ALGEBRAIC ASPECTS OF 2D SINGULAR SYSTEMS

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Abstract The paper investigates the behaviour $\mathcal{B}$ of a singular 2D system on a half plane. Some connections between the matrices appearing in the updating equations and the restrictions of $\mathcal{B}$ to the separation sets are presented.

1 Introduction

Consider a 2D system given by the following equation

$$\overline{E}\overline{x}(h+1,k+1) = \overline{A}\overline{x}(h,k+1) + \overline{B}\overline{x}(h+1,k) \tag{1}$$

where $\overline{x} \in \mathbf{R}^n$ and $\overline{E}, \overline{A}, \overline{B}$ are $q \times n$ matrices with entries in $\mathbf{R}$.

Clearly, if rank$\overline{E} = n$, (1) can be reduced to the equation of an unforced nonsingular 2D system [1], as follows

$$\begin{aligned} \overline{x}(h+1,k+1) &= (\overline{E}^T\overline{E})^{-1}\overline{E}^T\overline{A}\overline{x}(h,k+1) \\ &\quad +(\overline{E}^T\overline{E})^{-1}\overline{E}^T\overline{B}\overline{x}(h+1,k) \end{aligned} \tag{2}$$

If rank$\overline{E} = r < n$, we are allowed to introduce two nonsingular matrices $Q \in \mathbf{R}^{q\times q}$ and $N \in \mathbf{R}^{n\times n}$, such that

$$Q\overline{E}N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

So, letting

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N^{-1}\overline{x},\ Q\overline{A}N = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ Q\overline{B}N = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{4}$$

equation (1) can be rewritten as follows

$$\begin{aligned} x_1(h+1,k+1) &= A_{11}x_1(h,k+1) + B_{11}x_1(h+1,k) \\ &\quad + A_{12}x_2(h,k+1) + B_{12}x_2(h+1,k) \\ 0 &= A_{21}x_1(h,k+1) + B_{21}x_1(h+1,k) \\ &\quad + A_{22}x_2(h,k+1) + B_{22}x_2(h+1,k) \end{aligned} \tag{5}$$

In the particular case when $A_{21}, B_{21}, A_{22}, B_{22}$ are simultaneously zero, x_2 can be viewed as an $n-r$ dimensional input and (5) provides the state updating equation of a nonsingular 2D system. More generally, however, x_2 is the direct sum of exhogenous variables (i.e. inputs), and auxiliary variables that induce some dynamical constraints on the system trajectories, and (5) can be considered a singular 2D system, as studied in [2].

This paper constitutes a preliminary report on a research still in progress, concerning the analytical structure of the trajectories of system (5) in the half plane $\mathcal{H} = \{(h,k) : h+k \geq 0\}$. No "a priori" assumption is made on which components of x_2 can be given the role of exhogenous variables. Following the philosophy that underlies the behavioural approach by J. Willems and P. Rocha [3-5], the nature of the input functions is determined "a posteriori", after establishing what variables are constrained by equations (5).

2 An algebraic approach via duality

All signals x that will be considered in this paper are sequences indexed on the half plane $\mathcal{H}$ and taking values in some finite dimensional $\mathbf{R}$-vector space $x : \mathcal{H} \longrightarrow \mathbf{R}^n : (h,k) \longmapsto x(h,k)$. The single step updating structure (5) makes it convenient to introduce a partition of $\mathcal{H}$ into a countable family of separation sets $\mathcal{S}^i = \{(h,k) : h+k = i\}$, $i = 0,1,\dots$ and to associate with x a formal power series

$$\mathcal{X} = \sum_{i=0}^{+\infty}\sum_{j=-\infty}^{+\infty} x(i+j,i-j)\xi^{-j}\lambda^{-i} \tag{6}$$

So, the "bilateral" formal power series $\mathcal{X}^i = \sum_{j=-\infty}^{+\infty} x(i+j,i-j)\xi^{-j}$, $i = 0,1,\dots$ are associated with the restrictions of the signal x to the separation sets $\mathcal{S}^i$, $i = 0,1,\dots$.

Let denote by F^n and G^n respectively the spaces of polynomials in ξ, ξ^{-1}, λ and of formal power series in $\xi, \xi^{-1}, \lambda^{-1}$, with coefficients in $\mathbf{R}^n$. Introduce in $F^n \times G^n$ a nondegenerate bilinear function $\langle\cdot,\cdot\rangle_n$ that associates with a polynomial $p = \sum_{i=0}^{\ell}\sum_{j=-m}^{m} p_{ij}\xi^j\lambda^i$ in F^n and a series $\mathcal{X} = \sum_{i=0}^{+\infty}\sum_{j=-\infty}^{+\infty} x(i+j,i-j)\xi^{-j}\lambda^{-i}$ in G^n the coefficient of the constant term in the Cauchy product $p^T\mathcal{X}$

$$\langle p, \mathcal{X}\rangle_n = \sum_{i=0}^{\ell}\sum_{j=-m}^{m} p_{ij}^T x(i,j). \tag{7}$$

Every series $\mathcal{X}$ in G^q induces a linear function $\varphi_\mathcal{X}$ on F^n, defined by $\varphi_\mathcal{X} : p \longmapsto \langle p, \mathcal{X}\rangle_n$. Moreover, the linear mapping that associates $\mathcal{X}$ with the linear function $\varphi_\mathcal{X}$ is an isomorphism of G^n onto the space $\mathcal{L}[F^n]$ of linear functions on F^n and, consequently, each series in G^n (or , equivalently, each signal $x : \mathcal{H} \to \mathbf{R}^n$) can be identified with an element of the algebraic dual space $\mathcal{L}[F^n]$. This accounts for the possibility of expressing many features of signal spaces with support in $\mathcal{H}$ in terms of properties of suitable subspaces of F^n.

Let $M(\xi,\xi^{-1},\lambda)$ be a $q \times n$ matrix with entries in $\mathbf{R}[\xi,\xi^{-1},\lambda]$ and consider the linear mappings

$$\begin{aligned} \mu :\ & F^q \longrightarrow F^n \ :\ p \longmapsto M^T(\xi,\xi^{-1},\lambda)p \\ \mu^* :\ & G^n \longrightarrow G^q \ :\ \mathcal{X} \longmapsto M(\xi,\xi^{-1},\sigma)\mathcal{X} \end{aligned} \tag{8}$$

Here σ is the shift operator in G^1

$$\sum w(i+j,i-j)\xi^{-j}\lambda^{-i} \overset{\sigma}{\longmapsto} \sum w(i+1+j,i+1-j)\xi^{-j}\lambda^{-i} \tag{9}$$

and μ and μ^* are dual mappings [5], as $\langle\mu p, \mathcal{X}\rangle_n = \langle p, \mu^*\mathcal{X}\rangle_q$ holds for all $\mathcal{X}$ in G^n and p in F^q. We therefore have

$$\ker\mu^* = (\mathrm{im}\mu)^\perp, \tag{10}$$

where imμ denotes the $\mathbf{R}[\xi,\xi^{-1},\lambda]$-module generated by the columns of the matrix $M^T(\xi,\xi^{-1},\lambda)$.

In order to analyze the trajectories of system (5), we introduce the following series

$$\mathcal{X}_\ell = \sum_{i=0}^{+\infty}\sum_{j=-\infty}^{+\infty} x_\ell(i+j,i-j)\xi^{-j}\lambda^{-i}, \quad \ell = 1,2 \tag{11}$$

and the matrix

$$M(\xi,\sigma) = \begin{bmatrix} \sigma I - A_{11} - B_{11}\xi & -A_{12} - B_{12}\xi \\ -A_{21} - B_{21}\xi & -A_{22} - B_{22}\xi \end{bmatrix} \tag{12}$$

The constraints induced on x by equation (5) are expressed as

$$M(\xi,\sigma)\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = 0$$

Therefore the behaviour of (5) can be viewed as the kernel of the linear operator μ^* or, alternatively, as the orthogonal subspace to the $\mathbf{R}[\xi,\xi^{-1},\lambda]$-module $\mathcal{M}$ generated by the columns of the matrix $M^T(\xi,\lambda)$:

$$\mathcal{B} = \left\{\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \in G^n \ :\ M(\xi,\sigma)\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = 0\right\} = \ker\mu^* = \mathcal{M}^\perp \tag{13}$$

In our context an important consequence stems directly from the fact that G^n is the algebraic dual $\mathcal{L}[F^n]$, namely

$$\mathcal{B}^\perp = (\mathcal{M}^\perp)^\perp = \mathcal{M} \tag{14}$$

Actually (14) shows that the module $\mathcal{M}$ is uniquely determined by $\mathcal{B}$, so that $\mathcal{B}$ can be described as the kernel of some matrix $\overline{M}(\xi,\sigma)$ if and only if the columns of both $M^T(\xi,\lambda)$ and $\overline{M}^T(\xi,\lambda)$ generate the same $\mathbf{R}[\xi,\xi^{-1},\lambda]$-module.

The duality theory provides also an useful tool for analyzing the restrictions $\mathcal{B}^{[0,k]}$ of the behaviour $\mathcal{B}$ to the sets $\mathcal{S}^0 \cup \mathcal{S}^1 \cup \cdots \cup \mathcal{S}^k$. This is easily seen by considering the linear mappings

$$\begin{array}{ccccc} F_k^n & \xrightarrow{i} & F^n & \xrightarrow{\overline{\pi}} & F^n/\mathrm{im}\mu \\ G^n/\sigma^k G^n & \xleftarrow{\pi} & G^n & \xleftarrow{\overline{i}} & \ker\mu^* \end{array} \tag{15}$$

where F_k^n is the $\mathbf{R}[\xi,\xi^{-1}]$-submodule of the polynomial columns in F^n having degree less than or equal to k in the indeterminate λ, $G^n/\sigma^k G^n$ is (isomorphic to) the $\mathbf{R}[\xi,\xi^{-1}]$-submodule obtained by truncating in each series of G^n all terms with degree greater than k w.r. to λ^{-1}, the maps i and $\overline{i}$ are canonical injections, π and $\overline{\pi}$ are canonical projections.

Obviously $G^n/\sigma^k G^n$ is isomorphic with the space $\mathcal{L}[F_k^n]$ of linear functions on F_k^n. Moreover $F^n/\mathrm{im}\mu$ is isomorphic with a direct complement of $\mathrm{im}\mu$ in F^n, and using the duality theory on direct decompositions [6] gives $\ker\mu^* = (\mathrm{im}\mu)^\perp \cong \mathcal{L}[F^n/\mathrm{im}\mu]$. The first and the last space on the second row of (15) can be viewed as the algebraic duals of the corresponding spaces on the first row and the maps $\overline{\pi}\circ i$, $\pi\circ\overline{i}$ in (15) are dual linear maps w.r. to the bilinear function induced on the pairs $(F_k^n,\ G^n/\sigma^k G^n)$ and $(F^n/\mathrm{im}\mu,\ \ker\mu^*)$. Consequently the restriction $\mathcal{B}^{[0,k]}$ is given by

$$\mathcal{B}^{[0,k]} \cong \mathrm{im}(\pi\circ\overline{i}) = \ker(\overline{\pi}\circ i)^\perp \tag{16}$$

The above relation characterizes $\mathcal{B}^{[0,k]}$ as the subspace of all signals with support in $\mathcal{S}^0\cup\mathcal{S}^1\cup\cdots\cup\mathcal{S}^k$ and values in $\mathbf{R}^n$ that correspond to formal power series $\sum_{i=0}^k \mathcal{X}^i\lambda^{-i}$ satisfying the orthogonality condition

$$\langle\sum_{i=0}^k c_i(\xi)\lambda^i\ ,\ \sum_{i=0}^k \mathcal{X}^i\lambda^{-i}\rangle_n = [c_0(\xi)\ c_1(\xi)\ \cdots\ c_k(\xi)]\begin{bmatrix}\mathcal{X}^0\\ \mathcal{X}^1\\ \vdots\\ \mathcal{X}^k\end{bmatrix} = 0 \tag{17}$$

for all polynomial vectors $\sum_{i=0}^k c_i(\xi)\lambda^i$ in the $\mathbf{R}[\xi,\xi^{-1},\lambda]$-module $\mathrm{im}\mu$.

The $\mathbf{R}[\xi,\xi^{-1}]$-submodule of $\mathbf{R}^{1\times n(k+1)}[\xi,\xi^{-1}]$ whose elements are the rows $[c_0(\xi)\ c_1(\xi)\ \cdots\ c_k(\xi)]$ that satisfy the condition $\sum_{i=1}^k c_i(\xi)\lambda^i \in \mathrm{im}\mu$ is finitely generated. Therefore there exists a polynomial matrix $C^{[0,k]}(\xi)$ with $n(k+1)$ columns such that $\mathcal{B}^{[0,k]} = \ker C^{[0,k]}(\xi)$.

In the next section we shall take advantage of the particular structure of $M(\xi,\sigma)$ given by (12), when determing the $\mathbf{R}[\xi,\xi^{-1}]$-submodule $\mathcal{B}^{[0,k]}$.

3 Computation of trajectories

The following lemma directly provides a matrix $C^{[0,1]}(\xi)$ whose rows are given in terms of submatrices A_{ij} and B_{ij} that appear in the partition (12). The proof is based on Cayley-Hamilton theorem and can be found in [7].

Lemma Let A_{ij} and B_{ij} be as in (12) and define the polynomial matrices:

$$\mathcal{A}_{ij} := A_{ij} + B_{ij}\xi,\quad i,j = 1,2$$

$$[\,C_0(\xi)\quad C_1(\xi)\,] = \left[\begin{array}{cc|cc} \mathcal{A}_{21} & \mathcal{A}_{22} & 0 & 0 \\ \mathcal{A}_{11} & \mathcal{A}_{12} & I_r & 0 \\ 0 & 0 & \mathcal{A}_{21} & \mathcal{A}_{22} \\ 0 & 0 & \mathcal{A}_{21}\mathcal{A}_{11} & \mathcal{A}_{21}\mathcal{A}_{12} \\ 0 & 0 & \mathcal{A}_{21}\mathcal{A}_{11}^2 & \mathcal{A}_{21}\mathcal{A}_{11}\mathcal{A}_{12} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \mathcal{A}_{21}\mathcal{A}_{11}^r & \mathcal{A}_{21}\mathcal{A}_{11}^{r-1}\mathcal{A}_{12} \end{array}\right]$$

Then $C^{[0,1]}(\xi) = [\,C_0(\xi)\ \mid\ C_1(\xi)\,]$ and $\mathcal{B}^{[0,1]} = \ker C^{[0,1]}(\xi)$ or, equivalently,

$$\begin{bmatrix}\mathcal{X}^0\\ \mathcal{X}^1\end{bmatrix} \in \mathcal{B}^{[0,1]} \Leftrightarrow C_0\mathcal{X}^0 = -C_1\mathcal{X}^1 \tag{18}$$

Premultiplying both C_0 and C_1 by a suitable unimodular matrix U, one gets

$$UC_0 = \begin{bmatrix} D_0' \\ D_0 \\ 0 \end{bmatrix},\quad -UC_1 = \begin{bmatrix} D_1 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

where both D_0 and D_1 have full row rank. Just rewriting (18) as

$$\begin{aligned} D_0\mathcal{X}^0 &= 0 \\ D_1\mathcal{X}^1 &= D_0'\mathcal{X}^0 \end{aligned} \tag{20}$$

we easily see that all solutions of equation (20.1) can be viewed as restrictionsof admissible trajectories to the separation set $\mathcal{S}^0$. In fact D_1 has full row rank and, therefore, given any $\mathcal{X}^0$, eq. (20.2) can be fulfilled by suitably chosen values of $\mathcal{X}^1$.

We are now in a position for establishing the following

Theorem A signal $\mathcal{X} = \sum_{i=0}^{+\infty}\mathcal{X}^i\lambda^{-i}$ belongs to $\mathcal{B}$ if and only if $\mathcal{X}^i$ satisfy the following equations

$$\begin{aligned} &D_0\mathcal{X}^0 = 0 \\ &\begin{bmatrix} D_1 \\ D_0 \end{bmatrix}\mathcal{X}^{i+1} = \begin{bmatrix} D_0' \\ 0 \end{bmatrix}\mathcal{X}^i,\quad i = 0,1,\ldots \end{aligned} \tag{21}$$

PROOF Suppose that $\mathcal{X}$ satisfies (21). Then we have

$$C^{[0,1]}(\xi)\begin{bmatrix}\mathcal{X}^i \\ \mathcal{X}^{i+1}\end{bmatrix} = 0,\quad i = 0,1,\ldots \tag{22}$$

which implies

$$\mathcal{X}^i + \lambda^{-1}\mathcal{X}^{i+1} \in \mathcal{B}^{[0,1]},\quad i = 0,1,\ldots \tag{23}$$

The degree of all columns in $M^T(\xi,\lambda)$ w.r.to λ is less than or equal to one. So, any such column can be written as $c_0(\xi)+\lambda c_1(\xi)$ and we have

$$\begin{aligned} &\langle(c_0(\xi)+\lambda c_1(\xi))\lambda^i\xi^j, \mathcal{X}\rangle_n \\ =&\langle(c_0(\xi)+\lambda c_1(\xi))\xi^j, \mathcal{X}^i+\lambda^{-1}\mathcal{X}^{i+1}+\ldots\rangle_n \\ =&\langle(c_0(\xi)+\lambda c_1(\xi))\xi^j, \mathcal{X}^i+\lambda^{-1}\mathcal{X}^{i+1}\rangle_n = 0 \end{aligned} \tag{24}$$

as a consequence of (16) and (23). This shows that $\mathcal{X}$ is orthogonal to $\mathrm{im}\mu$ and therefore $\mathcal{X}\in\mathcal{B}$. The converse is obvious.

Equations (21) provide a recursive procedure for generating the system trajectories. Moreover, the difference $n - \mathrm{rank}\begin{bmatrix} D_1 \\ D_0 \end{bmatrix}$ gives the number of free variables that appear in system (5), i.e. the variables that can be arbitrarily chosen on all separation sets $\mathcal{S}^i$.

4 References

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