State realization of 2D finite dimensional autonomous systems

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Abstract This paper deals with the state space realization of autonomous autoregressive 2D systems in the context of behavioral approach. Any arbitrary autoregressive 2D system Σ can be viewed as the sum of an externally controllable subsystem Σ^c with an autonomous one Σ^a. Hence the state space realizations of Σ can be obtained by separately realizing Σ^c and Σ^a. A procedure for realizing externally controllable systems in state/driving-variable form has already been presented in [8]. Thus, in order to solve the general realization problem, it is enough to investigate the autonomous case. Here we consider in particular finite dimensional autonomous systems and derive a realization procedure based on Gröbner bases representations of such systems.

Keywords : autonomous systems; (externally) controllable systems; state/driving-variable realization; Gröbner basis.

1 Introduction

The realization of 2D systems by means of state space models is an important question which still waits for a complete solution. A major problem in this area concernsthe realization of noncausal systems. As well-known, 2D input/state/output models proposed in [1] and [2,3] realize precisely the class of quarter plane rational transfer function. A possible approach to the realization of noncausal transfer function is to consider singular versions of the above mentioned models, as done in [4,5]. However, these input/state/output models the disadvantage of not being in recursive form.

An alternative to the singular model approach is to consider state/driving variables models, where the joint input-output trajectories are recursively computed from the values of an auxiliary free variables via the state. It is shown in [8] that every externally controllable autoregressive (AR) 2D system can be realized in state/driving variables form independently of its causality structure. It still remains, however, to analyse the realization of arbitrary (i.e., not necessarily externally controllable) 2D systems. In this paper we reduce such a problem to the realization of autonomous systems. Indeed, we show that every AR system Σ can be viewed as the sum of an externally controllable system Σ^c and an autonomous one, Σ^a and, therefore, the realization of Σ can be obtained by realizing separately Σ^c and Σ^a. Since a procedure for realizing externally controllable systems is given in [8], it will be enough to derive a realization procedure for autonomous systems. Here we consider, in particular, the case of finite dimensional autonomous AR systems with q (real-valued) variables defined over $\mathbf{Z}^2$. It turns out that, in this case, the set $\mathcal{B}$ of admissible system trajectories can be represented as $\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}$, with σ_1 and σ_2 the 2D shifts and $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ a factor right coprime 2D polynomial matrix. This implies that the polynomial module generated by the rows of R is characterized by a pair of commuting invertible matrices. Basing on these matrices a state space realization of the original system can be easily obtained.

2 Autonomous 2D systems

In this paper we follow the behavioral approach to dynamical systems introduced in [9] and [6]. This means that we characterize a 2D system by means of its behaviour, which consists of the set of all the signals which are compatible with the system laws. Moreover, we do not start with a given input/output structure, i.e., the system signals are stacked together in a (multivariate) signal w instead of being divided into inputs u and outputs y. A 2D system Σ with q real valued variables defined over $\mathbf{Z}^2$ and with behaviour $\mathcal{B} \subseteq \{w : \mathbf{Z}^2 \to \mathbf{R}^q\}$ will be denoted by $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$.

In order to define the notion of autonomy we introduce the following nomenclature. A subset $S \subseteq \mathbf{Z}^2$ is said to be a 2D-unbounded set if there exist unbounded $\mathbf{Z}$-intervals I_1 and I_2 such that $I_1 \times I_2 \subseteq S$.

Definition 1 $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is an autonomous 2D system if there exists a subset $T \subseteq \mathbf{Z}^2$ such that $\mathbf{Z}^2 \setminus T$ is 2D unbounded and satisfies the following condition

$$\{w_1, w_2 \in \mathcal{B} \text{ and } w_1|_T = w_2|_T\} \Rightarrow \{w_1 = w_2\}$$

So, intuitively, a system is autonomous if the evolution of its traiectories is completely specified by what occurs in a restrict part of the domain $\mathbf{Z}^2$.

In the sequel we will be interested in the class of autoregressive 2D systems. $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is said to be an autoregressive (AR) system if there exists a (two sided) 2D polinomial matrix $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\},$$

with σ_1 and σ_2 the 2D shift operators. These are respectively defined by:

$$\sigma_1 w(t_1, t_2) = w(t_1 + 1, t_2)$$
$$\sigma_2 w(t_1, t_2) = w(t_1, t_2 + 1)$$

for all $w : \mathbf{Z}^2 \to \mathbf{R}^q$ and all $(t_1, t_2) \in \mathbf{Z}^2$. As stated in proposition 2.1, for autoregressive systems the autonomy is equivalent with the absence of free variables.

Notation Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be a system in the variables $(w_1, \ldots, w_q)^T := w$. The variable w_i, $i \in \{1, \ldots, q\}$, is a free variable if, for every $\alpha : \mathbf{Z}^2 \to \mathbf{R}$, there exists some $w \in \mathcal{B}$ such that $w_i = \alpha$. Similarly, a vector $(w_{i_1}, \ldots, w_{i_l})^T$, with $i_j \in J \subseteq \{1, \ldots, q\}$ and $i_j \neq i_k$ if $j \neq k$, is a vector of free variables if for every $\beta : \mathbf{Z}^2 \to \mathbf{R}^l$ there exists $w \in \mathcal{B}$ such that $(w_{i_1}, \ldots, w_{i_l})^T = \beta$. The number of free variables in a system Σ is defined as the maximum dimension of a vector of free variables in Σ.

Lemma 2.1 Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be an autoregressive 2D system such that $\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}$. Then:

1. The number l of free variables in Σ is $q - \operatorname{rank}(R)$.
2. If $l > 0$, there exists a nonzero $q \times l$ polynomial matrix $M(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that $\mathcal{B} \supseteq \operatorname{im} M(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$, where $M(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ is viewed as an operator from $\{v : \mathbf{Z}^2 \to \mathbf{R}^l\}$ into $\{w : \mathbf{Z}^2 \to \mathbf{R}^q\}$. Moreover, im M has l free variables.

PROOF: For the first statement we refer to [6, Chapter 3, Proposition 13]. As for (2), note that R can be factorized as $R = FP$ with P

a full rank factor left prime polynomial matrix of size $l \times q$. Consequently $\mathcal{B} \supseteq \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid P(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\} =: \ker P$. Finally, it follows from [8, Theorem 1] that there exists a $q \times l$ nonzero polynomial matrix M such that $\ker P = \operatorname{im} M$.

Proposition 2.1 An autoregressive system $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is autonomous if and only if it has no free variables.

PROOF: (i) Suppose that Σ is a system without free variables. Then, the method for chosing the initial conditions proposed in [7, Section 4.2] implies the following. There exists a subset I of $\mathbf{Z}^2$ of the form $I = ([-L, L] \times \mathbf{Z}) \cup (\mathbf{Z} \times [-L, L])$, for some positive integer L, such that every trajectory $w \in \mathcal{B}$ is completely determined by its initial values $w|_I$. In other words, if $w', w" \in \mathcal{B}$ and $w'|_I = w"|_I$, then $w' = w"$. This clearly means that Σ is autonomous.

(ii)Assume that Σ has $l > 0$ free variables and let M be as in lemma 2.1. Denote respectively by $\overline{m}_i$ and $\underline{m}_i$ the maximum and the minimum of the exponents of z_i in the entries of M, and define the extent of M as $e(M) := \sqrt{2} \max\{\overline{m}_1 - \underline{m}_1, \overline{m}_2 - \underline{m}_2\}$. Further, denote the Euclidean distance by $d(\cdot, \cdot)$. Given any subset $T \subseteq \mathbf{Z}^2$ such that $\mathbf{Z}^2 \setminus T$ is 2D-unbounded, define two trajectories v' and $v" \in \{v : \mathbf{Z}^2 \to \mathbf{R}^q\}$ in the following way. The trajectory v' is simply the zero trajectory. As for $v"$, we require that $v"(t_1, t_2) = 0$ if $d((t_1, t_2), T) \leq e(M)$; for (t_1, t_2) with $d((t_1, t_2), T) > e(M)$, we define $v"(t_1, t_2)$ in such a way that $Mv" \not\equiv 0$. Note that this is possible since the value of $Mv"$ at a point $(t_1^*, t_2^*) \in \mathbf{Z}$ depends only on the values of $v"$ at the points (t_1, t_2) such that $d((t_1, t_2), (t_1^*, t_2^*)) \geq e(M)$. Let now $w' := Mv' = 0$ and $w" := Mv"$. Clearly, $w', w" \in \mathcal{B}$. Moreover $w'|_T = w"|_T = 0$, and $w" \neq 0 = w'$. This shows that Σ is not autonomous.

Clearly, the set $\{w : \mathbf{Z}^2 \to \mathbf{R}^q\}$ of $\mathbf{R}^q$-valued functions defined on $\mathbf{Z}^2$ is a real vector space with respect to functions (pointwise) addition and (pointwise) scalar multiplication. A 2D system $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is said to be linear if $\mathcal{B}$ is a linear subspace of $\{w : \mathbf{Z}^2 \to \mathbf{R}^q\}$. A linear 2D system is said to be finite dimensional if its behaviour $\mathcal{B}$ is a finite dimensional vector space, otherwise Σ is said to be infinite dimensional.

Contrary to the one dimensional case, where autonomous linear systems are necessarily finite dimensional, autonomous 2D systems may be infinite dimensional. The following proposition characterizes the autonomy and the finite dimensionality properties of AR 2D systems.

Proposition 2.2 Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be an AR 2D system with behaviour $\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}$, where $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ is a 2D polynomial matrix. Then Σ is autonomous if and only if R has full column rank. Moreover Σ is finite dimensional if and only if R is right factor prime.

PROOF The first part of the result is an immediate consequence of lemma 2.1 and proposition 2.1. The second statement follows from [5, Theorem 3.8].

An extreme example of non-autonomous systems is the class of (externally) controllable systems. For these systems, the evolution of the trajectories outside a restricted part T of the domain eventually becomes independent of what occurs in T. Formally, we define (external) controllability as follows.

Definition 2 A 2D system $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is (externally) controllable if the following condition holds. There exists a positive real number ρ such that, for all $T_1, T_2 \subseteq \mathbf{Z}^2$ with $d(T_1, T_2) > \rho$ and for all $w_1, w_2 \in \mathcal{B}$, there exists $w \in \mathcal{B}$ such that $w|_{T_i} = w_i|_{T_i}, i = 1, 2$. Here $d(T_1, T_2)$ denotes the euclidean distance between the sets T_1 and T_2.

Remark We refer to the above notion of controllability as to external controllability in order to make a distinction from the classical notion, which applies to state space realization. Our definition is given at an external level, as it only refers to the (external) system variables $w \in \mathcal{B}$. However, when no possibility of confusion arises, we will simply refer to it as controllability.

The next result provides a characterization of controllability for autoregressive systems.

Proposition 2.3 [6] Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be an AR 2D system. Then Σ is controllable if and only if there exists a factor left-prime (two-sided) polynomial matrix $P(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid P(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}.$$

An interesting feature of controllability and autonomy is the fact that these are complementary properties, in the sense that an arbitrary AR system can be viewed as the sum of a controllable system with an autonomous one.

Notation Given two systems $\Sigma_i = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}_i), i = 1, 2$, the sum $\Sigma_1 + \Sigma_2$ of Σ_1 and Σ_2 is defined as $\Sigma_1 + \Sigma_2 := (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$, where $\mathcal{B} := \mathcal{B}_1 + \mathcal{B}_2$

Proposition 2.4 Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ a 2D system. Then there exist AR systems $\Sigma^c = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^c)$ and $\Sigma^a = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^a)$ such that:

1. Σ^c is controllable,
2. Σ^a is autonomous, and
3. $\Sigma = \Sigma^c + \Sigma^a$.

PROOF: Let $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}.$$

Then there exist polynomial matrices $F(z_1, z_2, z_1^{-1}, z_2^{-1})$, with full column rank, and $P(z_1, z_2, z_1^{-1}, z_2^{-1})$, with full row rank, such that $R = FP$. Without loss of generality we can assume that P is factor left-prime (otherwise the non-trivial left factors of P can be extracted and included in F) and that, moreover, $P = [P_1 \ P_2]$, with P_1 a square and full rank matrix. Define

$$\mathcal{B}^c = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid Pw = 0\},$$

$$\mathcal{B}^a = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, w_2 = 0 \text{ and } FP_1 w_1 = 0\},$$

$\Sigma^c = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^c)$ and $\Sigma^a = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^a)$. Clearly, by proposition 2.3, Σ^c is controllable and, by proposition 2.2, Σ^a is autonomous. We will show that $\Sigma = \Sigma^c + \Sigma^a$. Since both $\mathcal{B}^c$ and $\mathcal{B}^a$ are subspaces of $\mathcal{B}$,

$$\mathcal{B}^c + \mathcal{B}^a \subseteq \mathcal{B}.$$

In order to prove the reciprocal inclusion, assume that $w \in \mathcal{B}$ is given. Let $w^a := \begin{bmatrix} w_1^a \\ w_2^a \end{bmatrix}$, with $w_2^a := 0$ and w_1^a such that $P_1 w_1^a = Pw$ (note that $P_1(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ is a surjective operator, as P_1 has full row rank). Further define $w^c := w - w^a$. Now, since $FP_1 w_1^a = FPw = Rw = 0$, it follows that $w^a \in \mathcal{B}^a$. Moreover,

$$Pw^c = Pw - Pw^a = Pw - [P_1 \ P_2] \begin{bmatrix} w_1^a \\ w_2^a \end{bmatrix} = Pw - P_1 w_1^a = 0$$

and hence $w^c \in \mathcal{B}^c$. So, $w = w^c + w^a$ with $w^c \in \mathcal{B}^c$ and $w^a \in \mathcal{B}^a$, proving that $\mathcal{B} \subseteq \mathcal{B}^c + \mathcal{B}^a$. This yields the desired result.

Remark Note that the above sum $\mathcal{B} = \mathcal{B}^c + \mathcal{B}^a$ (and hence $\Sigma = \Sigma^c + \Sigma^a$) is not necessarily a direct sum, i.e. we may have $\mathcal{B}^c \cap \mathcal{B}^a \neq \{0\}$. However, it can be shown that $\mathcal{B} = \mathcal{B}^c \oplus \mathcal{B}^a$ if and only if in the decomposition $R = FP$ the matrix P is zero-left-prime. Moreover, it is not difficult to prove that $\Sigma = \Sigma_1^c + \Sigma_1^a = \Sigma_2^c + \Sigma_2^a$ imply $\Sigma_1^c = \Sigma_2^c$, i.e. in the decomposition $\Sigma = \Sigma^c + \Sigma^a$ the system Σ^c is unique. In fact, it turns out that Σ^c is the largest controllable subsystem of Σ. Curiously, this uniqueness does not necessarily hold for Σ^a. This is illustrated in the example below.

Example Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^3, \mathcal{B})$ with

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}.$$

where

$$R(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} z_1 - 1 & 0 & (z_1 - 1)(z_1 + z_2) \\ 0 & z_2 - 1 & (z_2 - 1)(z_2 - z_1) \end{bmatrix}.$$

Then, clearly, R can be decomposed as $R = FP$ with

$$F(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} z_1 - 1 & 0 \\ 0 & z_2 - 1 \end{bmatrix}$$

and

$$P(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} 1 & 0 & z_1 + z_2 \\ 0 & 1 & z_2 - z_1 \end{bmatrix}.$$

Constructing Σ^c and Σ^a as in the proof of the proposition 2.4 yields $\Sigma^c = (\mathbf{Z}^2, \mathbf{R}^3, \mathcal{B}^c)$ with

$$\mathcal{B}^c := \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid Pw = 0\},$$

and $\Sigma^a = (\mathbf{Z}^2, \mathbf{R}^3, \mathcal{B}^a)$ with

$$\mathcal{B}^a := \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, w_3 = 0, F \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0\}.$$

So, $\Sigma = \Sigma^a + \Sigma^c$. Let now $\bar{\Sigma}^a = (\mathbf{Z}^2, \mathbf{R}^3, \bar{\mathcal{B}}^a)$, with

$$\bar{\mathcal{B}}^a := \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid w_2 = 0, F\bar{P} \begin{bmatrix} w_1 \\ w_3 \end{bmatrix} = 0\}$$

and

$$\bar{P}(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} 1 & z_1 + z_2 \\ 0 & z_1 - z_2 \end{bmatrix}.$$

Applying the same reasoning as in the proof of proposition 2.4, it is easily shown that also $\bar{\Sigma}^a + \Sigma^c = \Sigma$. So, in the decomposition $\Sigma = \Sigma^a + \Sigma^c$ the autonomous subsystem is not unique.

The above decomposition of an arbitrary AR 2D system Σ into the sum of a controllable part Σ^c and an autonomous part Σ^a can be used to obtain state space realizations of Σ by separately realizing Σ^c and Σ^a. As shown in [8], every controllable AR system admits a realization of the form

$$\begin{cases} S(\sigma_1\sigma_2^{-1})\mathbf{x} & = 0 \\ \sigma_1\sigma_2\mathbf{x} & = A_1\sigma_1\mathbf{x} + A_2\sigma_2\mathbf{x} + B_1\sigma_1 v + B_2\sigma_2 v \\ w & = C\mathbf{x} + Dv \end{cases} \qquad (2.1)$$

with $\mathbf{x}$ the state, v an auxiliary variable, called the driving-variable, and where (2.1.1) can be regarded as a constraint on the admissible initial state along a diagonal line. Thus, in order to investigate the realizability of an arbitrary 2D system $\Sigma = \Sigma^c + \Sigma^a$ by the state model (2.1), it is enough to focus on the realizability of the autonomous part Σ^a. This problem will be considered in the next section.

3 Autonomous finite dimensional systems

We shall assume throughout that R is a full column rank, factor right prime matrix, with elements in the polynomial ring $\mathbf{R}[z_1, z_2, z_1^{-1}, z_2^{-1}] := A_\pm$. Moreover, for sake of simplicity, we shall first restrict to scalar behaviours, by assuming that $R = [r_1 \ \ r_2 \ \ \cdots \ \ r_t]^T$ a column vector, and successively extend the results to the general case.

3.1 Scalar case

In the subsequent discussion a significant role will be played by some connections between the ideals in $A_\pm$ and the ideals in $A_+ := \mathbf{R}[z_1, z_2]$ and by a characterization of the system behaviour based on the algebraic properties of dual spaces. The connections are provided by the following map

$$|\cdot| : A_\pm \to A_+ : p \mapsto |p| := z_1^{-i} z_2^{-j} p$$

where i and j are the minimum degrees of p w.r. to the variables z_1 and z_2 respectively. More precisely, if

$$p = \sum_{h,k \in \mathbf{Z}} p_{hk} z_1^h z_2^k$$

then

$$i := \min\{h \in \mathbf{Z} | \exists k \in \mathbf{Z}, p_{hk} \neq 0\}$$

$$j := \min\{k \in \mathbf{Z} | \exists h \in \mathbf{Z}, p_{hk} \neq 0\}.$$

In case $p = 0$, we define $|p| = 0$. Clearly, for every nonzero p, $|p|$ includes a monomial in z_1 and a monomial in z_2 with nonzero coefficients.

Denote by $I_\pm := (r_1, r_2, \ldots, r_t)_\pm$ the ideal in $A_\pm$ generated by $r_1, r_2, \ldots, r_t$ and by $I_+ := (|r_1|, |r_2|, \ldots, |r_t|)_+$ the ideal generated in A_+ by $|r_1|, |r_2|, \ldots, |r_t|$. Note that I_+ depends on the particular set of generators $r_1, r_2, \ldots, r_t$ in $I_\pm$.

Lemma 3.1 i) $p \in I_\pm$ if and only if there exists a pair of integers (i, j) such that $z_1^i z_2^j p \in I_+$.

ii) $A_\pm/I_\pm$ is finite dimensional if and only if the same holds for A_+/I_+.

PROOF i) is obvious. As far as ii) is concerned, suppose first that A_+/I_+ is a nonzero finite dimensional space. This implies that I_+, and hence $I_\pm$, include two nonzero polynomials $f(z_1)$ and $g(z_2)$, with $\deg f = \phi > 0$ and $\deg g = \gamma > 0$. It is easily seen that the cosets $[z_1^i z_2^j] := z_1^i z_2^j + I_\pm$, $0 \leq i < \phi$, $0 \leq j < \gamma$ constitute a finite set of generators for the quotient space $A_\pm/I_\pm$.

Viceversa, suppose that $A_\pm/I_\pm$ is a nonzero finite dimensional space and assume, by contradiction, $\dim A_+/I_+ = \infty$. Then there exists a polynomial $d \in A_+$ which is a common factor of $|r_1|, |r_2|, \ldots, |r_t|$ and is noninvertible in $A_\pm$ (since the $|r_i|$s cannot exhibit a factor like $z_1^i z_2^j$). As a consequence, $I_\pm \subseteq (d)_\pm$, where $(d)_\pm$ is the principal ideal of $A_\pm$ generated by d. To get a contradiction, it remains to prove that $A_\pm/(d)_\pm$ is infinite dimensional. Since d must include at least two nonzero monomials and these exhibit different degrees w.r. to z_1 or z_2, it is easy to see that $\{z_2^i, i \in \mathbf{Z}\}$ in the first case, and $\{z_1^i, i \in \mathbf{Z}\}$ in the second, are linearly independent sets, modulo $(d)_\pm$.

In the following we shall identify in the obvious way the "universe" $\mathbf{R}^{\mathbf{Z}\times\mathbf{Z}}$ of all signals defined on $\mathbf{Z} \times \mathbf{Z}$ with the space of the formal power series in $z_1, z_2, z_1^{-1}, z_2^{-1}$ and introduce a bilinear function

$$\langle \cdot, \cdot \rangle : \mathcal{U} \times A_\pm \to \mathbf{R},$$

by assuming that $\langle p, w \rangle$, $w \in \mathcal{U}$, $p = \sum_{i,j \in \mathbf{Z}} a_{ij} z_1^i z_2^j$, is the constant term in the Cauchy product $w\tilde{p}$, where $\tilde{p} = \sum_{i,j \in \mathbf{Z}} a_{ij} z_1^{-i} z_2^{-j}$. In this way, the space of the bilateral series with support in $\mathbf{Z} \times \mathbf{Z}$ is isomorphic to the algebraic dual of $A_\pm$. Moreover, since

$$w \in \mathcal{B} \Leftrightarrow \langle r, w \rangle = 0, \qquad \forall r \in I_\pm,$$

$\mathcal{B}$ can be identified with the orthogonal complement of $I_\pm$. We therefore have

$$\mathcal{B} = I_\pm^\perp, \qquad \mathcal{B}^\perp = I_\pm^{\perp\perp} = I_\pm$$

Lemma 3.2 If the matrix R is right factor prime, then $\mathcal{B}$ and $A_\pm/I_\pm$ are finite dimensional isomorphic vector spaces.

PROOF Since R is right factor prime, A_+/I_+ is finite dimensional. Therefore, by lemma 3.1, $A_\pm/I_\pm$ is finite dimensional too. Let now $\{[p_1], [p_2], \ldots, [p_n]\}$ be a basis of $A_\pm/I_\pm$ and consider the linear map

$$\psi : \mathcal{B} \to A_\pm/I_\pm : w \mapsto \sum_{i=1}^n \langle p_i, w \rangle [p_i]$$

If $[\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n]^T \in \mathbf{R}^n$ is orthogonal to $[\langle p_1, w \rangle \ \langle p_2, w \rangle \ \ldots \ \langle p_n, w \rangle]^T$ for all $w \in \mathcal{B}$, then

$$\langle \sum \alpha_i p_i, w \rangle = 0,$$

$\forall w \in \mathcal{B}$ and

$$\sum \alpha_i p_i \in \mathcal{B}^\perp = I_\pm$$

This implies $\alpha_i = 0$, $i = 1, 2, \ldots, n$ and consequently $[\langle p_1, w \rangle \ \langle p_2, w \rangle \ldots \langle p_n, w \rangle]^T$ span $\mathbf{R}^n$ as w varies over $\mathcal{B}$. Therefore ψ is surjective.

Suppose now that $w \in \mathcal{B}$ satisfies $\psi(w) = 0$ or, equivalently, $\langle w, p_i \rangle = 0$, $i = 1, 2, \ldots n$. Since every $p \in A_\pm$ can be expressed as

$$\sum_{i=1}^n \alpha_i p_i + r, \ r \in I_\pm,$$

we have

$$\langle w, p \rangle = \langle w, r \rangle = 0, \ \forall p \in A_\pm \Rightarrow w = 0.$$

Therefore ψ is injective.

Once a basis $\{[p_1], [p_2], \ldots, [p_n]\}$ has been chosen in $A_\pm/I_\pm$, and consider the dual basis $\{w_1, w_2, \ldots, w_n\}$ in $\mathcal{B}$ w.r. to the bilinear function $\langle [p], w \rangle := \langle p, w \rangle$. We have $\langle [p_i], w_j \rangle = \delta_{ij}$.

Consider the commutative linear maps

$$Z_1 : A_\pm/I_\pm \to A_\pm/I_\pm : [p] \mapsto [z_1 p]$$

$$Z_2 : A_\pm/I_\pm \to A_\pm/I_\pm : [p] \mapsto [z_2 p]$$

and the adjoint maps Z_1' and Z_2' in the dual space, defined by

$$\langle [p], Z_1' w \rangle := \langle [z_1 p], w \rangle$$

$$\langle [p], Z_2' w\rangle := \langle [z_2 p], w\rangle$$

Since $\langle [z_i p], w\rangle = (w\widehat{z_i p}, 1) = \langle [p], z_i^{-1} w\rangle$, $i = 1,2$, we have that $Z_1' w = z_1^{-1} w$ and $Z_2' w = z_2^{-1} w$.

Let $N_i = [n_{hk}^{(i)}]$, $i = 1,2$, be the matrices of the linear transformations Z_i w.r. to the basis $\{[p_1],[p_2],\dots,[p_n]\}$. Then $n_{hk}^{(i)} = \langle [z_i p_k], w_h\rangle$. Moreover, the matrices of the maps Z_1' and Z_2' w.r. to the dual basis are N_1^T and N_2^T respectively. In fact, letting $Z_i' w_j = \sum_h t_{hj}^{(i)} w_h$, $i = 1,2$, we have

$$\langle [p_k], Z_i' w_j\rangle = \sum_h t_{hj}^{(i)} \langle [p_k], w_h\rangle = t_{kj}^{(i)} \tag{3.1}$$

and, using the duality,

$$\langle [p_k], Z_i' w_j\rangle = \langle [z_i p_k], w_j\rangle = \sum_r n_{rk}^{(i)} \langle p_r, w_j\rangle = n_{jk}^{(i)} \tag{3.2}$$

Comparing (3.1) and (3.2) gives the result.

Let $\mathbf{x}(0,0) \in \mathbf{R}^n$ denote the vector in $\mathbf{R}^n$ whose elements are the components of w with respect to the dual basis

$$\mathbf{x}(0,0) = \begin{bmatrix} \langle [p_1], w\rangle \\ \langle [p_2], w\rangle \\ \dots \\ \langle [p_n], w\rangle \end{bmatrix}$$

and $C := [\,c_1 \quad c_2 \quad \dots \quad c_n\,]$ denote the row vector of components of $[1]$ w.r.to the basis in $A_\pm/I_\pm$. The computation of the coefficients of the series w (i.e. the signal values) is quite simple:

$$(z_1^{-h} z_2^{-k} w, 1) = \langle [1], (Z_1')^h (Z_2')^k w\rangle = C (N_1^T)^h (N_2^T)^k \mathbf{x}(0,0)$$

Once a "state vector"

$$\mathbf{x}(h,k) = \begin{bmatrix} \langle [p_1], (Z_1')^h (Z_2')^k w\rangle \\ \langle [p_2], (Z_1')^h (Z_2')^k w\rangle \\ \dots \\ \langle [p_n], (Z_1')^h (Z_2')^k w\rangle \end{bmatrix}$$

has been attached to every point in $\mathbf{Z} \times \mathbf{Z}$, the state and the signal values at (h,k) are uniquely determined by $\mathbf{x}(0,0)$ according to the equations

$$\begin{aligned} \mathbf{x}(h,k) &= (N_1^T)^h (N_2^T)^k \mathbf{x}(0,0) \\ w(h,k) &= C\mathbf{x}(h,k) \end{aligned}$$

When p_j, $j = 1,2,\dots n$ are monic monomials, i.e. $p_j = z_1^{\mu_j} z_2^{\nu_j}$, $j = 1,2,\dots,n$, the element w_j of the dual basis is the unique element of $\mathcal{B}$ taking the values 1 at (μ_j,ν_j) and 0 at (μ_i,ν_i), $i = 1,2,\dots,j-1,j+1,\dots,n$. Moreover, for every $(h,k) \in \mathbf{Z}\times\mathbf{Z}$, the components of the state vector $\mathbf{x}(h,k)$ are the values of w at $\{(\mu_1+h,\nu_1+k),(\mu_2+h,\nu_2+k),\dots,(\mu_n+h,\nu_n+k)\}$.

In the following subsection we shall apply the Gröbner basis algorithm [10] to the construction of a monomial basis in $A_\pm/I_\pm$ and to the computation of the matrices N_1^T and N_2^T that provide the state updating structure.

3.2 Computational Methods

Let $\mathcal{G}_+ = \{g_1, g_2, \dots, g_h\}$, $g_i \in A_+$, be a Gröbner basis of the ideal I_+ and suppose that $\{q_1 = 1, q_2, \dots, q_m\}$ is the set of monic monomials that are not multiple of the leading power products of any of the polynomials in $\mathcal{G}_+$. Then $\{q_j + I_+\}_{j=1,2,\dots,m}$ is a basis of A_+/I_+ and it is easy to compute a pair of commuting matrices M_1 and M_2 that represent the linear transformations

$$\phi_i : A_+/I_+ \to A_+/I_+ : q + I_+ \to z_i q + I_+, \; i = 1,2$$

w.r.to that basis. In [11] it has been shown that the annihilating polynomials of M_1 and M_2 are exactly the polynomials of the ideal I_+, i.e.

$$p(M_1, M_2) = 0 \Leftrightarrow p \in I_+$$

The construction of the invertible matrices N_1 and N_2 introduced in the previous section can be viewed as a further step of the above algorithm and sheds some light on the connections between $I_\pm$ and I_+.

Let μ be a positive integer such that

$$L_i = \operatorname{Im} M_i^\mu = \operatorname{Im} M_i^{\mu+1}, \quad i = 1,2$$

Then $L = L_1 \cap L_2$ is an M_1-M_2-invariant subspace such that $M_1 L = M_2 L = L$. Therefore the subspace $\mathcal{L}$ of A_+/I_+ spanned by

$$[z_1^{\nu_1} z_2^{\nu_2} q_1], [z_1^{\nu_1} z_2^{\nu_2} q_2], \dots, [z_1^{\nu_1} z_2^{\nu_2} q_m], \quad \nu_1 \ge \mu,\ \nu_2 \ge \mu \tag{3.3}$$

does not depend on ν_1 and ν_2.

Let $\nu_1 = \nu_2 = \mu$ and let S be a boolean matrix that selects in (3.3) a maximal independent subset

$$\{[z_1^\mu z_2^\mu q_1]\,[z_1^\mu z_2^\mu q_2]\,\dots\,[z_1^\mu z_2^\mu q_m]\}S = \{[z_1^\mu z_2^\mu q_{i_1}]\,[z_1^\mu z_2^\mu q_{i_2}]\dots[z_1^\mu z_2^\mu q_{i_n}]\}$$

Note that, because of the assumption on L,

$$\begin{aligned} &\{[z_1^{\mu+h} z_2^{\mu+k} q_1]\,[z_1^{\mu+h} z_2^{\mu+k} q_2]\,\dots\,[z_1^{\mu+h} z_2^{\mu+k} q_m]\}S \\ &= \{[z_1^{\mu+h} z_2^{\mu+k} q_{i_1}]\,[z_1^{\mu+h} z_2^{\mu+k} q_{i_2}]\dots[z_1^{\mu+h} z_2^{\mu+k} q_{i_n}]\} \end{aligned}$$

still provides a basis of $\mathcal{L}$, for any pair of nonnegative h and k.

Proposition 3.1 The polynomials $q_{i_1}, q_{i_2}, \dots, q_{i_n}$ constitute a basis of $A_\pm$, modulo $I_\pm$.

PROOF Suppose that $\sum_{h=1}^n \alpha_h q_{i_h}$ is in $I_\pm$.
Then there exists a positive integer ν such that $\sum_{h=1}^n \alpha_h q_{i_h} z_1^\nu z_2^\nu$ and, *a fortiori*, $\sum_{h=1}^n \alpha_h q_{i_h} z_1^{\mu+\nu} z_2^{\mu+\nu}$ belong to I_+.
Since the polynomials

$$q_{i_1} z_1^{\mu+\nu} z_2^{\mu+\nu}, q_{i_2} z_1^{\mu+\nu} z_2^{\mu+\nu}, \dots, q_{i_n} z_1^{\mu+\nu} z_2^{\mu+\nu}$$

are linearly independent modulo I_+, we have $\alpha_h = 0$, $h = 1,2,\dots,n$, and q_{i_h}, $h = 1,2,\dots,n$ are linearly independent modulo $I_\pm$.

Consider now a polynomial $p \in A_\pm$. Then there exists a positive integer ν such that $z_1^\nu z_2^\nu p \in A_+$. Therefore

$$\begin{aligned} (z_1^\nu z_2^\nu p) z_1^\mu z_2^\mu &= \sum_{h=1}^n \alpha_h q_{ih} z_1^\mu z_2^\mu = \sum_{h=1}^n \beta_h q_{ih} z_1^{\mu+\nu} z_2^{\mu+\nu} \mod I_+ \\ &= \sum_{h=1}^n \beta_h q_{ih} z_1^{\mu+\nu} z_2^{\mu+\nu} \mod I_\pm \end{aligned}$$

Multiplying both sides by $z_1^{-\mu-\nu} z_2^{-\mu-\nu}$ we have

$$p = \sum_{h=1}^n \beta_q q_{ih} \mod I_\pm,$$

showing that the polynomials q_{ih} generate $A_\pm$ modulo $I_\pm$.

Once $[q_{i_1}], [q_{i_1}], \dots, [q_{i_n}]$ has been selected as a basis in $A_\pm/I_\pm$, we are interested in constructing the commuting invertible matrices that represent Z_1 and Z_2. Consider first the matrices N_1 and N_2 that represent the restrictions of ϕ_1 and ϕ_2 to $\mathcal{L}$. We have

$$\begin{aligned} &\phi_1\{[z_1^\mu z_2^\mu q_{i_1}], [z_1^\mu z_2^\mu q_{i_2}], \dots, [z_1^\mu z_2^\mu q_{i_n}]\} \\ &= \phi_1\{[z_1^\mu z_2^\mu q_1], [z_1^\mu z_2^\mu q_2], \dots, [z_1^\mu z_2^\mu q_m]\}S \\ &= \phi_1\{[q_1], [q_2], \dots, [q_m]\} M_1^\mu M_2^\mu S \\ &= \{[q_1], [q_2], \dots, [q_m]\} M_1^{\mu+1} M_2^\mu S \end{aligned}$$

On the other hand

$$\begin{aligned} &\phi_1\{[z_1^\mu z_2^\mu q_{i_1}], [z_1^\mu z_2^\mu q_{i_2}], \dots, [z_1^\mu z_2^\mu q_{i_n}]\} \\ &:= \{[z_1^\mu z_2^\mu q_{i_1}], [z_1^\mu z_2^\mu q_{i_2}], \dots, [z_1^\mu z_2^\mu q_{i_n}]\} N_1 \\ &= \{[q_1], [q_2], \dots, [q_m]\} M_1^\mu M_2^\mu S N_1 \end{aligned}$$

We therefore have $M_1^{\mu+1} M_2^\mu S = M_1^\mu M_2^\mu S N_1$. Since $M_1^\mu M_2^\mu S$ has full column rank, letting

$$H := (M_1^T)^\mu (M_2^T)^\mu M_1^\mu M_2^\mu$$

we obtain

$$N_1 = (S^T H S)^{-1} (S^T H M_1 S)$$

and, similarly,

$$N_2 = (S^T H S)^{-1} (S^T H M_2 S)$$

3.3 Vector case

Suppose now that R is a $t \times q$ full column rank right prime matrix, describing a q variables behaviour. All concepts previously introduced for the scalar case have an immediate extension to the vector case. Let

$$A^q_+ := \mathbf{R}^{1\times q}[z_1, z_2]$$

$$A^q_\pm := \mathbf{R}^{1\times q}[z_1, z_2, z_1^{-1}, z_2^{-1}]$$

and define the map

$$|\cdot| : A^q_\pm \to A^q_+ : r \mapsto |r| := z_1^i z_2^j r,$$

where i and j are the minimum degrees of r w.r. to the indeterminates z_1 and z_2 respectively. In case $p = 0$, we define $|p| = 0$.

Let $\mathcal{M}_\pm := (r_1, \ldots, r_t)_\pm$ be the module in $A^q_\pm$ generated by the rows of R and $\mathcal{M}_+ := (|r_1|, \ldots, |r_t|)_+$ the module in A^q_+ generated by the rows of the matrix

$$\bar{R} := \begin{bmatrix} |r_1| \\ \vdots \\ |r_t| \end{bmatrix} = \Lambda R$$

where $\Lambda := \operatorname{diag}\{z_1^{\nu_1} z_2^{\mu_1}, \ldots, z_1^{\nu_t} z_2^{\mu_t}\}$ and ν_i and μ_i satisfy $z_1^{\mu_i} z_2^{\nu_i} r_i = |r_i|$, $i = 1, \ldots, t$.

Lemma 3.3

(i) A row r belongs to $\mathcal{M}_\pm$ if and only if there exists a pair of integers (i,j) such that $z_1^i z_2^j r$ is in $\mathcal{M}_+$.

(ii) $A^q_\pm/\mathcal{M}_\pm$ is finite dimensional if and only if $A^q_+/\mathcal{M}_+$ is finite dimensional.

PROOF: (i) Obvious.

(ii) Suppose that $A^q_+/\mathcal{M}_+$ is finite dimensional. This implies that there exist polynomials $f_i(z_1)$ and $g_i(z_2)$, $i = 1, \ldots, q$, such that

$$f_i(z_1) e_i^T \in \mathcal{M}_+$$

$$g_i(z_2) e_i^T \in \mathcal{M}_+$$

where e_i is the element of the canonical basis of $\mathbf{R}^q$ with 1 in position i. It is easily seen that, if $\chi_i := \deg f_i$ and $\gamma_i := \deg g_i$, then

$$\bigcup_{i=1,\ldots,q} \{z_1^h z_2^k e_i^T + \mathcal{M}_+, 0 \le h < \chi_i, 0 \le k < \gamma_i\}$$

constitutes a set of generators for $A^q_\pm/\mathcal{M}_\pm$.

Conversely suppose that $A^q_\pm/\mathcal{M}_\pm$ is finite dimensional and let D be any right factor of $\bar{R}$

$$\bar{R} = \hat{R} D.$$

It follows that $R = \Lambda^{-1}\bar{R} = \Lambda^{-1}\hat{R}D$, where Λ^{-1} is still a polynomial matrix with elements in $A_\pm$. Since the module $\mathcal{M}_\pm(D)$ generated by the rows of D, satisfies $\mathcal{M}_\pm \subseteq \mathcal{M}_\pm(D)$, we have

$$\dim A^q_\pm/\mathcal{M}_\pm \ \geq \ \dim A^q_\pm/\mathcal{M}_\pm(D)$$

and $A^q_\pm/\mathcal{M}_\pm(D)$ is finite dimensional. Therefore there exist polynomials $f_i(z_1)$ and $g_i(z_2), i = 1, \ldots, q$, such that

$$f_i(z_1) e_i^T \in \mathcal{M}_\pm(D)$$

$$g_i(z_2) e_i^T \in \mathcal{M}_\pm(D)$$

and, consequently, there exist polynomial matrices H and K such that

$$HD = \operatorname{diag}\{f_1(z_1), \ldots, f_q(z_1)\} \qquad (3.4)$$

$$KD = \operatorname{diag}\{g_1(z_2), \ldots, g_q(z_2)\} \qquad (3.5).$$

(3.4) implies that det D is a polynomial in z_1 and (3.5) implies that det D is a polynomial in z_2. Therefore D is unimodular and $\bar{R}$ is right factor prime. Therefore $A^q_+/\mathcal{M}_+$ is finite dimensional.

Denote now by $\mathcal{U}^q$ the universe of formal power series in z_1, z_2, z_1^{-1}, z_2^{-1} with q components and introduce a bilinear function $\langle\cdot,\cdot\rangle$ in $A^q_\pm \times \mathcal{U}^q$ letting

$$\langle r, w\rangle = (\tilde{r}w, 1).$$

Note that if r is the polynomial row $r := \sum_{i,j\in\mathbf{Z}} r_{ij} z_1^i z_2^j$, $\tilde{r}$ is given by $\sum_{i,j\in\mathbf{Z}} r_{ij} z_1^{-i} z_2^{-j}$ and $\langle r, w\rangle$ is the constant term in the Cauchy product $\tilde{r}w$. Then $\mathcal{U}^q$ is isomorphic to the algebraic dual of $A^q_\pm$ and we still have

$$\mathcal{B} = \mathcal{M}_\pm^\perp$$

$$\mathcal{B}^\perp = \mathcal{M}_\pm^{\perp\perp} = \mathcal{M}_\pm.$$

Moreover $\mathcal{B}$ and $A^q_\pm/\mathcal{M}_\pm$ are finite dimensional isomorphic vector spaces.

As in the scalar case, if N_1 and N_2 are the matrices of the linear trasformations

$$Z_1 : A^q_\pm/\mathcal{M}_\pm \to A^q_\pm/\mathcal{M}_\pm : [r] \mapsto [z_1 r]$$

$$Z_2 : A^q_\pm/\mathcal{M}_\pm \to A^q_\pm/\mathcal{M}_\pm : [r] \mapsto [z_2 r]$$

w.r. to the basis $[r_1], \ldots, [r_n]$ of $A^q_\pm/\mathcal{M}_\pm$ and if the state vector relative to $w \in \mathcal{B}$ is defined as

$$\mathbf{x}(h,k) := \begin{bmatrix} \langle Z_1^h Z_2^k [r_1] \ , \ w\rangle \\ \vdots \\ \langle Z_1^h Z_2^k [r_n] \ , \ w\rangle \end{bmatrix},$$

then

$$\mathbf{x}(h,k) := (N_1^T)^h (N_2^T)^k \mathbf{x}(0,0).$$

Moreover, if C is a $q \times n$ constant matrix such that

$$\begin{bmatrix} [e_1^T] \\ \vdots \\ [e_q^T] \end{bmatrix} = C \begin{bmatrix} [r_1] \\ \vdots \\ [r_n] \end{bmatrix},$$

then $w(h,k) = C\mathbf{x}(h,k)$.

By applying the theory of Gröbner basis over modules [12], a basis in $A^q_+/\mathcal{M}_+$ with elements of the type $z_1^h z_2^k e_i^T + \mathcal{M}_+$ can be easily obtained. After computing the matrices M_1 and M_2 that represent the transformations

$$\phi_i : A^q_+/\mathcal{M}_+ \to A^q_+/\mathcal{M}_+ : r + \mathcal{M}_+ \mapsto z_i r + \mathcal{M}_+, \qquad i = 1, 2$$

w.r. to that basis, the procedure for extracting N_1 and N_2 from M_1 and M_2 is the same we introduced in the scalar case.

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