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RESIDUAL GENERATION AND FAULT DETECTION IN 2D FILTERS

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ABSTRACT The parity checks of a 2D system are shown to constitute a free module over the ring of polynomials in two variables, whose generators are easily computed from a suitable MFD of the system. Given a specific parity check, an efficient realization technique is presented, that provides a state space model of the corresponding residual generator. Finally, some connections with 2D observers theory are discussed.

1. Introduction

Over the past decade several contributions to the problem of failure detection have been presented in the literature [1-4].

All failure detection methods considered so far use redundancy among the measured variables of the plant. Redundancy relations fall in two classes: *direct redundancy* exploits the relationships among instantaneous outputs of sensors, while *temporal redundancy* takes advantage of the relationships among the histories of sensor outputs and actuator inputs. In both cases, the signal generated by the detection process - the *residual* - depends on the difference between the measured and expected values of the plant output. In the absence of a failure, a zero residual should testify the normal behaviour of the plant.

This paper deals with the solution of the failure detection problem for 2D systems. 2D systems constitute a relatively recent area of research, and the results concerning the 2D failure detection problem are quite scarce in the literature [9]. The input and output signals that are processed in 2D failure detection are defined on the discrete plane $\mathbf{Z} \times \mathbf{Z}$ or, more frequently, on a suitable half-plane of $\mathbf{Z} \times \mathbf{Z}$. Moreover, since quarter plane causality is assumed, the output value at (i,j) only depends on the input values and initial conditions of the system on the set $\{(h,k) : h \leq i, k \leq j\}$.

Failure detection based on direct redundancy only keeps into account the outputs of the sensors at the single point (i,j). In this case the causal structure of the system is not relevant and the detection problem can be tackled along the same lines as in the 1D case.

Viceversa, when considering temporal redundancy, the difference between the causality structures calls for a specific treatment of the 2D case. As one can expect, because of the shift invariance property the residual generation process is naturally represented by doubly indexed MA models. Consequently 2D residual generators can be implemented by 2D systems that realize MA models .

The paper is organized as follows. In section 2 we analyze the structure of the redundancy relations that underlie 2D parity cheks. This leads to a representation of parity checks as elements of a free module over the ring of polynomials in two variables, whose structure is completely specified by a finite set of generators computed from the matrix fraction description (MFD) of the system. In section 3 we assume that a specific parity check has been given and we present an efficient state space realization procedure of the corresponding residual generator. In the last section some connections between 2D observers theory and the residual generator structure are investigated.

2. 2D parity relations

Consider a 2D system (*plant*), represented by the state model [6]

$$\begin{aligned} \mathbf{x}(h+1,k+1) =& A_1\mathbf{x}(h,k+1) + A_2\mathbf{x}(h+1,k) \\ &+ B_1\mathbf{u}(h,k+1) + B_2\mathbf{u}(h+1,k) \\ \mathbf{y}(h,k) =& C\mathbf{x}(h,k) + D\mathbf{u}(h,k) \end{aligned} \quad (1)$$

where $\mathbf{x}$ is an n-dimensional *local state* vector, $\mathbf{u}$ is an m-dimensional vector of known inputs, $\mathbf{y}$ is a p-dimensional vector of measured outputs and A_1, A_2, B_1, B_2, C, D are matrices of appropriate dimensions. Assume further that C is full rank, which rules out direct redundancy among the instantaneous values of the sensors.

The *transfer matrix* of (1) is given by

$$W(z_1,z_2) = C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) + D \quad (2)$$

A *parity relation* is a linear combination of a finite *window* of present and lagged values of $\mathbf{u}$ and $\mathbf{y}$, that is identically zero for any location of the data window in the discrete plane if no failures occur in (1). Therefore the parity criterion is invariant with respect to two-dimensional shifts and hence is associated with a 2D moving average model.

Let us first assume that the plant undergoes a free state evolution starting from an initial *global state*

$$\mathcal{X}_0 = \sum_{i \in \mathbf{Z}} \mathbf{x}(i,-i)z_1^i z_2^{-i}$$

Denote by

$$Y(z_1,z_2) = \sum_{i+j \geq 0} \mathbf{y}(i,j)z_1^i z_2^j = C(I - A_1z_1 - A_2z_2)^{-1}\mathcal{X}_0 \quad (3)$$

the formal power series associated with the output values in the half plane $\{(i,j) : i+j \geq 0\}$ and, for any (i,j) and $\nu \geq 0$, introduce the $(1+2+\ldots+(\nu+1))p$-dimensional vector

$$\mathbf{y}_\nu(i,j) = \left[\mathbf{y}^T(i-\nu,j)\ \mathbf{y}^T(i-\nu+1,j-1)\ \ldots \right.$$
$$\left. \mathbf{y}^T(i,j-\nu)\ldots\mathbf{y}^T(i-1,j)\ \mathbf{y}^T(i,j-1)\ \mathbf{y}^T(i,j)\right]^T \tag{4}$$

Clearly $\mathbf{y}_\nu(i,j)$ represents the output data contained in a ν-th order triangular window with vertices $(i,j),(i-\nu,j)$ and $(i,j-\nu)$.

Let

$$q^T(z_1,z_2) = [\,q_1(z_1,z_2)\quad q_2(z_1,z_2)\quad \ldots q_p(z_1,z_2)\,]$$

be a polynomial row vector and let $Q_0+Q_1\ldots+Q_s$ be its representation as sum of homogeneous terms. Assume now that

$$p^T(z_1,z_2) = q^T(z_1,z_2)C(I-A_1z_1-A_2z_2)^{-1} \tag{5}$$

be a polynomial matrix of degree $\nu-1$, so that the degree of $q^T(z_1,z_2)C$ and, by the rank assumption on C, the degree s of $q^T(z_1,z_2)$ cannot exceed ν.

Since the degree of the nonzero homogeneous terms of $q^T(z_1,z_2)Y(z_1,z_2)$ is less than or equal to $\nu-1$, we have

$$Q_sY_{\nu-s+k}+\ldots Q_1Y_{\nu-1+k}+Q_0Y_{\nu+k}=0$$

for any $k\geq 0$. Denoting by $\mathbf{q}_{ij}^T$ the coefficients of $z_1^iz_2^j$ in Q_{i+j}, the $(1+2+\ldots+(\nu+1))p$-dimensional real vector

$$\mathbf{v} = \left[0\ \ldots\ 0 \mid \mathbf{q}_{s0}^T\ \mathbf{q}_{s-1,1}^T\cdots\mathbf{q}_{0s}^T \mid \ldots \mathbf{q}_{10}^T\ \mathbf{q}_{01}^T \mid \mathbf{q}_{00}^T\right]^T \tag{6}$$

satisfies

$$\mathbf{v}^T\mathbf{y}_\nu(i,\nu+k-i)=0, \tag{7}$$

i.e. is orthogonal to the free output vector $\mathbf{y}_\nu(i,\nu+k-i)$ for any $k\geq 0$ and for any i. Clearly $\mathbf{v}^T$ provides a parity check on the free outputs space, in the sense that, if the product (7) is different from 0, a failure has occurred in the plant.

The converse is also true; that is, given a $(1+2+\ldots+(\nu+1))p$-dimensional vector $\mathbf{v}$, orthogonal to the free output vectors $\mathbf{y}_\nu(i,\nu+k-i)$ for any $k\geq 0$ and any i, its entries can be viewed as the coefficients of a polynomial vector $q^T(z_1,z_2)\in\mathbf{R}[z_1,z_2]^p$ and $q^T(z_1,z_2)C(I-A_1z_1-A_2z_2)^{-1}$ is a polynomial matrix.

So doing, we have obtained a complete characterization of the parity checks which can be performed on the output data contained in a ν-th order triangular window. As ν varies, the above polynomial vectors $q^T(z_1,z_2)$ constitute a free module of $\mathbf{R}[z_1,z_2]^p$ which can be characterized starting from a l.c.MFD $M^{-1}(z_1,z_2)N(z_1,z_2)$ of $C(I-A_1z_1-A_2z_2)^{-1}$. The parity checks consist of the polynomial row vectors $q^T(z_1,z_2)$ which make

$$q^T(z_1,z_2)M^{-1}(z_1,z_2)N(z_1,z_2)$$

to be polynomial. Since $M^{-1}N$ is coprime, by Lemma 5.3 in [7] this is equivalent to the requirement that

$$q^T(z_1,z_2)M^{-1}(z_1,z_2)$$

is polynomial, i.e. that $q^T(z_1,z_2)$ belongs to the free module S generated by the rows of $M(z_1,z_2)$.

The parity checks previously introduced apply also when the input of the plant is different from zero. This of course requires that the free output evolution should have been previously reconstructed from the actual input and output functions. In this case the formal power series that represents the free output is given by

$$\begin{bmatrix} I & -W(z_1,z_2)\end{bmatrix}\begin{bmatrix}Y(z_1,z_2)\\ U(z_1,z_2)\end{bmatrix} \tag{8}$$

with $U(z_1,z_2)=\sum_{i+j\geq 0}\mathbf{u}(i,j)z_1^iz_2^j$.

If $q^T(z_1,z_2)$ is any row polynomial vector in S, the coefficients $\mathbf{r}(i,j)$ of the series resulting from the discrete convolution

$$q^T(z_1,z_2)\begin{bmatrix} I & -W(z_1,z_2)\end{bmatrix}\begin{bmatrix}Y(z_1,z_2)\\ U(z_1,z_2)\end{bmatrix} \tag{9}$$

are zero whenever $i+j\geq\nu$, for some positive integer ν. So, the above convolution represents a residual generation process, in the sense that $\mathbf{r}(i,j)\neq 0$ for $i+j\geq\nu$ indicates that some failure occurred in the system.

3. Realization of 2D residual generators

The aim of this section is to implement the residual generation process, introduced at the end of section 2, by means of a 2D dynamical system driven by the inputs and the outputs of the plant.

Let $q^T(z_1,z_2)$ be a parity check for (2), so that the matrix (5) and, consequently, $q^T(z_1,z_2)W(z_1,z_2)$ are polynomial. The application of the parity check to the formal power series (8) representing the free output evolution reduces to apply the row vector

$$g^T(z_1,z_2)=q^T(z_1,z_2)\begin{bmatrix} I & -W(z_1,z_2)\end{bmatrix} \tag{10}$$

to the output and input data vector $\begin{bmatrix}Y(z_1,z_2)\\ U(z_1,z_2)\end{bmatrix}$. So the residual $\mathbf{r}(h,k)$ can be viewed as the output of a 2D system $\Sigma_g=(F_1,F_2,G_1,G_2,H,J)$:

$$\begin{aligned}\mathbf{x}'(h+1,k+1) =& F_1\mathbf{x}'(h,k+1)+F_2\mathbf{x}'(h+1,k)\\ &+G_1\begin{bmatrix}\mathbf{y}(h,k+1)\\ \mathbf{u}(h,k+1)\end{bmatrix}+G_2\begin{bmatrix}\mathbf{y}(h+1,k)\\ \mathbf{u}(h+1,k)\end{bmatrix}\\ \mathbf{r}(h,k) =& H\mathbf{x}'(h,k)+J\begin{bmatrix}\mathbf{y}(h,k)\\ \mathbf{u}(h,k)\end{bmatrix}\end{aligned} \tag{11}$$

driven by $\mathbf{y}(h,k)$ and $\mathbf{u}(h,k)$ and realizing the polynomial vector $g^T(z_1,z_2)$.

Actually the residual $\mathbf{r}(h,k)$ generated by Σ_g is the sum of a forced term, that provides the expected parity check on the pair $\mathbf{y}(h,k)$ and $\mathbf{u}(h,k)$, and a second term, that depends on the initial conditions of Σ_g, which are in general unknown. However, since $g^T(z_1,z_2)$ is a polynomial vector, we can assume that the matrices F_1 and F_2 satisfy the condition $\det(I-F_1z_1-F_2z_2)=1$. In this way Σ_g is a finite memory dynamical system [8] and the (undesired) second term vanishes in a finite number of steps.

Assuming that the parity check $q^T(z_1,z_2)$ is applied to output data belonging to the half plane $\{(i,j) : i+j \geq 0\}$, the degree hypothesis on $p^T(z_1,z_2)$ implies that data processing should be extended at least up to the terms appearing on the ν-th diagonal $\{(i,j) : i+j = \nu\}$. Hence the parity check is reliable from the ν-th diagonal onwards. We aim to prove that Σ_g can be realized in such a way that the transient of $\mathbf{r}(h,k)$ due to nonzero initial conditions on Σ_g vanishes on the ν-th diagonal. This shows that the existence of a nonzero initial global state $\mathcal{X}'_0$ for Σ_g does not impair the performance of the residual generator.

Consider preliminarly a polynomial transfer matrix $L(z_1,z_2)$ of degree $\nu > 0$. Whatever realization we refer to, a pulse in $(0,0)$ gives rise to a nonzero output and hence to nonzero local states on the ν-th diagonal. So, bearing in mind that the state updating equation introduces a single step delay between inputs and states, there exist values of $\mathbf{x}'(0,0)$ leading to nonzero local states on the $(\nu-1)$th diagonal. The following lemma states that there exist realizations of $L(z_1,z_2)$ whose free state evolution is zero on the diagonals with indices greater than $\nu-1$. So, by the above argument, these realizations exhibit a minimum lenght dynamical memory.

LEMMA [9] *The polynomial transfer matrix*

$$L(z_1,z_2) = \sum_{i+j\leq\nu} L_{ij} z_1^i z_2^j, \quad L_{ij} \in \mathbf{R}^{p\times m}, \quad \nu > 0$$

can be realized by a 2D system $\Sigma_g = (F_1,F_2,G_1,G_2,\ H,J)$ *whose free state evolution*

$$X'(z_1,z_2) = \sum_{h+k\geq 0} \mathbf{x}'(h,k) z_1^h z_2^k = (I - F_1z_1 - F_2z_2)^{-1}\mathcal{X}'_0$$

satisfies the condition $\mathbf{x}'(h,k) = 0$ *when* $h+k \geq \nu$.

We are now in a position to prove the main result of this section.

THEOREM 1 *Let* $q^T(z_1,z_2) \in \mathcal{S}$ *and assume that* (5) *is a polynomial row vector with degree* $\nu - 1$. *Then the parity check associated with* $q^T(z_1,z_2)$ *can be implemented by a residual generator* Σ_g *whose unforced motion* $\mathbf{x}'(h,k)$ *vanishes for* $h+k \geq \nu$.

PROOF The row vector $g^T(z_1,z_2)$ given in (10) is the transfer matrix of the residual generator. By the lemma above, there exists a realization Σ_g of $g^T(z_1,z_2)$ having a free state evolution which satisfies $\mathbf{x}'(h,k) = 0$ for $h+k \geq \deg g^T(z_1,z_2)$. So we are reduced to prove that $\deg g^T(z_1,z_2) \leq \nu$.

By (10), the degree of the polynomial matrix $g^T(z_1,z_2)$ is the maximum between $\deg q^T(z_1,z_2)$ and $\deg q^T(z_1,z_2)\cdot$ $\cdot[C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) + D]$.

Now the assumption $\deg p^T(z_1,z_2) = \nu - 1$ implies

$$\deg q^T(z_1,z_2)C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) \leq \nu$$

Furthermore, by the full rank assumption on C, we have

$$q^T(z_1,z_2) = p^T(z_1,z_2)(I - A_1z_1 - A_2z_2)C^T(CC^T)^{-1}$$

which gives $\deg q^T(z_1,z_2) \leq \nu$. Therefore $\deg g^T(z_1,z_2)$ is less than or equal to ν. $\nabla\nabla$

As a consequence of the theorem above, the dynamical system Σ_g constitutes the best residual generator we can expect when implementing the parity check associated with $q^T(z_1,z_2)$. In fact the free evolution of Σ_g vanishes on the diagonals $\mathcal{C}_i = \{(h,k) : h+k = i\}$, for all $i \geq \nu$. On the other hand we process output and input values of the plant Σ that are located on the diagonals $\mathcal{C}_i$ for all $i \geq 0$. Since the parity check utilizes a data set that belongs to $\nu+1$ consecutive diagonals, the output values of Σ_g on $\mathcal{C}_\nu$ constitute the first set of residuals which are significant for the parity check.

4. Connections with 2D observers theory

A residual generator can be regarded as a dynamic device that, when connected to the plant inputs and outputs, generates a signal that goes to zero in a finite number of steps if the plant operation is correct. Within this context, it is apparent that the philosophy underlying the realization of 2D residual generators should be closely connected to 2D observers theory, as presented in [5].

When 2D dead-beat observers are considered, the connection is provided by the *innovation* signal, i.e. the difference between the plant output and the estimated output $C\hat{X}(z_1,z_2) + DU(z_1,z_1)$, which constitutes itself a parity check. To see this, note that the existence of a dead-beat observer is equivalent to the possibility of solving the Bézout equation

$$Q(z_1,z_2)(I - A_1z_1 - A_2z_2) + P(z_1,z_2)C = I_n \qquad (12)$$

over the polynomial ring $\mathbf{R}[z_1,z_2]$. When a polynomial solution (P,Q) is available, we construct a 2D transfer matrix

$$\hat{W}(z_1,z_2) = \begin{bmatrix} P & Q(B_1z_1 + B_2z_2) - PD \end{bmatrix} \qquad (13)$$

and any finite memory realization $\hat{\Sigma} = (\hat{A}_1,\hat{A}_2,\hat{B}_1,\hat{B}_2,\hat{C},\hat{D})$ of $\hat{W}$ is a dead-beat observer of Σ. So, assuming $\mathcal{X}_0$ as the initial global state of $\hat{\Sigma}$, the innovation

$$\begin{aligned} Y(z_1,z_2) &- C\hat{X}(z_1,z_2) - DU(z_1,z_2) \\ &= CQ\mathcal{X}_0 - C(I - \hat{A}_1z_1 - \hat{A}_2z_2)^{-1}\hat{\mathcal{X}}_0 \end{aligned} \qquad (14)$$

goes to zero in a finite number of steps. Clearly the 2D system $\Sigma_g = (\hat{A}_1,\hat{A}_2,\hat{B}_1,\hat{B}_2,C\hat{C},C\hat{D} - [I \quad -D])$ obtained by modifying the readout equation of the observer, is a residual generator.

A deeper insight into the problem of connecting 2D observers and residual generators via Bézout equation can be gained if a modified transfer matrix is introduced before proceeding to a finite memory realization. In fact, consider the transfer matrix between the plant and the innovation signals

$$\begin{aligned} C\hat{W} &- [I \quad -D] \\ &= [CP - I \quad CQ(B_1z_1 + B_2z_2) - CPD + D] \end{aligned} \qquad (15)$$

Clearly, to guarantee that (15) is polynomial we need to obtain a solution (P,Q) of (12), such that both CP and CQ are polynomial matrices. This shows that the polyno-

mial solutions of (12), which give the transfer matrices of dead beat observers, provide only a special class of residual generators. In fact, using suitable solutions of (12), residual generators can be realized even in cases when the Bézout equation has no solutions on the ring of stable rational functions, i.e. when Σ does not admit asymptotic observers. In the remaining part of this section we shall prove that
i) solutions (P, Q) of (12) always exist such that CP and CQ are polynomial
ii) a suitable solution of this kind provides the parity checks generator M introduced in Sec.2.

In the following, $\mathcal{A}$ will be a shorthand notation for $A_1z_1 + A_2z_2$ and, modulo a coordinate change in the local state space, C will be assumed as $[\,I_p \quad 0\,]$. By partitioning Q, P and $\mathcal{A}$ conformably, eqn.(12) becomes

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I - \mathcal{A}_{11} & -\mathcal{A}_{12} \\ -\mathcal{A}_{21} & I - \mathcal{A}_{22} \end{bmatrix} + \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} [\,I \quad 0\,] = I_n \quad (16)$$

and consequently CQ and CP will be polynomial if Q_{11}, Q_{12} and P_1 are. It is clear from the structure of (16) that the triples (Q_{21}, Q_{22}, P_2) and (Q_{11}, Q_{12}, P_1) can be determined independently each other and that the equations of the first triple are always solvable on the field of rational functions.

Obtaining the second triple requires to solve the system

$$Q_{11}(I - \mathcal{A}_{11}) - Q_{12}\mathcal{A}_{21} + P_1 = I \quad (17)$$

$$-Q_{11}\mathcal{A}_{12} + Q_{12}(I - \mathcal{A}_{22}) = 0 \quad (18)$$

over the ring $\mathbb{R}[z_1, z_2]$. The results are summarized in the following theorem

THEOREM 2 *Let* $Q_{11}^{-1}Q_{12}$ *be a left coprime MFD of* $\mathcal{A}_{12}(I - \mathcal{A}_{22})^{-1}$*, such that* $Q_{11}(0,0) = I$ *and let* $P_1 := I - Q_{11}(I - \mathcal{A}_{11}) - Q_{12}\mathcal{A}_{21}$. *Then*
i) Q_{11}, Q_{12}, P_1 *solve* (17) *and* (18)
ii) the rows of the matrix $M := I - P_1$ *generate the* $\mathbb{R}[z_1, z_2]$*-module of parity checks.*

PROOF First of all, notice that every l.c. MFD $Q_{11}^{-1}Q_{12}$ of $\mathcal{A}_{12}(I - \mathcal{A}_{22})^{-1}$ satisfies $\det Q_{11}(z_1, z_2) \mid \det(I - \mathcal{A}_{11})$ and hence $\det Q_{11}(0,0) \neq 0$. Premultiplying $Q_{11}(z_1, z_2)$ and $Q_{12}(z_1, z_2)$ by $Q_{11}(0,0)^{-1}$ gives a MFD that fulfills the hypothesis of the theorem. Moreover Q_{11}, Q_{12} and P_1 solve (17) and (18).

To prove ii), define $N := [\,Q_{11} \quad Q_{12}\,]$. Then $M^{-1}N$ is a left coprime MFD of $C(I - \mathcal{A})^{-1}$. In fact $M(0,0) = I - P_1(0,0) = I$, so that $M(z_1, z_2)$ is invertible , and

$$\begin{aligned} MC - N(I - \mathcal{A}) & \\ = [\,I - P_1 - Q_{11}(I - \mathcal{A}_{11}) - Q_{12}\mathcal{A}_{21} & \\ Q_{11}\mathcal{A}_{12} - Q_{12}(I - \mathcal{A}_{22})\,] &= 0 \end{aligned}$$

Finally, any left common factor of M and N is also a left common factor of Q_{11} and Q_{12}, which implies that M and N are left coprime. $\nabla\nabla$

References

1. E.Y.Chow and A.S.Willsky *Analytical redundancy and the design of robust failure detection systems* IEEE Trans. Aut. Contr., AC-29, 7, 603-14, 1984

2. X.C.Lou, A.S.Willsky and G.C.Verghese *Optimally robust redundancy relations for failure detection in uncertain systems* Automatica, 22, 3, 333-44, 1986

3. M.A.Massoumnia *A geometric approach to the synthesis of failure detection filters* IEEE Trans. Aut. Contr., AC-31, 9, 839-46, 1986

4. R.Isermann *Process fault detection based on modeling and estimation methods. A survey* Automatica, 20, 4, 387-404, 1984

5. M.Bisiacco *On the structure of 2D observers* IEEE Trans. Aut. Contr. ,AC-31, 7, 676-80, 1986

6. E.Fornasini and G.Marchesini *Doubly-indexed dynamical systems: state space models and structural properties* Math. Sys. Th., 12, 59-72, 1978

7. S.Kung, B.Lévy, M.Morf and T.Kailath *New results in 2D systems theory. Part 1* Proc IEEE, 65, 6, 861-72, 1977

8. M.Bisiacco *State and output feedback stabilizability of 2D systems* IEEE Trans. Circ. Sys., CAS-32, 12, 1246-54, 1985

9. E.Fornasini and M.Marchesini *Failure detection in 2D systems* CTAT, 4, 4,381-93, 1988