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Abstract—**Necessary and sufficient conditions for the existence of a decoupling bicausal precompensator for multivariable 2-D systems are derived in state space and frequency domains. In general, the decoupling problem for 2-D systems can be solved by feedback compensators if suitable injectivity assumptions are introduced on the input-state matrices. The structure of dynamic compensators is derived for this case and the 2-D decoupling problem with stability is solved.**

I. Introduction

Since the late sixties, the decoupling problem constitutes one of the most attractive research topics in multivariable 1-D systems theory. Besides several appealing consequences in the applications, the interest in this field relies on the analytical tools that have been introduced in developing the underlying theory. The decoupling schemes considered in the literature have different characteristics. These include the topology of the interconnections (based on the use of precompensators, feedback compensators, or compound strategies), the dynamical characteristics of the subsystems that enter in the interconnections, the use of state-space or input–output models and, finally, the algebraic structures (fields, rings) which provide the framework where the systems are defined [1]–[5]. In most applications, we are required to solve at the same time the decoupling and the stabilization problems. In these cases, state or output feedbacks have to be considered and only those schemes that include dynamic compensators become relevant to the solution.

2-D systems provide input–output and state-space models representing physical processes which depend on two independent variables. In some cases, one of these variables is time and the other represents a spatial dimension (as in the study of some classes of distributed parameter systems and delay differential systems), while for other problems, such as image processing, none of the independent variables can be sought of as time. Typically, they apply to two-dimensional data processing in several fields, such as seismology, X-ray image enhancement, image deblurring, digital picture processing, etc. Also, 2-D systems constitute a natural framework for modeling multivariable networks, large scale systems obtained by interconnecting many subsytems and, in general, physical processes where both space and time have to be taken into account [6], [7].

Recently, the feedback control theory of 2-D systems attracted the interest of research people and a great deal of attention has been devoted to problems related to stabilization and characterization of closed-loop characteristic polynomials [8]–[11]. Moreover, the systematic application of 2-D polynomial matrices techniques allowed us to extend the original single-input single-output analysis to include multivariable 2-D systems.

In this note, we aim to analyze how 2-D compensators apply to noninteracting control of multivariable 2-D systems and to find necessary and sufficient conditions for the existence of a feedback law that makes the closed-loop transfer matrix diagonal and nonsingular. We shall tackle this problem using MFD's in two variables, applied to input–output and state-space models. It is worthwhile to remark that several equivalent strategies, based on bicausal precompensators, static precompensators and compensators, static precompensators and dynamic compensators can be implemented in generating noninteracting controls for 1-D systems. As we shall see, in the case of 2-D systems these strategies are not equivalent, since they allow decoupling of different classes of systems.

The state equation of a multivariable 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ having m inputs and m outputs is given by

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h+1, k) + A_2 x(h, k+1) \\ &\quad + B_1 u(h+1, k) + B_2 u(h, k+1) \\ y(h, k) &= Cx(h, k) + Du(h, k) \end{aligned} \tag{1.1}$$

where u and y are the m-dimensional vectors of input and output values, x is an n-dimensional local state vector, and A_1, A_2, B_1, B_2, C, D are matrices of appropriate dimensions. In the following, we shall adopt the standard convention that a scalar sequence $\{s(h, k)\}$ with nonnegative indexes h, k is associated with a formal power series $\Sigma s(h, k) z_1^h z_2^k$ having nonnegative powers in z_1 and z_2. According to this convention, a proper (strictly proper) rational function can be represented as a quotient $p(z_1, z_2)/q(z_1, z_2)$ of coprime polynomials with $q(0, 0) \neq 0$ ($q(0, 0) \neq 0$ and $p(0, 0) = 0$).

Therefore, the transfer matrix of Σ is the $m \times m$ rational matrix

$$W(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D \tag{1.2}$$

whose entries are proper rational functions in two variables. The system (1.1) is called *strictly proper* if $D = 0$ and *bicausal* if D is an invertible matrix. It is immediate to see that Σ is strictly proper if $W(0, 0) = 0$ and bicausal if $W(0, 0)$ is an invertible matrix.

Because of the structure of 2-D systems, a number of different state feedback schemes is allowed. The simplest of these is represented by the static control law

$$u(h, k) = Kx(h, k), \qquad K \in R^{m \times n}. \tag{1.3}$$

Comparing to static state feedback in 1-D theory, the possibilities of modifying the dynamical behavior by applying (1.3) are much poorer [12].

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we obtain

$$W = \bar{N}Q_0^{-1}(I + D_S)^{-1}. \qquad (2.5)$$

Denoting by $\bar{n}_{ij}$, $(\Delta_1)_{ij}$, and $(N_S)_{ij}$ the (i,j)-indexed element of $\bar{N}$, Δ_1, and N_S, respectively, it is easy to see that $\bar{n}_{ii} = (\Delta_1)_{ii}(1 + (N_S)_{ii}) \neq 0$ and $\bar{n}_{ij} = (\Delta_1)_{ii}(N_S)_{ij}$ if $i \neq j$ so that $o(\bar{n}_{ii}) < o(\bar{n}_{ij})$ $i \neq j$. This implies that $\bar{N}$ is diagonally ordered and so $\bar{n}_{ij}/\bar{n}_{ii} = (N_S)_{ij}/(1 + (N_S)_{ii})$, $i \neq j$ are strictly proper. Hence, from 1) we have that there exists an MFD with structure (2.4) that satisfies 2). Actually, we shall prove that all MFD's having structure (2.4) satisfy 2).

To this end, reduce (2.5) to a right coprime MFD by eliminating a g.c.r.d. of $\bar{N}Q_0$ and $(I + D_S)$. It is not restrictive to assume that the g.c.r.d. can be written as $I + F_S$, with $F_S(0, 0) = 0$ and the corresponding factorization is

$$\bar{N}Q_0 = (\tilde{N}Q_0)(I + F_S),\ I + D_S = (I + \tilde{D}_S)(I + F_S) \quad (2.6)$$

with $\tilde{D}_S(0, 0) = 0$. In this way, we get a right coprime MFD

$$W = (\tilde{N}Q_0)(I + \tilde{D}_S)^{-1}. \qquad (2.7)$$

Since $\bar{N} = \tilde{N}(I + Q_0F_SQ_0^{-1})$, $\tilde{N}$ satisfies property 2). Finally, consider any MFD of W having a structure (2.4). By the coprimeness of (2.7), there exists a g.c.r.d. G of N and $(I + D_S)$ such that

$$N = \tilde{N}Q_0G \qquad (2.8)$$

$$I + D_S = (I + \tilde{D}_S)G \qquad (2.9)$$

hold. It is straightforward to see from (2.9) that G can be written as $(I + G_S)$, with $G_S(0, 0) = 0$, so that $\bar{N} = NQ_0^{-1} = \tilde{N}(I + Q_0G_SQ_0^{-1})$ satisfy property 2).

Vice versa, assume that property 2) holds for an MFD having structure (2.4) and rewrite $\bar{N}$ as

$$\bar{N} = \begin{bmatrix} \bar{n}_{11} & & & \\ & \bar{n}_{22} & & \\ & & \ddots & \\ & & & \bar{n}_{mm} \end{bmatrix} \cdot \begin{bmatrix} 1 & \bar{n}_{12}/\bar{n}_{11} & \cdots & \bar{n}_{1m}/\bar{n}_{11} \\ \bar{n}_{21}/\bar{n}_{22} & 1 & \cdots & \bar{n}_{2m}/\bar{n}_{22} \\ & \cdots\cdots & & \\ \bar{n}_{m1}/\bar{n}_{mm} & \bar{n}_{m2}/\bar{n}_{mm} & \cdots & 1 \end{bmatrix}.$$

Then, assuming $\Delta = \mathrm{diag}\{\bar{n}_{11}\bar{n}_{22} \cdots \bar{n}_{mm}\}$ and

$$V = \left(I + \begin{bmatrix} 0 & \bar{n}_{12}/\bar{n}_{11} & \cdots & \bar{n}_{1m}/\bar{n}_{11} \\ \bar{n}_{21}/\bar{n}_{22} & 0 & \cdots & \bar{n}_{2m}/\bar{n}_{22} \\ & \cdots\cdots & & \\ \bar{n}_{m1}/\bar{n}_{mm} & \bar{n}_{m2}/\bar{n}_{mm} & \cdots & 1 \end{bmatrix} \right) \cdot Q_0^{-1}(I + D_S)^{-1}$$

the transfer matrix factorizes in the following form:

$$W = \bar{N}Q_0^{-1}(I + D_S)^{-1} = \Delta V. \qquad (2.10)$$

Since the constant term of the series expansion of V is given by the nonsingular matrix Q_0^{-1}, it turns out that V is bicausal. Consequently, property 1) holds. ■

The aforementioned proposition contains a criterion for the existence of a decoupling bicausal precompensator. Actually, from (2.10) we have $WV^{-1} = \Delta$, so that V^{-1} also provides the transfer matrix of the precompensator. In order to exploit this result, it is worthy to notice that we need an algorithm for deciding, starting from N, whether there exists a constant matrix Q_0 such that $\bar{N} := NQ_0$ satisfies property 2). To this purpose, denote by ν_i the minimum order of the polynomials in the ith row of N and by r_i the row vector constituted by the homogeneous polynomials of degree ν_i that belong to the ith row of N.

Letting $R^T = [r_1^T r_2^T \cdots r_m^T]$, decompose N as $N = R + S$, so that the elements of the ith row of S are polynomials of order greater than ν_i.

Now observe that, if property 2) holds for some Q_0, the matrix $\bar{R}$ defined by

$$\bar{R} := RQ_0 \qquad (2.11)$$

is diagonal and nonsingular. This follows directly from the fact that $\bar{n}_{ij}/\bar{n}_{ii}$ is strictly proper rational for $i \neq j$. On the other hand, the property that Q_0 diagonalizes R in (2.11) determines Q_0 modulo a constant nonsingular diagonal right factor. Consequently, once we have obtained a matrix Q_0 that diagonalizes R, the whole set of diagonalizing matrices satisfies property 2) if and only if Q_0 does. Finally, the diagonalizability of R in (2.11) implies that the structure of the ith row r_i is

$$r_i = \bar{R}_{ii}[p_{i1}\, p_{i2} \cdots p_{im}] \qquad i = 1, 2, \cdots, m \quad (2.12)$$

where $\bar{R}_{ii}$ is the homogeneous polynomial in the (i, i)th position of $\bar{R}$ and $[p_{i1}\, p_{i2} \cdots p_{im}]$ is the ith row of Q_0^{-1}. Based on the previous remarks, we have an algorithm for checking property 2) in Proposition 1 which consists of the following steps.

Step 1: For $i = 1, 2, \cdots m$, extract the homogeneous row r_i from the ith row of R.

Step 2: For $i = 1, 2, \cdots m$, decompose r_i in the form (2.12).

Step 3: Check if $[p_{ij}]$ is an invertible matrix. Then set $Q_0 = [p_{ij}]^{-1}$.

III. Decoupling Bicausal Precompensators: State Variable Approach

In this section, we are concerned with the existence of a decoupling bicausal precompensator for a strictly proper 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ represented by the state updating equations (1.1). As we shall see, the conditions that will be derived are only partially reminiscent of those obtained in [1] for 1-D state-space systems. In fact, the 1-D decouplability condition can be expressed as a rank condition on a constant matrix, which allows us to construct a decoupling static feedback law, while the decoupling compensators for 2-D systems are dynamical systems and the decouplability condition is expressed in terms of algebraic properties of a polynomial matrix in two indeterminates.

To shorten our notations, we write $\mathscr{A} := A_1z_1 + A_2z_2$ and $\mathscr{B} := B_1z_1 + B_2z_2$.

As a preliminary remark, we note that the integers $\nu_1, \nu_2, \cdots, \nu_m$ introduced in the previous section to define R can be obtained from the state model assuming

$$d_i := \min\{j: C_i\mathscr{A}^j\mathscr{B} \neq 0,\ j = 0, 1, \cdots, n-1\};\ \nu_i := d_i + 1.$$

Clearly, the existence of $d_1, d_2, \cdots, d_m$ is guaranteed if and only if the system transfer matrix $W(z_1, z_2)$ is nonsingular. Actually, the nonsingularity of W is necessary to solve the decoupling problem and in the sequel this condition will be always assumed.

Proposition 2: Let M_0 be the $m \times m$ 2-D polynomial matrix given by

$$M_0 = \begin{bmatrix} C_1\mathscr{A}^{d_1}\mathscr{B} \\ C_2\mathscr{A}^{d_2}\mathscr{B} \\ \vdots \\ C_m\mathscr{A}^{d_m}\mathscr{B} \end{bmatrix}. \qquad (3.1)$$

Then the system can be decoupled by a decoupling bicausal precompensator if and only if: 1) there exists a constant nonsingular matrix Q_0 such that $M_0Q_0 = \mathrm{diag}\{\epsilon_1\epsilon_2 \cdots \epsilon_m\}$, where ϵ_i, $i = 1, 2, \cdots, m$ are homogeneous 2-D polynomials of degree $d_i + 1$; and 2) $M_0^{-1}C(I - \mathscr{A})^{-1}\mathscr{B}$ is proper rational.

Proof: Assume that 1) and 2) hold. It is immediate to see that $p := \det M_0$ is a homogeneous polynomial of degree $m + \Sigma_i d_i$ and that the ith column of adj M_0 is a homogeneous polynomial vector of degree $m - 1 + \Sigma_{j=i} d_j$, $i = 1, 2, \cdots, m$.

Consider the following series expansion of the transfer matrix:

$$C(I - \mathscr{A})^{-1}\mathscr{B} = \begin{bmatrix} C_1 \mathscr{A}^{d_1}\mathscr{B} \\ C_2 \mathscr{A}^{d_2}\mathscr{B} \\ \cdots \\ C_m \mathscr{A}^{d_m}\mathscr{B} \end{bmatrix} + \begin{bmatrix} C_1 \mathscr{A}^{d_1+1}\mathscr{B} \\ C_2 \mathscr{A}^{d_2+1}\mathscr{B} \\ \cdots \\ C_m \mathscr{A}^{d_m+1}\mathscr{B} \end{bmatrix} + \cdots = M_0 + M_1 + \cdots \tag{3.2}$$

and premultiply both sides by M_0^{-1}. We obtain

$$M_0^{-1} C(I - \mathscr{A})^{-1}\mathscr{B} = p^{-1}\left[(\text{adj } M_0) M_0 + (\text{adj } M_0) M_1 + \cdots\right]. \tag{3.3}$$

The degrees of the nonzero polynomials in the matrices (adj $M_0)M_r := [p_{ij}^{(r)}]$, $r = 0, 1, \cdots$ are given by $\deg p_{ij}^{(r)} = m + r + \Sigma_i d_i$.

By Assumption 2), the left-hand side of (3.3) admits a power series expansion

$$M_0^{-1} C(I - \mathscr{A})^{-1}\mathscr{B} = \mathscr{P}_0 + \mathscr{P}_1 + \cdots \tag{3.4}$$

where $\mathscr{P}_i$ are homogeneous matrices of degree i. Comparing (3.3) and (3.4) and equating the homogeneous terms of the same degree, we have (adj $M_0)M_0 = p\mathscr{P}_0$ and, hence, $\mathscr{P}_0 = p^{-1}(\text{adj } M_0)M_0 = I$. This implies that the matrix

$$M_0^{-1} C(I - \mathscr{A})^{-1}\mathscr{B} = I + \mathscr{P}_1 + \mathscr{P}_2 + \cdots \tag{3.5}$$

is a bicausal transfer matrix. Recalling Assumption 1), by (3.5) we have

$$C(I - \mathscr{A})^{-1}\mathscr{B}[I + \mathscr{P}_1 + \mathscr{P}_2 + \cdots]^{-1} Q_0 = M_0 Q_0 = \text{diag}\{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\}$$

which shows that $[I + \mathscr{P}_1 + \mathscr{P}_2 + \cdots]^{-1} Q_0$ is a decoupling bicausal compensator.

Conversely, suppose that there exists a decoupling bicausal precompensator with transfer matrix $U = U_0 + U_S$, where U_0 is a nonsingular constant matrix and U_S is a strictly proper rational matrix. Then

$$C(I - \mathscr{A})^{-1}\mathscr{B}(U_0 + U_S) = \text{diag}\{\delta_1, \delta_2, \cdots, \delta_m\} \tag{3.6}$$

where δ_i are proper rational functions.

Denote by ϵ_i the homogeneous polynomial of minimum degree in the series expansion of δ_i and equate the minimum degree homogeneous rows on both sides of (3.6). We obtain

$$\begin{bmatrix} C_1 \mathscr{A}^{d_1}\mathscr{B} \\ C_2 \mathscr{A}^{d_2}\mathscr{B} \\ \vdots \\ C_m \mathscr{A}^{d_m}\mathscr{B} \end{bmatrix} U_0 = \text{diag}\{\epsilon_1 \epsilon_2 \ \cdots \ \epsilon_m\}.$$

Therefore, the property 1) holds with $Q_0 = U_0$.

It remains to prove property 2). Consider an MFD of the transfer matrix given by

$$C(I - \mathscr{A})^{-1}\mathscr{B} = N(I + D_S)^{-1} \tag{3.7}$$

where $\bar{N} = NQ_0$ satisfies condition 2) of Proposition 1 and $D_S(0, 0) = 0$. Let $\Delta := \text{diag}\{\bar{n}_{11}\bar{n}_{22} \cdots \bar{n}_{mm}\}$ and rewrite $\bar{N}$ in the form $\bar{N} = \Delta + P$. Then the aforementioned condition implies that in $\bar{N} = \Delta(I + \Delta^{-1}P)$ the matrix fraction $\Delta^{-1}P$ is strictly proper. By (3.2) and (3.7) we have

$$\begin{aligned} M_0 + M_1 + M_2 + \cdots &= \bar{N}Q_0^{-1}(I + D_S)^{-1} = \bar{N}(I + Q_0^{-1}D_S Q_0)^{-1} Q_0^{-1} \\ &= \Delta(I + \Delta^{-1}P)(I + Q_0^{-1}D_S Q_0)^{-1} Q_0^{-1} \end{aligned} \tag{3.8}$$

and hence $M_0 = \Delta Q_0^{-1}$. Thus, premultiplying (3.8) by M_0^{-1}, we see that the matrix

$$M_0^{-1} C(I - \mathscr{A})^{-1}\mathscr{B} = Q_0(I + \Delta^{-1}P)(I + Q_0^{-1}D_S Q_0)^{-1} Q_0^{-1}$$

is proper rational.

IV. Dynamic Feedback Decoupling

To carry through the analysis of the feedback decoupling scheme for a strictly proper 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$, an important remark is that the application of dynamic state feedback together with static precompensation produces transfer matrices that can be obtained also using suitable bicausal precompensators.

In fact, let $K(z_1, z_2)$ and Q_0 be the transfer matrices of the compensator and the precompensator, respectively. Then the transfer matrix of the closed-loop system is given by

$$W(z_1, z_2)\left[(I - K(z_1, z_2)(I - \mathscr{A})^{-1}\mathscr{B})^{-1} Q_0\right] \tag{4.1}$$

and the term in square brackets can be viewed as the transfer matrix of a bicausal precompensator.

A significant difference with respect to 1-D system is that, given a bicausal precompensator $U(z_1, z_2)$, the 2-D transfer matrix WU need not be implementable using a dynamic state feedback compensator and a static precompensator.

This is illustrated by the following example. Consider the system $\Sigma = (A_1, A_2, B_1, B_2, C)$ given by

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the transfer matrix of Σ

$$W(z_1, z_2) = \begin{bmatrix} z_1 & 0 \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} 1 & z_2 \\ z_2 & 1 \end{bmatrix}^{-1}$$

satisfies condition 2) of Proposition 1, there exists a decoupling bicausal precompensator. Its transfer matrix is easily computed and is given by

$$U(z_1, z_2) = \begin{bmatrix} 1 & z_2 \\ z_2 & 1 \end{bmatrix}.$$

However, we cannot decouple the system adopting state feedback and static precompensation, since, in this case, it is not possible to find a constant nonsingular Q_0 and a proper rational $K(z_1, z_2)$ that make the matrix (4.1) diagonal.

Our aim is now to find structural conditions on the matrices of the state model (1.1) which guarantee that a decoupling bicausal compensator $U(z_1, z_2)$ can be replaced by a feedback compensator $K(z_1, z_2)$ and, possibly, a static precompensator Q_0.

These conditions correspond to assuming that, for any bicausal $U(z_1, z_2)$, there exist proper rational $K(z_1, z_2)$ and nonsingular Q_0 that solve

$$W(z_1, z_2)U(z_1, z_2) = C(I - \mathscr{A} - \mathscr{B}K(z_1, z_2))^{-1}\mathscr{B}Q_0. \tag{4.2}$$

For this, substitute in (4.2) $U_0 + U_S$ for U and the series expansion $C\mathscr{B} + C\mathscr{A}\mathscr{B} + C\mathscr{A}^2\mathscr{B} + \cdots$ for W. Then, looking for a solution of (4.2) in terms of $K(z_1, z_2)$ and Q_0, one starts by choosing $Q_0 = U_0$. Since

$$\begin{aligned} C(I - \mathscr{A})^{-1}\mathscr{B}(I + U_S U_0^{-1}) &= C[I - \mathscr{A} - \mathscr{B}K(z_1, z_2)]^{-1}\mathscr{B} \\ &= C(I - \mathscr{A})^{-1}\mathscr{B}[I - K(z_1, z_2)(I - \mathscr{A})^{-1}\mathscr{B}]^{-1} \end{aligned} \tag{4.3}$$

we are reduced to solving the following equation in $K(z_1, z_2)$:

$$(I + U_S U_0^{-1})^{-1} = I - K(z_1, z_2)(I - \mathscr{A})^{-1}\mathscr{B}. \quad (4.4)$$

Introduce a realization $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}, I)$ of the left-hand side of (4.4) and let

$$\tilde{K}(z_1, z_2) := K(z_1, z_2)(I - \mathscr{A})^{-1}, \quad \tilde{\mathscr{A}} = \tilde{A}_1 z_1 + \tilde{A}_2 z_2,$$
$$\tilde{\mathscr{B}} = \tilde{B}_1 z_1 + \tilde{B}_2 z_2.$$

Therefore, the search for a causal $K(z_1, z_2)$ that solves (4.4) reduces to find a causal $\tilde{K}(z_1, z_2)$ that solves

$$\tilde{C}(I - \tilde{\mathscr{A}})^{-1}\tilde{\mathscr{B}} = -\tilde{K}(z_1, z_2)\mathscr{B}. \quad (4.5)$$

We shall show that, if the matrix $[B_1 \mid B_2]$ is injective, there exists a causal solution of (4.5). In fact, in this case, there exists a left inverse L of $[B_1 \mid B_2]$ and the matrix $F := [\tilde{B}_1 \mid \tilde{B}_2]L$ satisfies

$$F[B_1 \mid B_2] = [\tilde{B}_1 \mid \tilde{B}_2] \quad (4.6)$$

or equivalently, $F(B_1 z_1 + B_2 z_2) = \tilde{B}_1 z_1 + \tilde{B}_2 z_2$.

This implies that a solution of (4.5) is given by the proper rational matrix $\tilde{K}(z_1, z_2) = -\tilde{C}(I - \tilde{\mathscr{A}})^{-1}F$ so that

$$K(z_1, z_2) = -\tilde{C}(I - \tilde{\mathscr{A}})^{-1}F(I - \mathscr{A}) \quad (4.7)$$

constitutes the transfer matrix of a causal decoupling compensator.

The injectivity of $[B_1 \mid B_2]$ does not impose any constraint on the structure of the transfer matrix, since any 2-D transfer matrix admits a realization with $[B_1 \mid B_2]$ injective [9]. So, when dealing with the decoupling problem starting from the state-space equations, we will always assume $[B_1 \mid B_2]$ to be injective.

Example: Consider the 2-D system Σ given by

$$A_1 = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The transfer matrix of Σ is

$$W(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 - B_2 z_2)$$
$$= \begin{bmatrix} \dfrac{z_1 + z_1^2 + z_1^2 z_2 - z_1 z_2^2}{1 - z_1^2 - z_1 z_2} & \dfrac{2z_1 - z_1^2 - z_1 z_2}{1 - z_1^2 - z_1 z_2} \\ -z_2^2 & z_2 \end{bmatrix}.$$

The matrix

$$M_0 = C\mathscr{B} = \begin{bmatrix} z_1 & 2z_1 \\ 0 & z_2 \end{bmatrix}$$

satisfies properties 1) and 2) of Proposition 2. In fact, assuming

$$Q_0 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

the matrix $M_0 Q_0$ is diagonal and $M_0^{-1}W$ is proper rational.

A decoupling bicausal precompensator is then given by

$$\hat{U} = W^{-1}M_0 Q_0 = \frac{1}{1 + z_1 + 2z_2 + 2z_2^2} \cdot \begin{bmatrix} 1 - z_1^2 - z_1 z_2 & -2 + z_1 + z_2 \\ z_2(1 - z_1^2 - z_1 z_2) & 1 + z_1 + z_1 z_2 - z_2^2 \end{bmatrix}.$$

In order to simplify computations, it will be convenient to modify the precompensator by postmultiplying its transfer matrix by a diagonal bicausal factor

$$T = \begin{bmatrix} \dfrac{1}{1 - z_1^2 - z_1 z_2} & 0 \\ 0 & 1 \end{bmatrix}.$$

In this way, $U = \hat{U}T$ is still a decoupling bicausal precompensator for Σ and

$$U^{-1} = \begin{bmatrix} 1 + z_1 + z_1 z_2 - z_2^2 & 2 - z_1 - z_2 \\ -z_2 & 1 \end{bmatrix}$$

is a polynomial matrix. The feedback decoupling scheme includes $U_0 = Q_0$ as static precompensator. For applying (4.7) to obtain the transfer matrix $K(z_1, z_2)$ of a causal dynamic compensator, a realization $(\tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}, I)$ of

$$U_0 U^{-1} = \begin{bmatrix} 1 + z_1 + 2z_2 + z_1 z_2 - z_2^2 & -z_1 - z_2 \\ -z_2 & 1 \end{bmatrix}$$

is needed. It is immediate to verify that the realization given by

$$\tilde{A}_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \tilde{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \tilde{B}_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$
$$\tilde{B}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

satisfies $\tilde{C}(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2)^{-1}(\tilde{B}_1 z_1 + \tilde{B}_2 z_2) + I = U_0 U^{-1}$.

In this example, the matrix F is given by

$$F = [\tilde{B}_1 \mid \tilde{B}_2][B_1 \mid B_2]^{-1} = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

so that

$$K(z_1, z_2) = -\tilde{C}(I - \tilde{\mathscr{A}})^{-1}F(I - \mathscr{A})$$
$$= \begin{bmatrix} z_1 + 2z_2 - 1 & z_1 + 2 & 2z_2 - 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

V. Stable Decoupling

In the previous section, we have shown how to realize a noninteracting control of a 2-D system using dynamic compensators. In general, the decoupled system we obtain need not be internally stable for all choices of the decoupling compensator. So, it is important to decide if, given a 2-D system, there exists stabilizing decoupling compensators and then to have procedures for their construction.

In this section, we will show that, if the system is stabilizable and there exists a noninteracting control, it is possible to select a decoupling compensator which stabilizes the closed-loop system.

More precisely, let the system (1.1) satisfy the following conditions:

1) Stabilizability Condition: The matrix $[I - A_1 z_1 - A_2 z_2 \; B_1 z_1 + B_2 z_2]$ has full rank for all (z_1, z_2) belonging to the unitary polydisk $P_1 = \{(z_1, z_2): |z_1| \leq 1, |z_2| \leq 1\}$.

2) Decouplability Condition: 1) and 2) of Proposition 2.

Then the class of stabilizing compensators contains compensators which are decoupling [for the system (1.1)]. This property depends on the following facts.

1) The stabilization by state feedback preserves the decouplability of the system.

2) If an internally stable system is decouplable, it can be decoupled without losing internal stability.

The proof of the first fact is immediate, since the matrix M_0 defined by (3.1) and relative to the original system coincides with the matrix M_0 that corresponds to the closed-loop system. So that both systems satisfy Condition 1) of Proposition 2. As far as Condition 2) of Proposition 2 is concerned, it is sufficient to note that the transfer matrix W_F of the feedback system differs from $C(I - \mathscr{A})^{-1}\mathscr{B}$ in a bicausal multiplicative factor, so $M^{-1}W_F$ is proper rational since $M^{-1}C(I - \mathscr{A})^{-1}\mathscr{B}$ is.

To prove the second part, observe that, if U is a decoupling bicausal precompensator, $\Delta := WU$ is a proper rational diagonal matrix. Now let h be the l.c.m. of the denominators of the elements of U^{-1}. Then also Uh^{-1} is a bicausal decoupling precompensator and $W(Uh^{-1}) = \Delta h^{-1}$.

Consequently, we can assume in (4.4) that the matrix $(I - U_S U_0^{-1})^{-1} = U_0 U^{-1}$ is polynomial and that its realization $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}, \tilde{D})$ satisfies the condition $\det(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2) = 1$. In this case, the transfer matrix $K(z_1, z_2)$ of the compensator, given by (4.7), is a polynomial matrix.

In order to obtain an internally stable closed-loop system, we construct a coprime realization of $K(z_1, z_2)$, i.e., a 2-D system $\bar{\Sigma} = (\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C}, \bar{D})$, where both matrices

$$\begin{bmatrix} I - \bar{A}_1 z_1 - \bar{A}_2 z_2 & \bar{B}_1 z_1 + \bar{B}_2 z_2 \end{bmatrix}$$

and

$$\begin{bmatrix} I - \bar{A}_1 z_1 - \bar{A}_2 z_2 \\ \bar{C} \end{bmatrix}$$

are full rank for any (z_1, z_2) in $C \times C$ [9], [11].

The state-space model resulting from the feedback connection of Σ and $\bar{\Sigma}$ is internally stable, as a consequence of the following properties:

- The plant Σ is internally stable.
- The compensator $\bar{\Sigma}$ is a coprime realization of a polynomial matrix. Therefore, $\det(I - \bar{A}_1 z_1 - \bar{A}_2 z_2) = 1$, which implies that $\bar{\Sigma}$ is internally stable.
- The closed-loop system is externally stable, since its transfer matrix is the product of the stable matrix $W(z_1, z_2)$ and the polynomial matrix $U(z_1, z_2)$.

VI. Conclusions

In this note, we have tackled the problem of constructing a noninteracting control for an m-inputs n-outputs 2-D system. Necessary and sufficient conditions for the existence of a decoupling bicausal precompensator have been derived both in state space and in frequency domains. We have examined how to implement the action of a bicausal precompensator by means of a state feedback compensator. Differently from the 1-D theory, bicausal precompensation is not equivalent either to static or to dynamic state feedback, unless some further assumptions are introduced on the state-space model.

These assumptions are also sufficient for showing that the 2-D stable decoupling problem can be solved using dynamic compensation.

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