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A note on the state space realization of 2D FIR transfer functions

E. Fornasini and S. Zampieri

Department of Electronics and Computer Science, 6/a Via Gradenigo, 35131 Padova, Italy

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Abstract: Some properties of minimal state space realizations of 2D FIR transfer functions are investigated. It is shown that hidden modes are allowed in minimal realization and, consequently, there exist internally unstable minimal realizations of FIR transfer functions.

Keywords: 2D systems; minimal realization; FIR filters; nilpotent matrices.

1. Introduction

It is well known that many analytical procedures for control design are based on state space representations of systems dynamics. In the 2D case, any linear stationary input output map, represented by a proper rational transfer function $W(z_1, z_2)$, can be converted to an equivalent system of first order equations with the following structure [2]:

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h, k+1) + A_2 x(h+1, k) \\ &\quad + B_1 u(h, k+1) + B_2 u(h+1, k), \end{aligned} \tag{1a}$$

$$y(h, k) = Cx(h, k) + Du(h, k). \tag{1b}$$

System (1) is usually denoted as

$$\Sigma = (A_1, A_2, B_1, B_2, C, D)$$

and called a (state space) realization of $W(z_1, z_2)$. Generally speaking, it should be expected that any constraint we assume on the structure of the pair (A_1, A_2) is reflected in a restriction on the class of transfer functions which can be realized by (1), and vice versa.

In this paper we shall be concerned with two properties of 2D systems and 2D input/output maps that are strongly connected with each other. The first one is the so called 'finite memory' property [1] of the state space model. A 2D system is finite memory if, for any arbitrary initial set of local states,

$$\mathscr{X}_0 = \{ x(i, -i),\ i \in \mathbb{Z} \} \tag{2}$$

the state free evolution goes to zero in a finite number of steps. This property depends only on the structure of the pair (A_1, A_2) and requires that the 2D characteristic polynomial of the system satisfies

$$\det(I - A_1 z_1 - A_2 z_2) = 1. \tag{3}$$

The second property is the finite impulse response (FIR) of the input/output map or, equivalently, the polynomial character of $W(z_1, z_2)$.

The connections between these properties are very well understood in the 1D situation, where

(i) finite memory reduces to the nilpotency of the matrix A,

(ii) finite memory implies that the transfer function is FIR,

(iii) minimal realizations of FIR transfer functions are finite memory.

Only proposition (ii) has an immediate extension to 2D systems, as can be easily seen from the expression

$$W(z_1, z_2) = \frac{C \operatorname{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2)}{\det(I - A_1 z_1 - A_2 z_2)} + D. \tag{4}$$

The objective pursued in Section 2 is to investigate some connections between the finite memory property of a 2D state space model and the structure of its pair (A_1, A_2), while in Section 3 we shall cope with the minimal realization problem of FIR transfer functions in two variables. One char-

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acteristic feature of 2D minimal realizations is that the numerator and the denominator of (4) need not be coprime polynomials. Therefore we expect that the characteristic polynomial of a minimal 2D state model may be a multiple of the transfer function denominator.

In that case the FIR property of a transfer function does not imply the internal stability of its minimal realizations.

2. Finite memory 2D systems

The spectral properties of matrices A_1 and A_2 are strongly connected with the finite memory property of a 2D system. However, the problem of obtaining a complete characterization of matrix structures that satisfy equation (3) has not yet been solved.

While (3) implies in an obvious way the nilpotency of A_1 and A_2, the converse is not true, since the nilpotency of A_1 and A_2 does not imply finite memory. To show this, just take

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

A stronger condition on A_1 and A_2 corresponds to requiring that the multiplicative semigroup S generated by A_1 and A_2 consists of nilpotent elements. In this case, by a theorem due to Levitzki [5], the elements of S can be simultaneously put in strict triangular form, i.e. zeros on and below the main diagonal. The above assumption on A_1 and A_2 is now sufficient, but not necessary, for obtaining a finite memory 2D system.

In fact, consider the following pair of nilpotent matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \tag{5}$$

It is immediate to check that

$$\det(I - A_1 z_1 - A_2 z_2) = 1.$$

However the multiplicative semigroup generated by A_1 and A_2 includes

$$A_1 A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is not nilpotent.

Remark. If the dimension of A_1 and A_2 is 2, then the finite memory condition (3) implies (and hence is equivalent to) the nilpotency of all elements of S. Actually, the case $A_1 = 0$ is trivial; so there is no restriction in assuming that A_1 is a Jordan block

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{6}$$

and A_2 any 2×2 nilpotent matrix

$$A_2 = \begin{bmatrix} u \\ v \end{bmatrix} [\alpha \quad \beta], \quad u\alpha + v\beta = 0. \tag{7}$$

We therefore have

$$\det(I - A_1 z_1 - A_2 z_2) = 1 - \alpha v z_1 z_2.$$

Thus finite memory gives $\alpha v = 0$, and $u\alpha + v\beta = 0$ implies $\beta v = \alpha u = 0$. We conclude that the structure of A_2 is as follows:

$$A_2 = \begin{bmatrix} 0 & u\beta \\ 0 & 0 \end{bmatrix}$$

and S consists of nilpotent elements.

Using some properties of the family of matrices $\{A_1^{i \sqcup\!\sqcup j} A_2\}$, generated recursively [3] according to

$$A_1^{i \sqcup\!\sqcup 0} A_2 = A_1^i, \quad A_1^{0 \sqcup\!\sqcup j} A_2 = A_2^j, \tag{8a}$$

$$A_1^{i \sqcup\!\sqcup j} A_2 = A_1 \left(A_1^{i-1 \sqcup\!\sqcup j} A_2 \right) + A_2 \left(A_1^{i \sqcup\!\sqcup j-1} A_2 \right) \quad \text{if } i,\ j > 0, \tag{8b}$$

a necessary and sufficient condition for finite memory can be stated as follows. Consider first the identity

$$(I - A_1 z_1 - A_2 z_2)^{-1} = \sum_{i,j=0}^{\infty} A_1^{i \sqcup\!\sqcup j} A_2 z_1^i z_2^j = \frac{\operatorname{adj}(I - A_1 z_1 - A_2 z_2)}{\det(I - A_1 z_1 - A_2 z_2)}. \tag{9}$$

The finite memory condition (3) implies that

$$\sum_{i,j} A_1^{i \sqcup\!\sqcup j} A_2 z_1^i z_2^j$$

is a polynomial of degree less than n, where n is the dimension of A_1 and A_2 and, a fortiori,

$$A_1^{i \sqcup\!\sqcup j} A_2 = 0, \quad i + j = n. \tag{10}$$

Vice versa, using the recurring equations (8), we see that conditions (10) give

$$A_1{}^{i}\sqcup\!\sqcup^{j}A_2 = 0, \quad i+j \geq n,$$

which in turn imply that $(I - A_1 z_1 - A_2 z_2)^{-1}$ is a polynomial and

$$\det(I - A_1 z_1 - A_2 z_2) = 1.$$

This proves that (10) constitutes a criterion for finite memory.

Neither condition (3) nor criterion (10) yield a set of canonical forms for characterizing those pairs (A_1, A_2) that exhibit the finite memory property.

However, if we confine ourselves to pairs of matrices with dimension 3, a finite set of canonical forms (with respect to similarity transformations) can be computed directly. Since these will be needed in the next section, we shall sketch here their construction.

Referring to the ranks of A_1 and A_2, we classify all possible cases as follows:

Case 1: (rank A_1) (rank A_2) $= 0$.

One of the matrices is 0 and the other one can be reduced to a nilpotent matrix in Jordan form. No further constraint arises when considering equation (3).

Case 2: (rank A_1) (rank A_2) $= 1$.

Both matrices have rank 1. There is no restriction in assuming that A_1 is in Jordan form,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} := J^{(1)},$$

and A_2 is a generic 3×3 nilpotent matrix of rank 1,

$$A_2 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} [\alpha \quad \beta \quad \gamma], \quad \alpha u + \beta v + \gamma w = 0. \tag{11}$$

Similarity transformations induced by

$$T = \left[\begin{array}{c|cc} t_{11} & t_{12} & t_{13} \\ \hline 0 & t_{11} & 0 \\ 0 & t_{32} & t_{33} \end{array}\right], \quad t_{11} t_{33} \neq 0,$$

preserve the structure of $J^{(1)}$ and reduce A_2 to one of the following structures:

$$A_2' = \begin{bmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ w & 0 & 0 \end{bmatrix} \neq 0, \tag{12a}$$

$$A_2'' = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \neq 0, \tag{12b}$$

$$A_2''' = \begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & w & 0 \end{bmatrix} \neq 0. \tag{12c}$$

Finally, equation (3) introduces a further constraint on A_2', so that the admissible pairs are

$$A_1 = J^{(1)}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ w & 0 & 0 \end{bmatrix} \neq 0, \tag{13a}$$

$$A_1 = J^{(1)}, \quad A_2 = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \neq 0, \tag{13b}$$

$$A_1 = J^{(1)}, \quad A_2 = \begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & w & 0 \end{bmatrix} \neq 0, \tag{13c}$$

Case 3: (rank A_1) (rank A_2) $= 2$.

Suppose that A_1 is of rank 2 (the case rank $A_2 = 2$ can be dealt with by symmetry), reduced to its Jordan form

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} := J^{(2)},$$

and A_2 is a generic 3×3 nilpotent matrix of rank 1. Similarity transformations induced by

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{11} & t_{12} \\ 0 & 0 & t_{11} \end{bmatrix}, \quad t_{11} \neq 0, \tag{14}$$

preserve the structure of $J^{(2)}$, while reducing A_2 to one of the structures given in (12). The finite memory condition implies then that the admissible pairs are only two, namely

$$A_1 = J^{(2)}, \quad A_2 = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \neq 0, \tag{15a}$$

$$A_1 = J^{(2)}, \quad A_2 = \begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0. \tag{15b}$$

Case 4: (rank A_1) (rank A_2) = 4.

Assume $A_1 = J^{(2)}$ and let A_2 be a generic 3×3 nilpotent matrix of rank 2,

$$A_2 = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \quad r_i \in \mathbb{R}^{1\times 3}. \tag{16}$$

If r_1 belongs to the row span of r_2 and r_3, then there is no restriction in assuming $r_1 = 0$. In fact in this case a similarity transformation exists, as induced by a suitable matrix (14), that annihilates the first row of A_2.

If r_1 does not belong to the row span of r_2 and r_3, then, using a similarity transformation as above, we may assume that either $r_2 = 0$ (and $r_3 \neq 0$) or $r_3 = 0$ (and $r_2 \neq 0$).

In case $r_1 = 0$, the finite memory condition puts further constraints on A_2, that reduces to

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -u & 0 & 0 \\ 0 & u & 0 \end{bmatrix} \neq 0. \tag{17}$$

In case $r_2 = 0$, there is no way of satisfying the finite memory condition, while in case $r_3 = 0$ the matrix A_2 reduces to

$$A_2 = \begin{bmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{bmatrix}, \quad u, w \neq 0. \tag{18}$$

3. Realization of FIR transfer functions

Obtaining a finite memory realization of a polynomial transfer function is quite easy if we are not required to obtain a low dimensional system. Actually a finite memory realization of the monomial $q_{ij} z_1^i z_2^j$ is given by

$$A_1 = \left[\begin{array}{cccc|c} 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 0 & \\ \hline & & 0 & & 0 \end{array}\right], \quad B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{19a}$$

with $i \times i$ left upper matrix,

$$A_2 = \left[\begin{array}{c|cccc} 0 & & 0 & & \\ & 1 & & & \\ \hline & 0 & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \\ & 0 & & & 0 \end{array}\right], \quad B_2 = 0, \tag{19b}$$

with $j \times j$ right lower matrix,

$$C = [0 \quad 0 \quad \cdots \quad \cdots \quad 1]. \tag{19c}$$

in case $i > 0$, and by a nilpotent 1D system in case $i = 0$.

So in realizing a 2D polynomial

$$\sum_{i,j} q_{ij} z_1^i z_2^j$$

it is sufficient to use direct sums, that preserve the finite memory property [4].

However if we look for *minimal* realizations of FIR transfer functions, in general it is not true that these are necessarily finite memory (which makes a remarkable difference with the 1D case). In this section we aim to show that this phenomenon holds independently of the (real or complex) field where the matrix elements take their values and implies that minimal realizations of FIR transfer functions need not be internally stable.

To prove our statement, we shall go through the following steps:

1. We show that there exists a 2D polynomial $n(z_1, z_2)$ of degree 3, that cannot be realized in dimension 3.

2. We construct an (infinite memory) realization of dimension 3 for the transfer function $n(z_1, z_2)/(1 + z_2)$ and a series connection of such a realization with a realization of dimension 1 for $1 + z_2$. The realization of $n(z_1, z_2)$ obtained in this way is minimal, internally unstable and exhibits pole/zero cancellations.

Step 1. Assume that A_1 and A_2 belong to $\mathbb{C}^{3\times 3}$ and are the state updating matrices of a finite memory 2D system.

For all matrices

$$C = [c_1 \quad c_2 \quad c_3]$$

in $\mathbb{C}^{1\times 3}$ and B_1, B_2 in $\mathbb{C}^{3\times 1}$, the system transfer function is a polynomial

$$W(z_1, z_2) = C \operatorname{adj}(I - A_1z_1 - A_2z_2)(B_1z_1 + B_2z_2) \qquad (20)$$

of degree not greater than 3, that we shall rewrite as

$$W(z_1, z_2) = p_1(z_1, z_2) + p_2(z_1, z_2) + p_3(z_1, z_2),$$

where p_i are homogenous forms of degree i, $i = 1, 2, 3$.

Referring to the cases considered in Section 2, we see that, independently of the choice of B_1, B_2 and C, the polynomials p_i must satisfy the following constraints:

Case 1: p_3 belongs either to the principal ideal (z_1^2) or to the principal ideal (z_2^2).

Case 2: p_3 belongs to the principal ideal (z_1z_2).

Case 3: if $p_3 = q^3$, where q is a first order form, then either $q = z_1$, or $q = z_2$

Case 4: if $p_3 = q^3$, where q is a first order form, then q is also a factor of p_2.

The above constraints show that the polynomial

$$n(z_1, z_2) = (z_1 + z_2)^3 + z_2^2 + z_2 \qquad (21)$$

cannot be realized by a finite memory third order 2D system. Therefore, if we look for a third order 2D realization of $n(z_1, z_2)$, a system with

$$\det(I - A_1z_1 - A_2z_2) \neq 1$$

would be needed. However in that case pole/zero cancellations between

$$\det(I - A_1z_1 - A_2z_2)$$

and

$$C \operatorname{adj}(I - A_1z_1 - A_2z_2)(B_1z_1 + B_2z_2)$$

must occur, and therefore the degree of the latter must be greater than or equal to 4, which is impossible for third order systems.

Consequently, the dimension of any state space realization of $n(z_1, z_2)$ is greater than 3.

Step 2. Consider the following 2D systems:

$$\Sigma_1 = (A_1, A_2, B_1, B_2, C)$$

with

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0],$$

and

$$\Sigma_2 = (F_1, F_2, G_1, G_2, H, D)$$

with

$$F_1 = F_2 = 0, \quad G_1 = 0, \quad G_2 = H = 1, \quad D = 1.$$

They realize $n(z_1, z_2)/(1 + z_2)$ and $1 + z_2$ respectively. Then the series connection of Σ_2 and Σ_1 is a strictly proper 2D system

$$\hat{\Sigma} = (\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2, \hat{C})$$

with

$$\hat{A}_1 = \begin{bmatrix} F_1 & 0 \\ B_1H & A_1 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} F_2 & 0 \\ B_2H & A_2 \end{bmatrix} \qquad (22a)$$

$$\hat{B}_1 = \begin{bmatrix} G_1 \\ B_1D \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} G_2 \\ B_2D \end{bmatrix}, \quad \hat{C} = [0 \quad C], \qquad (22b)$$

which provides a fourth order, and hence a minimal realization of $n(z_1, z_2)$.

Since the characteristic polynomials of $\hat{\Sigma}$ is

$$\det(I - \hat{A}_1z_1 - \hat{A}_2z_2) = 1 + z_2, \qquad (23)$$

the realization $\hat{\Sigma}$ above is not finite memory.

The above example allows to point out some interesting consequences:

(i) Minimal realizations of FIR transfer functions need not be finite memory.

(ii) Pole/zero cancellations are allowed in minimal realizations. Actually in $\hat{\Sigma}$ we have

$$\hat{C} \operatorname{adj}(I - \hat{A}_1z_1 - \hat{A}_2z_2)(\hat{B}_1z_1 + \hat{B}_2z_2) = n(z_1, z_2)(1 + z_2),$$

$$\det(I - \hat{A}_1z_1 - \hat{A}_2z_2) = 1 + z_2.$$

(iii) minimal realizations of FIR (and hence BIBO stable) transfer functions may be unstable.

In fact in the above example the variety of $1+z_2$ intersects the unit closed polydisc

$$\mathscr{P}_1=\{(z_1, z_2): |z_1|\le 1, |z_2|\le 1\}.$$

Remark.

The connections of the above results (in particular the existence of pole/zero cancellations in 2D minimal realizations) with some conjectures that have been advanced in the literature are rather intriguing. This is partly due to the use of Roesser's model by the authors of the conjectures, that introduces heavy a priori constraints on the state space realization matrices, and to different notations adopted in 2D transfer functions.

As is well known, Roesser's model can be viewed as a particular case of model (1), where A_1, A_2, B_1, B_2 have the following structure:

$$A_1=\begin{bmatrix} A_{11} & A_{12}\\ 0 & 0\end{bmatrix},\quad B_1=\begin{bmatrix} B^{(1)}\\ 0\end{bmatrix}, \tag{24a}$$

$$A_2=\begin{bmatrix} 0 & 0\\ A_{21} & A_{22}\end{bmatrix},\quad B_2=\begin{bmatrix} 0\\ B^{(2)}\end{bmatrix}. \tag{24b}$$

Consequently, minimal realizations in Roesser's form often exhibit a larger dimension than unconstrained minimal realizations.

When a transfer function $W(z_1, z_2)$ is expressed as the ratio of two irreducible polynomials in z_1^{-1} and z_2^{-1}, i.e.

$$W(z_1, z_2)=\frac{h(z_1^{-1}, z_2^{-1})}{k(z_1^{-1}, z_2^{-1})}$$

with $\nu_1=\deg_{z_1^{-1}}k$, $\nu_2=\deg_{z_2^{-1}}k$, it is possible to construct state space models in Roesser's form (and, a fortiori, in general form (1)) with dimension $\nu_1+2\nu_2$ and $\nu_2+2\nu_1$. In all cases, Roesser's realizations cannot exhibit a dimension lower than $\nu_1+\nu_2$, and in [7] Sontag conjectured that $\nu_1+\nu_2$ is the dimension of minimal Roesser's realizations over the complex field. Note that this does by no means preclude the existence of unconstrained realizations with dimension lower than $\nu_1+\nu_2$. As an example, in our case we have

$$W(z_1, z_2)=\frac{(z_1^{-1}+z_2^{-1})^3+z_1^{-3}z_2^{-1}+z_1^{-3}z_2^{-2}}{z_1^{-3}z_2^{-3}}$$

and therefore $\nu_1+\nu_2=6$, whereas $W(z_1, z_2)$ admits an unconstrained realization of dimension 4.

Roesser's realizations of dimension greater than $\nu_1+\nu_2$ are always associated with pole/zero cancellations in the rational function

$$C\begin{bmatrix} z_1^{-1}I-A_{11} & A_{12}\\ -A_{21} & z_2^{-1}I-A_{22}\end{bmatrix}^{-1}\begin{bmatrix} B^{(1)}\\ B^{(2)}\end{bmatrix}+D.$$

Therefore, the conjectural nature of Sontag's statement and the existence of counterexamples when only real valued Roesser's realizations are allowed suggested the following remark by Lévy [6, p. 223]: "This seems to suggest that, unlike in the 1D case, the problem of finding irreducible realizations and the one of finding realizations of minimum state space size are not necessarily the same".

Whether or not Sontag's conjecture is true, it is clear that the problem of the existence of pole/zero cancellations in minimal realizations with general structure would remain still unsolved. Actually cancellations in minimal Roesser's models could be viewed as an artifact of the structure (24), that possibly disappears when the model (1) is used and the state space dimension becomes lower. On the other hand, even if Sontag's conjecture is true, we could conceivably think of minimal unconstrained realizations that exhibit pole/zero cancellations.

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