$ such that*

$$\|\hat{A}^t(\omega)\|_2\leqq M\lambda^t.$$

Proof. By the continuity assumption, there exists a real $\delta>0$ such that $\hat{A}(\omega)(1+\delta)$ is asymptotically stable for every ω.

Then the Lyapunov equation

$$\hat{P}(\omega)=I+\hat{A}^*(\omega)\hat{P}(\omega)\hat{A}(\omega)(1+\delta)^2$$

admits a unique positive-definite solution

$$\hat{P}(\omega)=\sum_{t=0}^{+\infty}\hat{A}^*(\omega)^t\hat{A}(\omega)^t(1+\delta)^{2t},$$

which is continuous in $[0,2\pi]$.

Let M^2 denote the maximum spectral radius of $\hat{P}(\omega)$ in $[0,2\pi]$ and $\lambda:=(1+\delta)^{-1}$. Then, for every v in C^n we have

$$v^*M^2v\geqq v^*\hat{P}(\omega)v=\sum_{t=0}^{\infty}\|(1+\delta)^t\hat{A}(\omega)^tv\|_2^2\geqq\lambda^{-2t}\|\hat{A}(\omega)^tv\|_2^2,$$

which proves our assertion. □

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