Realization and partial realization of 2D input/output maps

M.Bisiacco, E.Fornasini, S.Zampieri

Summary This communication discusses some aspects of the problem of constructing dynamical models of 2D input/output maps. In the first part a finite array of data is given, and a recursive algorithm is provided for generating the whole class of minimum degree transfer functions, whose series expansion fits the data window. In the second part a 2D input/output map is completely assigned, via a rational transfer function in two indeterminates. Some algebraic properties of its state space realizations are investigated, and related to the structure of noncommutative power series.

1. Introduction

The purpose of this paper is to outline some recent results, connected with the construction of irreducible rational functions in two variables that model a finite two dimension data array, and to point out perspectives and constraints in computing minimal state space realizations of such rational functions.

Questions relating to the recursive generation of different Padé approximations schemes of a finite 2D array have been actively studied by several authors [1-3]. The novelty of the algorithm presented in this paper is that all data available in the positive orthant are fitted by the power series expansion of the rational function. Once a rational input/output model of the data has been obtained, it is natural to ask for a state space realization $\Sigma = (A_1, A_2, B_1, B_2, C, D)$, given by

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) \\ &\quad + B_1 u(h,k+1) + B_2 u(h+1,k) \\ y(h,k) &= Cx(h,k) + Du(h,k) \end{aligned} \tag{1.1}$$

that exhibits some desirable properties.
Coprime (i.e. free of hidden modes) realizations, which are very useful when synthesizing 2D observers and controllers, have been analyzed in the second half of the last decade and are now well understood.

The situation is different if we look for minimal realizations, whose structure still constitutes the bottleneck of the 2D theory. At the moment counterexamples are available, that provide negative answers to many questions we could naively hope to solve by just extending 1D results. A stricking difference w.r. to the classical case is that hidden modes are allowed in minimal 2D realizations. This result encompasses many interesting consequences, ranging from the stability of minimal realizations to the existence of minimal realizations that are not modally controllable and reconstructible. In the perspective of this paper, an important consequence is that minimal state space realizations of a finite 2D array are allowed, whose characteristic polynomials cannot be obtained as denominators or rational irreducible 2D Padé approximants.

2. Rational representations of 2D finite data arrays

Suppose we are given a finite array of data with support in the discrete plane, and we look for an irreducible rational function in two variables, whose power series expansion fits the samples of the array. For sake of simplicity, we assume that the support of the array is included in $\mathbf{N}^2$ and data are entered by diagonals, according to the restriction to $\mathbf{N}^2$ of the total ordering relation $<_T$ in $\mathbf{Z}^2$ given by $(h_1, h_2) <_T (k_1, k_2)$ iff $(h_1 + h_2 < k_1 + k_2)$ or $(h_1 + h_2 = k_1 + k_2$ and $h_2 < k_2)$. Consequently, at every time instant there exists an integer pair $q = (q_1, q_2) \in \mathbf{N}^2$ such that the 2D partial sequence of data $A^q := \{A_p, p <_T q\}$ is already known. The acquisition of a new sample A_q updates the partial sequence into $A^{q \oplus 1}$, with

$$q \oplus 1 := \begin{cases} (q_1 + 1, q_2 + 1), & \text{if } q_1 \geq 1; \\ (q_2 + 1, 0), & \text{if } q_1 = 0 \end{cases} \tag{2.1}$$

A causal rational function $c(z_1^{-1}, z_2^{-1})/b(z_1^{-1}, z_2^{-1})$ constitutes a "rational representation" of A^q if the samples of A^q are fitted by the coefficients of the power series expansion

$$\frac{c(z_1^{-1}, z_2^{-1})}{b(z_1^{-1}, z_2^{-1})} = \sum_{p \in N^2} s_p z^p := \sum_{p_1, p_2 \in N} s_{p_1, p_2} z_1^{p_1} z_2^{p_2} \tag{2.2}$$

for all $p <_T q$. The denominators of all rational representations are termed "valid polynomials" of the partial sequence A^q .

Once a valid polynomial b of A^q has been computed, obtaining a numerator c is a trivial task, that essentially reduces to a convolution operation between A^q and b. Therefore, the parametrization of the valid polynomials set $\mathbf{V}(A^q)$ can be viewed as the crucial step of the rational representation problem. However, since $\mathbf{V}(A^q)$ is an infinite set, an efficient solution should single out, on the basis of some optimum property, a finite subset of valid polynomials. In the following, we shall concentrate on valid polynomials with minimum degree w.r. to a convenient degree definition for polynomials in two variables.

To that purpose, introduce in $\mathbf{Z} \times \mathbf{Z}$ the partial ordering given by the product of the orderings: $(r_1, r_2) \leq (s_1, s_2)$ iff $r_1 \leq s_1$, $r_2 \leq s_2$, and define the degree of a polynomial $b(z_1^{-1}, z_2^{-1})$ as the integers pair constituted by the degrees of that polynomial w.r. to the variables z_1^{-1} and z_2^{-1} . If $\deg b = (h_1, h_2)$, the coefficient of $z_1^{-h_1} z_2^{-h_2}$ needs not be different from zero. In case it is different from zero, b is called a δ-polynomial. Note that the denominator of any coprime causal rational function is a δ-polynomial.

Valid δ-polynomials can be characterized by a set of linear relations, as stated in the following Theorem

THEOREM 1 [3] *Let* $b = \sum_r b_r z^{-r} = \sum_{r_1, r_2} b_{r_1, r_2} z_1^{-r_1} z_2^{-r_2}$ *be a* δ-*polynomial. Then* b *is a valid polynomial for the partial sequence* A^q *if and only if*

$$b[A^q]_p := \sum_{o \leq i \leq \deg b} b_i A_{p+i} = 0 \tag{2.3}$$

for all $p \in S(\deg b, q)$, *where the set* $S(\cdot,\cdot)$ *is defined by*

$$S(x, y) := \{t \in \mathbf{Z}^2 : t + x \in \mathbf{N}^2, t + x <_T y, t \not\leq 0\} \tag{2.4}$$

A valid polynomial $b \in \mathbf{V}(A^q)$ is "minimal" with respect to the given set of data A^q if none valid polynomial $\bar{b}$ exists, such that $\deg \bar{b} < \deg b$. Minimal polynomials are denominators of coprime rational representations of A^q , which in turn belong to the set $\mathbf{V}_\delta(A^q)$ of valid δ-polynomials.

"Irreducible sets" of minimal polynomials provide the optimal finite subsets of valid polynomials we are looking for. Consider a set of minimal polynomials $\mathbf{I}_q \subset \mathbf{V}(A^q)$, obtained by selecting a representative in every subset of minimal polynomials having the same degree. Such $\mathbf{I}_q$ is called irreducible. Given a partial sequence A^q, the corresponding irreducible polynomial sets are non unique. What is unique, however, is the set of the degrees of the polynomials that belong to anyone of the irreducible sets.

The construction and the updating of an irreducible set of minimal polynomials is based on the following polynomial vector spaces $\mathbf{F}_s(A^q) := \{b \in \mathbf{R}[z_1^{-1}, z_2^{-1}] : \deg b \leq s, b[A^q]_p = 0, \forall p \in S(s, q)\}$, $s \in \mathbf{N}^2$. These spaces are very useful, since a δ-polynomial $b(z_1^{-1}, z_2^{-1})$ of degree s is a valid polynomial for the sequence A^q if and only if $b \in \mathbf{F}_s(A^q)$, and it is possible to single out a finite subfamily of the family $\{\mathbf{F}_s(A^q),\ s \in \mathbf{N}^2\}$ that allows to compute an irreducible set $\mathbf{I}_q$. Moreover, when a new sample A_q is obtained, updating $\mathbf{F}_s(A^q)$ into $\mathbf{F}_s(A^{q\oplus 1})$ is quite simple. Actually, in case $q \leq s$ we have $\mathbf{F}_s(A^q) = \mathbf{F}_s(A^{q\oplus 1})$;

if not, the definition of $\mathbf{F}_s(A^{q\oplus 1})$ involves one more linear constraint w.r. to that of $\mathbf{F}_s(A^q)$ and $\mathbf{F}_s(A^{q\oplus 1})$ is a subspace of $\mathbf{F}_s(A^q)$ with a generator set obtainable from a generator set of $\mathbf{F}_s(A^q)$ (see [3]).

The computational complexity of the procedure for updating the family $\{\mathbf{F}_s,\ s \in \mathbf{N}^2\}$ and for extracting $\mathbf{I}_q$ is very high. So it seems reasonable to look for alternative algorithms that avoid dealing directly with the structure of the above family. A detailed procedure has been designed in [3] where, for all q's, the infinite family is replaced by a finite set of polynomials. These polynomials still allow to compute $\mathbf{I}_q$ and are recursively updated when a new sample A_q is available.

3. Minimal state space realizations of rational representations

Suppose now that a rational representation $W(z_1, z_2) \in \mathbf{R}(z_1, z_2)$ of a finite array has been computed and we want to select an efficient state space realization of $W(z_1, z_2)$.

In setting up a structure analysis for a class of algebraic objects like 2D realizations of a rational transfer function $W(z_1, z_2)$, it is desirable to be able to recognize what special subclasses of 2D realizations are "simple" and to be able to measure the lack of "simplicity" in the general realization. One then wants some sort of passage from the general realization to these better behaved ones, that will contribute the target for all subsequent analysis efforts.

In principle, the situation is very similar in the 1D case, where, in order to study the set of all realizations of a 1D transfer function, we slice out of the set a certain piece - the subset of non-canonical realizations - in such a way that the piece being cut away is capable of description and at the same time the object resulting after the excision, i.e. the set of minimal realizations, is also capable of an easy and algebraically sound description. When dealing with 2D realizations, however, the dichotomy minimal/nonminimal does not provide a convenient partition of Real(W), the set of state space realizations of W. As a matter of fact, there is no concrete way for passing from nonminimal to minimal realizations and, on the other hand, 2D minimal realizations often do not constitute a set endowed with strong structural properties.

It is to circumvent these difficulties that we introduce noncommutative power series [4]. We shall see that every realization in Real(W) can be viewed as the matrix representation (MR) of a noncommutative power series σ whose commutative image is W. Furthermore all realizations that provide minimal MR's of the corresponding n.c. power series turn out to have a simple characterization in terms of rechability and observability matrices. On the other hand, if $\Sigma \in$ Real(W) does not constitute a minimal MR of a noncommutative power series, then linear algorithms apply so as to provide a minimal MR. So, a good insight into the structure of Real(W) can be gained if we previously discard all realizations that do not provide minimal MR's of n.c. power series. This elimination can be performed in a very simple way and the remaining subset of Real(W) is easily parametrized by a family of n.c. power series.

Consider the commutative power series expansion $W = \sum w_{i,j} z_1^i z_2^j$ and a noncommutative power series [5] in two non commuting variables $\sigma = \sum_{\pi \in \{\xi_1, \xi_2\}^*} (\sigma, \pi)\pi$, whose commutative image $\phi(\sigma)$ is $W(z_1, z_2)$. This amounts to say that the coefficients (σ, π) of (3.2) satisfy the following infinite set of linear equations

$$\sum_{\pi : \phi(\pi) = z_1^i z_2^j} (\sigma, \pi) = w_{i,j} \quad i, j \in \mathbf{N} \tag{3.1}$$

Suppose, moreover, that σ is recognizable, which implies the existence of matrices A_1, A_2, B_1, B_2, C, D of suitable dimensions, such that σ admits a MR with the following structure

$$\sigma = D + C(I - A_1\xi_1 - A_2\xi_2)^{-1}(B_1\xi_1 + B_2\xi_2) \tag{3.2}$$

Clearly we have that any MR of σ is a state space realization of W and, viceversa, any state space realization $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ of $W(z_1, z_2)$ induces a recognizable noncommutative series $\sigma = C(I - A_1\xi_1 - A_2\xi_2)^{-1}(B_1\xi_1 + B_2\xi_2) + D$ that satisfies (3.1). Therefore, denoting by $N(W)$ the set of all noncommutative recognizable power series that satisfy (3.1), we have that the map

$$\begin{aligned} \psi &:= \mathrm{Real}(W) \to N(W) := (A_1, A_2, B_1, B_2, C, D) \\ &\to \sigma = C(I - A_1\xi_1 - A_2\xi_2)^{-1}(B_1\xi_1 + B_2\xi_2) + D \end{aligned} \tag{3.3}$$

is surjective. The inverse mapping of ψ induces a partition of Real(W) into disjoint and non-empty sets with the following properties:

i) $\psi^{-1}(\sigma), \sigma \in N(W)$, is the set of all MR's of σ.

ii) Minimal MR's of any $\sigma \in N(W)$ are uniquely determined, modulo a similarity transformation and can be obtained using linear procedures, starting from any other MR of σ. Also, given $\sigma \in N(W)$, a modified version of Ho's algorithm allows to compute a minimal MR of σ.

iii) Preimages $\psi^{-1}(\sigma_1), \psi^{-1}(\sigma_2)$ of different series σ_1 and σ_2 in $N(W)$ possibly include minimal MR's with different dimensions. Actually it is not difficult to show that $N(W)$ includes noncommutative power series with minimal MR's of arbitrarily large dimensions.

iv) 2D realizations that constitute MR's of different noncommutative power series are nonequivalent.

The above approach to the analysis of Real(W) enlightens many interesting properties of 2D realizations. First of all, $W(z_1, z_2)$ needs not admit a unique minimal state space realization (modulo similarity). Actually the dimensions of minimal MR's of the series in $N(W)$ constitute a set $L \subset \mathbf{N}$, whose lower bound m is possibly attained by minimal MR's of many different series in $N(W)$.

EXAMPLE 1 The transfer function $W(z_1, z_2) = z_1 z_2/(1 - z_1 - z_2)^2$ admits Σ, given by

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0],$$

and $\bar{\Sigma}$, given by $(\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C}) = (A_2, A_1, B_2, B_1, C)$, as nonequivalent minimal realizations. Note that Σ and $\bar{\Sigma}$ provide minimal MR's of the following n.c.power series in $N(W)$

$$\sigma = (1 - \xi_1 - \xi_2)^{-1}\xi_1(1 - \xi_1 - \xi_2)^{-1}\xi_2 = \xi_1\xi_2 + \text{higher order terms}$$

$$\bar{\sigma} = (1 - \xi_1 - \xi_2)^{-1}\xi_2(1 - \xi_1 - \xi_2)^{-1}\xi_1 = \xi_2\xi_1 + \text{higher order terms,}$$

The dimension of a minimal MR of a noncommutative recognizable power series $\sigma \in \mathbf{R} << \xi_1, \xi_2 >>$ does not decrease if the matrices are allowed to take their values in $\mathbf{C}$, instead of $\mathbf{R}$. However the dimension m of minimal realizations of W possibly depends on the field. In fact the set $N(W)$ is bigger, and the lower bound m of the set L may decrease, if noncommutative power series take coefficients in $\mathbf{C}$ instead of $\mathbf{R}$.

EXAMPLE 2 Consider the polynomial transfer function $W(z_1, z_2) = z_1^2 + z_2^2$. When looking for a realization of dimension 2, elementary degree evaluations show that matrices A_1 and A_2 must satisfy $\det(I - A_1 z_1 - A_2 z_2) = 1$. This implies that A_1 and A_2 can be simultaneously put in structure upper triangular form and therefore

$$z_1^2 + z_2^2 = [c_1 \quad c_2] \begin{bmatrix} 1 & -\alpha_1 z_1 - \alpha_2 z_2 \\ 0 & 1 \end{bmatrix} (B_1 z_1 + B_2 z_2)$$

factorizes into the product of two first order forms. This requires $\mathbf{C}$ to be the ground field. The complex valued minimal realization

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ -i \end{bmatrix}, \; C = [1 \quad 0]$$

is a MR of $\sigma' = \xi_1^2 + i(\xi_2\xi_1 - \xi_1\xi_2) + \xi_2^2 \in \mathbf{C} < \xi_1, \xi_2 >$.

The real valued minimal realization given by

$$A_1 = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},$$

$$C = [1/2 \quad 1/2 \quad 0]$$

provides a minimal MR of $\sigma'' = \xi_1^2 + \xi_2^2 \in \mathbf{R} < \xi_1, \xi_2 >$.

Using noncommutative power series sheds also some light on a property of 2D hidden modes, that has been recently discovered. Given an irreducible 2D transfer function $n(z_1, z_2)/d(z_1, z_2)$ and a state space realization $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ of $W(z_1, z_2)$, hidden modes are the common factors of $C\text{adj}(I - A_1z_1 - A_2z_2)$ and $\det(I - A_1z_1 - A_2z_2)$.

Since in the 1D case hidden modes are associated with unreachable and/or unobservable parts, that prevent a system from being minimal, a naive extension of 1D theory would suggest that minimal 2D realizations are free of hidden modes. Actually cancellations of 2D polynomials between $C\text{adj}(I-A_1z_1-A_2z_2)(B_1z_1+B_2z_2)$ and the characteristic polynomial $\det(I - A_1z_1 - A_2z_2)$ are always connected with the existence of plane curves where one of the PBH controllability and reconstructibility matrices are not full rank, but this fact is not in contradiction with the minimality of the realization.

We shall sketch here an example, showing that the above intuition is false. It is based on a multistep procedure that can be summarized as follows:

i) express $W(z_1, z_2)$ as the product W_1W_2 of two irreducible transfer functions that exhibit the cancellation of some nonconstant polynomial $c(z_1, z_2)$.

ii) construct two minimal realization Σ_1 and Σ_2 of $W_1(z_1, z_2)$ and $W(z_1, z_2)$, respectively

iii) perform the series connection of Σ_1 and Σ_2. This provides a realization of $W(z_1, z_2)$, whose characteristic polynomial includes the factor $c(z_1, z_2)$.

REMARK The noncommutative power series σ_{12} that corresponds to the series connection of Σ_1 and Σ_2, is the product of the noncommutative power series σ_1 and σ_2 associated with Σ_1 and Σ_2 respectively. The cancellation of a commutative polynomial $c(z_1, z_2)$ in W_1W_2 needs not imply that cancellations arise in the product $\sigma_1\sigma_2 = \sigma_{12}$, when σ_1 and σ_2 are expressed via finite sums, products and inverses of noncommutative polynomials. This suggests that the series connection of Σ_1 and Σ_2 may be a minimal MR of σ_{12}, irrespective of cancellations in W_1W_2.

EXAMPLE 3 Consider the transfer function $W(z_1, z_2) = (z_1 + z_2)^3 + z_2^2 + z_2$ and its factorization into

$$W(z_1, z_2) = W_1(z_1, z_2)W_2(z_1, z_2) = \left[\frac{(z_1 + z_2)^3 + z_2^2 + z_2}{1 + z_2}\right][1 + z_2]$$

Starting from a minimal realization Σ_1 of $W_1(z_1, z_2)$, given by

$$\bar{A}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$\bar{C} = [\,1 \quad 0 \quad 0\,],$$

and a minimal realization Σ_2 of $W_2(z_1, z_2)$, given by $\hat{A}_1 = [0], \hat{A}_2 = [0], \hat{B}_1 = [0], \hat{B}_2 = [1], \hat{C} = [1], \hat{D} = [1]$, we compute the series connection of Σ_1 and Σ_2

$$A_1 = \begin{bmatrix} \hat{A}_1 & 0 \\ \bar{B}_1\hat{C} & \bar{A}_1 \end{bmatrix}, A_2 = \begin{bmatrix} \hat{A}_2 & 0 \\ \bar{B}_2\hat{C} & \bar{A}_2 \end{bmatrix}, B_1 = \begin{bmatrix} \hat{B}_1 \\ \bar{B}_1\hat{D} \end{bmatrix}, B_2 = \begin{bmatrix} \hat{B}_2 \\ \bar{B}_2\hat{D} \end{bmatrix},$$

$$C = [\,0 \quad \bar{C}\,]$$

The above system constitutes a minimal realization of W. The proof of this fact is rather long [11], and involves a detailed analysis of the matrix pairs A_1, A_2 in $\mathbf{C}^{3\times 3}$ that satisfy the finite memory condition $\det(I - A_1 z_1 - A_2 z_2) = 1$. Note that Σ_1 and Σ_2 provide minimal MR's of $\sigma_1 = (\xi_1 + \xi_2)^2(1 + \xi_2)^{-1}(\xi_1 + \xi_2) + \xi_2$ and $\sigma_2 = 1 + \xi_2$ respectively, and Σ provides a minimal MR of $\sigma = [(\xi_1 + \xi_2)^2(1 + \xi_2)^{-1}(\xi_1 + \xi_2) + \xi_2](1 + \xi_2)$. No cancellation arises in the above expression because of the noncommutativity of the factors $(1 + \xi_2)$ and $(\xi_1 + \xi_2)$.

References

1. N.K.Bose, S.Basu *Two-dimensional matrix Padé approximants* IEEE Trans.Aut.Contr. vol.AC-25, pp.509-14, 1980

2. S.Sakata *Partial realization of 2D discrete linear systems and 2D Padé approximation* Proc IEEE, vol 78, pp.604-613, 1990

3. S.Zampieri *An algorithm for 2D partial realization*,submitted for publication

4. E.Fornasini *On the relevance of noncommutative power series in spatial filters realization* IEEE Trans. Circ. Sys., vol CAS-25, pp.290-99, 1978

5. M.Fliess *Matrices de Hankel* J.Math. Pures Appl., vol.53, pp.197-222, 1974

6. E.Fornasini, S.Zampieri *A note on the state space realization of 2D FIR transfer functions*, submitted for publication

M.Bisiacco, E.Fornasini, S,Zampieri
Dept. of Informatics and Comp. Sci.
6/a via Gradenigo, 35131 PADOVA, Italy