STRUCTURE AND SOLUTIONS OF THE LQ OPTIMAL CONTROL PROBLEM FOR 2D SYSTEMS

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The LQ optimal control problem of 2D systems is addressed and solved in an l_2 environment. The optimal stabilizing control law does not preserve, in general, neither quarter plane nor weak causality. Some preliminary results on system and cost structures guaranteeing that the optimal feedback law is causal or weakly causal are discussed.

1. INTRODUCTION

This paper deals with 2D optimal control problems. The class of discrete time 2D systems we shall consider has as a prototype the linear model

$$x(h+1, k+1) = A_1\, x(h, k+1) + A_2\, x(h+1, k) + B_1\, u(h, k+1) + B_2\, u(h+1, k) \tag{1.1}$$

where $x(h, k) \in \mathbb{R}^n$ and $u(h, k) \in R^m$ are the local state and the input value at (h, k) and A_1, A_2, B_1, B_2 are real matrices of suitable dimensions [1].

Assuming that the local states $x(i, -i)$ have been assigned, we wish to choose the control sequence so that the system behaves in some desirable way. We have to settle two questions at the outset, namely what the control objective is and what sort of controls and initial conditions are to be allowed.

As far as the first question is concerned, the cost of controls will be specified precisely by a scalar performance criterion of the following form

$$\sum_{h+k \geqq 0} x^{\mathrm{T}}(h, k)\, Q\, x(h, k) + u^{\mathrm{T}}(h, k)\, R\, u(h, k) \tag{1.2}$$

with $R > 0$ and $Q \geqq 0$.

Denoting by

$$\mathscr{E}_t := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i + j = t\}$$

the tth separation set in $\mathbb{Z} \times \mathbb{Z}$ and by

$$\mathscr{U}_t := \{u(i, t - i)\}_t$$
$$\mathscr{X}_t := \{x(i, t - i)\}_t \quad \text{where} \quad t \in \mathbb{Z}\,,$$

the restrictions of $u(\cdot, \cdot)$ and $x(\cdot, \cdot)$ to $\mathscr{E}_t$, it is clear that, given any initial "global state" $\mathscr{X}_0$ on $\mathscr{E}_0$, $\mathscr{X}_t$ only depends on $\mathscr{U}_0, \mathscr{U}_1, \ldots, \mathscr{U}_{t-1}$ and the value of (1.2) is uniquely determined by the 2D control sequence $\{u(h, k)\}$ via the updating equation (1.1). So in what follows the cost functional (1.2) will be denoted as $J(u, \mathscr{X}_0)$. We now turn to the problem of admissible controls and initial global states. It is apparent from the structure of J that admissible input functions must belong to the space $l_2^{2D}(\mathbb{R}^m)$ of $\mathbb{R}^m$-valued sequences $u(\cdot, \cdot)$ defined on

$$Z_+^2 := \{(h, k) \in \mathbb{Z} \times \mathbb{Z}: h + k \geqq 0\} = \bigcup_{t \geqq 0} \mathscr{E}_t \tag{1.3}$$

and satisfying the finite norm condition

$$\|u(\cdot, \cdot)\|_2^2 := \sum_{h+k \geqq 0} u^{\mathrm{T}}(h, k)\, u(h, k) < \infty \tag{1.4}$$

Furthermore, we are only interested in state dynamics $x(\cdot, \cdot)$ that belong to $l_2^{2D}(\mathbb{R}^n)$. Although this condition is not necessary for guaranteeing the finiteness if J in case Q is singular, it fulfills the natural requirement of imposing a stable pattern on the admissible state evolutions.

In fact, $x(\cdot, \cdot) \in l_2^2(\mathbb{R}^n)$ implies that the associated global states $\mathscr{X}_t$ satisfy

$$\|\mathscr{X}_t\|_2^2 := \sum_{i=-\infty}^{+\infty} x^{\mathrm{T}}(-i, t + i)\, x(-i, t + i) < \infty \tag{1.5}$$

$$\sum_{t=0}^{\infty} \|\mathscr{X}_t\|_2^2 = \|x(\cdot, \cdot)\|_2^2 < \infty \tag{1.6}$$

showing that $\|\mathscr{X}_t\| \to 0$ as t goes to infinity.

Just putting $t = 0$ in (1.5), we argue that the allowable bilateral sequences of initial conditions must belong to $l_2(\mathbb{R}^n)$.

Within this framework, the 2D optimization problems can be stated as follows:

i) given $\mathscr{X}_0 \in l_2(\mathbb{R}^n)$, derive conditions for the existence and the uniqueness of an input $u(\cdot, \cdot) \in l_2^{2D}(\mathbb{R}^m)$ that minimizes the cost J

ii) whenever these conditions are satisfied, explicitly compute the optimal input and the corresponding value of J.

In Section 2 we will summarize from [2] the solution to these problems and outline the structure of the resulting control law. Under suitable natural assumptions on the matrices of (1.1) and (1.2), the optimal input sequence exists and is expressed as a linear feedback law that stabilizes the closed loop system. Although the closed loop solution is quite appealing from the control point of view, it conceals the serious drawback that the resulting system doesn't exhibit a 2D causal structure. Actually, preserving quarter plane causality and obtaining a control law that minimizes (1.2) constitute in general a pair or conflicting objectives.

In Section 3 we shall discuss the possibility of implementing the optimal control law by means of causal and weakly causal [3] feedback structures and discuss the important property that weakly causal feedbacks provide a family of suboptimal control strategies, that converge to the optimal one.

2. EXISTENCE AND STRUCTURE OF THE OPTIMAL CONTROL LAW

As well known, the existence and the uniqueness of a stabilizing optimal solution for the 1D infinite time least squares problem can be decided independently of the explicit computation of the control law. Indeed these questions can be settled a priori by analyzing a couple of rank conditions on the polynomial matrices

$$[I - Az \mid Bz] \quad \text{and} \quad \begin{bmatrix} I - Az \\ \hdashline Q \end{bmatrix}$$

We might expect that the corresponding polynomial matrices in two indeterminates play an analogous role in the 2D case. This is the gist of the following proposition. It gives conditions under which the optimization procedure admits a unique solution and supplies an input function that stabilizes the system, in the sense that the corresponding values of $x(h, k)$ converge to zero as $h + k$ goes to infinity.

Proposition 2.1 [2]. Consider the 2D system (1.1) and the cost functional (1.2). The following facts are equivalent:

1. for every global state $\mathscr{X}_0$ in $l_2(\mathbb{R}^n)$ there exists a (unique) l_2^{2D} solution of the optimal control problem, i.e. there exists an input sequence $u(\cdot, \cdot)$ in $l_2^{2D}(\mathbb{R}^m)$ such that $x(\cdot, \cdot)$ belongs to $l_2^{2D}(\mathbb{R}^n)$ and the corresponding value of J is minimized.
2. the 2D polynomial matrix

$$[I - A_1 z_1 - A_2 z_2 \mid B_1 z_1 + B_2 z_2] \tag{2.1}$$

has full rank on the set

$$\mathscr{M} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}: |z_1| = |z_2| \leqq 1\}$$

and the 2D polynomial matrix

$$\begin{bmatrix} Q \\ \hdashline I - A_1 z_1 - A_2 z_2 \end{bmatrix} \tag{2.2}$$

has full rank on the unit torus

$$\mathscr{T} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}: |z_1| = |z_2| = 1\}$$

Remark 1.1. In case $B_1 = B_2 = 0$, Huang's theorem [4] implies that (2.1) is full rank on $\mathscr{M}$ if and only if is full rank on $\mathscr{P}_1 = \{(z_1, z_2): |z_1| \leqq 1, |z_2| \leqq 1\}$. This property does not extend to more general situations. Indeed, assuming $A_1 = 1$, $A_2 = 2$, $B_1 = 1$, $B_2 = 0$, $R = Q = 1$, we see that the matrix (2.1) is full rank on $\mathscr{M}$. Yet (2.1) vanishes at $(0, \frac{1}{2}) \in \mathscr{P}_1$.

From now on we shall steadily assume that the rank conditions on (2.1) and (2.2) are fulfilled. Then, using a well established result of the LQ 1D theory, we are allowed to conclude that the ω-dependent algebraic Riccati equation (AREω)

$$\begin{aligned}\hat{P}(\omega) &= Q + \hat{A}^*(\omega)\,\hat{P}(\omega)\,\hat{A}(\omega) - \hat{A}^*(\omega)\,\hat{P}(\omega)\,\hat{B}(\omega)\\ &\quad [R + \hat{B}^*(\omega)\,\hat{P}(\omega)\,\hat{B}(\omega)]^{-1}\,\hat{B}^*(\omega)\,\hat{P}(\omega)\,\hat{A}(\omega)\,,\end{aligned} \tag{2.3}$$

with $\hat{A}(\omega) = A_1 + e^{i\omega} A_2$, $\hat{B}(\omega) = B_1 + e^{i\omega} B_2$, is pointwise solvable in $[0, 2\pi]$ and admits for every ω a stabilizing hermitian positive semidefinite solution. This solution provides a key tool for evaluating the minimum cost and computing an input function that minimizes (1.2).

Let us first start with a somewhat heuristic analysis of the system dynamics in terms of Fourier transforms. We assume that $\mathscr{X}_0$ and $\mathscr{U}_t$, $t = 0, 1, \ldots$, belong to l_2. Clearly all global states $\mathscr{X}_t$, $t = 1, 2, \ldots$, are in l_2 and the Fourier transforms

$$\begin{aligned}{}^\wedge\mathscr{U}_t(\omega) &= \sum_{h=-\infty}^{+\infty} u(t+h, -h)\, e^{-ih\omega}\\ {}^\wedge\mathscr{X}_t(\omega) &= \sum_{h=-\infty}^{+\infty} x(t+h, -h)\, e^{-ih\omega}\end{aligned} \tag{2.4}$$

have components in $L_2[0, 2\pi]$. Furthermore, equation (1.1) can be rewritten as a first order recursive equation

$${}^\wedge\mathscr{X}_{t+1}(\omega) = \hat{A}(\omega)\,{}^\wedge\mathscr{X}_t(\omega) + \hat{B}(\omega)\,{}^\wedge\mathscr{U}_t(\omega)\,, \tag{2.5}$$

which provides the global states updating in the ω-domain.

Suppose in addition that the l_2^{2D} norms of the input sequence $u(\cdot, \cdot)$ and of the state dynamics $x(\cdot, \cdot)$ are both finite, i.e.

$$\begin{aligned}\|u(\cdot, \cdot)\|_2^2 &= (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} {}^\wedge\mathscr{U}_t^*(\omega)\,{}^\wedge\mathscr{U}_t(\omega)\, d\omega < \infty\\ \|x(\cdot, \cdot)\|_2^2 &= (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} {}^\wedge\mathscr{X}_t^*(\omega)\,{}^\wedge\mathscr{X}_t(\omega)\, d\omega < \infty\end{aligned} \tag{2.6}$$

Then, by applying the squares completion method we are able to express the cost functional (1.2) as [2]

$$\begin{aligned}J &= (2\pi)^{-1} \int_0^{2\pi} {}^\wedge\mathscr{X}_0^*(\omega)\,\hat{P}(\omega)\,{}^\wedge\mathscr{X}(\omega)\, d\omega\\ &\quad + (2\pi)^{-1} \int_0^{2\pi} \sum_{t=0}^{\infty} {}^\wedge\mathscr{S}_t^*(\omega)\,[R + \hat{B}^*(\omega)\,P(\omega)\,\hat{B}(\omega)]\,{}^\wedge\mathscr{S}_t(\omega)\, d\omega\end{aligned} \tag{2.7}$$

where

$$\hat{K}(\omega) := -[R + \hat{B}^*(\omega)\,\hat{P}(\omega)\,\hat{B}(\omega)]^{-1}\,\hat{B}^*(\omega)\,\hat{P}(\omega)\,\hat{A}(\omega) \tag{2.8}$$

$${}^\wedge\mathscr{S}_t(\omega) = {}^\wedge\mathscr{U}_t(\omega) - \hat{K}(\omega)\,{}^\wedge\mathscr{X}_t(\omega) \tag{2.9}$$

It is easy to see that the minimum value of J

$$J_{\min} = (2\pi)^{-1} \int_0^{2\pi} {}^\wedge\mathscr{X}_0^*(\omega)\,\hat{P}(\omega)\,{}^\wedge\mathscr{X}_0(\omega)\, d\omega \tag{2.10}$$

is attained using the closed loop control given by

$$^{\wedge}\mathcal{U}_t(\omega) = \hat{K}(\omega)\,^{\wedge}\mathcal{X}_t(\omega) \tag{2.11}$$

The conclusions we have drawn so far depict the situation in a way that may convince of the intuitive reasonableness of the result. However, some caveats are in order, since the validity of the procedure heavily depends on the fact that (2.11) and (2.5) give rise to 2D sequences $u(\cdot, \cdot)$ and $x(\cdot, \cdot)$ that belong to l_2^{2D}.

More precisely, the solution of the optimal control problem outlined above makes sense if we are able to give a positive answer to the following questions:

i) does the matrix $\hat{K}(\omega)$ map the space $L_2[0, 2\pi]$ into itself? This requirement is necessary for guaranteeing that the feedback law (2.11) (reinterpreted in the time domain) always transforms an l_2 global state $\mathcal{X}_t$ into an l_2 input sequence $\mathcal{U}_t$.

ii) does the solution of (AREω) ensure the asymptotic stability of the closed loop system, in the sense that, for any $\mathcal{X}_0 \in l_2$, the resulting global states sequences $\{\mathcal{X}_t\}$ can be viewed as an element of l_2^{2D}?

To answer these questions, we need the following technical lemma.

Lemma 2.1 [2]. The stabilizing hermitian positive semidefinite solution $P(e^{i\omega}) := \hat{P}(\omega)$ of (AREω) extends to an analytic solution of the polynomial algebraic Riccati equation (AREz)

$$\begin{aligned} P(z) &= Q + (A_1^T + A_2^T z^{-1})\, P(z)\,(A_1 + A_2 z) \\ &- (A_1^T + A_2^T z^{-1})\, P(z)\,(B_1 + B_2 z)\,[R + (B_1^T + B_2^T z^{-1})\, P(z)\,(B_1 + B_2 z)]^{-1} \\ &(B_1^T + B_2^T z^{-1})\, P(z)\,(A_1 + A_2 z) \end{aligned} \tag{2.12}$$

in an open annulus that includes the unit circle γ_1.

A major consequence of viewing the pointwise solution $\hat{P}(\omega)$ of (AREω) as the restriction to γ_1 of an analytic matrix $P(z)$ is that $\hat{K}(\omega)$ enjoys the same property. Indeed the matrix

$$\begin{aligned} K(z) &= -[R + (B_1^T + B_2^T z^{-1})\, P(z)\,(B_1 + B_2 z)]^{-1} \\ &(B_1^T + B_2^T z^{-1})\, P(z)\,(A_1 + A_2 z) \end{aligned} \tag{2.13}$$

analytically extends $\hat{K}(\omega)$ in the annulus and therefore admits a Laurent power series expansion

$$K(z) = \sum_{h=-\infty}^{\infty} K_h z^h \tag{2.14}$$

This implies that the matrices K_h exponentially decay as $|h|$ increases and the feedback law (2.11) associates an input $^{\wedge}\mathcal{U}_t(\omega) \in L_2[0, 2\pi]$ to every global state $^{\wedge}\mathcal{X}_t(\omega) \in L_2[0, 2\pi]$. In time domain this is consistently expressed by

$$u(h, k) = \sum_{i=-\infty}^{+\infty} K_i\, x(h + i, k - i) \tag{2.15}$$

This positive answers the first question.

As far as the second question is concerned, note that the Lyapunov equation

$$\hat{V}(\omega) = I + \hat{\Gamma}^*(\omega)\,\hat{V}(\omega)\,\hat{\Gamma}(\omega)\,, \tag{2.16}$$

with $\hat{\Gamma}(\omega) := \hat{A}(\omega) + \hat{B}(\omega)\,\hat{K}(\omega)$, admits a unique positive definite solution, given by the sum of the following pointwise convergent series

$$\hat{V}(\omega) = \sum_{h=0}^{\infty} \hat{\Gamma}^*(\omega)^h\,\hat{\Gamma}(\omega)^h$$

Furthermore, the linearity of (2.16) and the uniqueness of its solution for every ω in $[0, 2\pi]$ imply that the matrix $\hat{V}(\omega)$ is a continuous function of ω and hence its spectral radius $\varrho(\omega)$ is uniformly bounded by some positive $\bar{\varrho}$.

Combining together all these properties and applying B. Levi's and Parseval's theorems one gets

$$\begin{aligned}
\|x(\cdot,\cdot)\|_2^2 &= \sum_{t=0}^{\infty} \|\mathscr{X}_t\|_2^2 \\
&= (2\pi)^{-1} \sum_{t=0}^{\infty} \int_0^{2\pi} {}^\wedge\mathscr{X}_0^*(\omega)\,\hat{\Gamma}^*(\omega)^t\,\hat{\Gamma}(\omega)^t\,{}^\wedge\mathscr{X}_0(\omega)\,\mathrm{d}\omega \\
&= (2\pi)^{-1} \int_0^{2\pi} {}^\wedge\mathscr{X}_0^*(\omega)\,\hat{V}(\omega)\,{}^\wedge\mathscr{X}_0(\omega)\,\mathrm{d}\omega \leqq \bar{\varrho}\|\mathscr{X}_0\|_2^2\,.
\end{aligned}$$

So $x(\cdot,\cdot)$ and, obviously, $u(\cdot,\cdot)$ belong to $l_2^{2\mathrm{D}}$.

Example 2.1. Consider once more the system of Remark 1.1. By solving the associated (AREω), one obtains a unique positive definite solution

$$\hat{P}(\omega) = \frac{(5 + 4\cos\omega) + \sqrt{(5 + 4\cos\omega)^2 + 4}}{2}$$

and the corresponding feedback matrix is

$$\hat{K}(\omega) = \frac{-(1 + 2\,\mathrm{e}^{\mathrm{i}\omega})\,[(3 + 4\cos\omega) + \sqrt{(5 + 4\cos\omega)^2 + 4}]}{2(5 + 4\cos\omega)}$$

Obviously the inverse Fourier transform of $\hat{K}(\omega)$ is an infinite support l_2-sequence. This shows that in general the computation of $u(h, k)$ depends on an infinite number of local states.

3. OPTIMAL CONTROL LAWS THAT INVOLVE A FINITE NUMBER OF LOCAL STATES

The control law (2.15) we obtained through the solution of (2.12) provides a state feedback that stabilizes (1.1) and minimizes (1.2). The input value at (h, k) depends in general on infinitely many local states $x(h - i, k + i)$, $i \in \mathbb{Z}$ (see Example 2.1). So, implementing (2.15) destroys the quarter plane causality of the original system and produces an half plane causal 2D system [3], whose updating equation requires in principle to cope with an infinite dimensional state vector.

Now it seems quite natural to ask whether there exist 2D systems and cost functionals that give rise to causal or weakly causal optimal control laws. In other terms, the problem we address in this section is to explore what conditions on (1.1) and (1.2) do guarantee that the stabilizing feedback matrix given by (2.13) belongs to $\mathbb{R}[z, z^{-1}]^{m \times n}$ and, in particular, to $\mathbb{R}^{m \times n}$. We shall provide only partial results on this subject. The feeling they give, however, is that the possibility of achieving 2D optimal control while preserving 2D weak causality is extremely rare. So, if preserving weak causality is the main issue, we must in general put up with suboptimal control laws.

Confining ourselves to single input 2D systems of dimension 1, we are able to get a complete classification of systems (1.1) and cost functions (1.2) that produce feedback laws with the following structure

$$u(h, k) = \sum_{i=-N}^{+N} K_i x(h - i, k + i) \tag{3.1}$$

for some integer N. Matrices in (1.1) and (1.2) are scalars, that will be henceforth denoted by the corresponding lower case letter a_1, a_2, b_1, b_2, q, r. The same convention will be adopted for all scalar quantities we will deal with in the sequel. So, letting

$$a(z) = a_1 + a_2 z \quad b(z) = b_1 + b_2 z\,, \tag{3.2}$$

the Riccati equation (2.12) and the corresponding feedback matrix are given by

$$p(z) = q + a(z)\,a(z^{-1}) - a(z)\,a(z^{-1})\,b(z)\,b(z^{-1})\,p^2(z) \times [r + b(z)\,b(z^{-1})\,p(z)]^{-1} \tag{3.3}$$

$$k(z) = -b(z^{-1})\,a(z)\,p(z)\,[r + b(z)\,b(z^{-1})\,p(z)]^{-1} \tag{3.4}$$

Since we are interested in feedback laws $k(z)$ whose Laurent series expansion has finite support, it seems useful to summarize here some properties of the ring of bilateral polynomials $\mathbb{R}[z, z^{-1}]$.

Given any nonzero polynomial

$$\tau(z, z^{-1}) = \sum_{i=M}^{N} \tau_i z^i\,, \quad \text{with} \quad \tau_N \tau_M \neq 0$$

the integer $\delta(\tau) := N - M$ is called the degree of τ. If $\tau = 0$, by definition deg $(\tau) = -\infty$. Clearly every polynomial of zero degree, i.e. every monomial az^i, $a \neq 0$, $i \in \mathbb{Z}$, is a unit of $\mathbb{R}[z, z^{-1}]$. Moreover, $\mathbb{R}[z, z^{-1}]$ is an euclidean domain and a fortiori an UFD.

If $\tau(z, z^{-1}) = \tau(z^{-1}, z)$, then $N = -M$ and the polynomial is termed reciprocal. As an element of $\mathbb{C}[z, z^{-1}]$, such a polynomial admits a (nonunique) factorization

$$\tau(z, z^{-1}) = t(z)\,t(z^{-1}) \tag{3.5}$$

where $t(z) \in \mathbb{C}[z]$ and deg $(t) = \delta(\tau)/2$.

Proposition 3.1. Consider a single input 2D system of dimension 1, that satisfies the conditions of Proposition 2.1. The stabilizing optimal feedback law $k(z)$ given by (3.3) and (3.4) belongs to $\mathbb{R}[z, z^{-1}]$ if and only if one of the following cases occurs:

1. $a(z)\, b(z) = 0$
2. $a(z)\, b(z)$ is an unit of $\mathbb{R}[z, z^{-1}]$
3. $q = 0$ and $|a_1| + |a_2| < 1$

Furthermore in these cases the allowable $k(z)$'s reduce to the following three simple structures:

$$k_0, \quad k_1 z, \quad k_{-1} z^{-1} \tag{3.6}$$

Proof. 1. If $a(z) = 0$ or $b(z) = 0$, one gets from (3.3) and (3.4)

$$p(z) = q/[1 - a(z)\, a(z^{-1})], \quad k(z) = 0 \tag{3.7}$$

2. If both $a(z)$ and $b(z)$ are units of $\mathbb{R}[z, z^{-1}]$, then $a(z)\, a(z^{-1})$ and $b(z)\, b(z^{-1})$ are both nonzero real constants and (3.3) reduces to a constant coefficients algebraic Riccati equation. Thus $p(z)$ is a nonnegative real number and $k(z)$ has structure (3.6).

3. If $q = 0$ and $b(z) \neq 0$, equation (3.3) admits two solutions:

$$p_1(z) = 0 \tag{3.8}$$

$$p_2(z) = \frac{r[a(z)\, a(z^{-1}) - 1]}{b(z)\, b(z^{-1})} \tag{3.9}$$

The internal stability assumption $|a_1| + |a_2| < 1$ directly implies that, for every ω in $[0, 2\pi]$, $p_2(e^{i\omega})$ is negative. Thus $p_1(z)$ is the stabilizing solution and $k(z) = 0$.

It remains to show that the above classification covers all possible cases, in the sense that, whenever none of 1, 2 and 3 is satisfied, the support of $k(z)$ is an infinite set. The proof is rather long and will be performed in several steps.

Lemma 3.1. If conditions 1, 2 and 3 do not hold, then

$$\begin{aligned} &\text{a) } p(z) \neq 0 \\ &\text{b) } k(z) \neq 0 \\ &\text{c) } a(z) + b(z)\, k(z) \neq 0 \end{aligned} \tag{3.10}$$

Proof. a) Assuming $p(z) = 0$ gives $q = 0$ and $k(z) = 0$. Since condition 3 does not hold, $|a_1| + |a_2| \geq 1$ and the zero feedback matrix cannot stabilize the system.

b) Assuming $k(z) = 0$ gives $p(z) = 0$ and case b) reduces to case a).

c) Assume $a(z) + b(z)\, k(z) = 0$. We apply (3.3) and (3.4) first, to obtain the following relation connecting $p(z)$ and $k(z)$

$$p(z) = q + a(z^{-1})\, p(z)\, [a(z) + b(z)\, k(z)] \tag{3.11}$$

It is clear from (3.11) that $p(z) = q$. Also, substituting q for $p(z)$ into (3.3) and recalling that $a(z) \neq 0$, we get $q = 0$. Thus case c) reduces to case a) too. □

Since $a(z) + b(z)\,k(z) \neq 0$, we are allowed to solve (3.4) with respect to $p(z)$, namely

$$p(z) = -\frac{r\,k(z)}{b(z^{-1})\,[a(z) + b(z)\,k(z)]} \tag{3.12}$$

Substituting (3.12) into (3.3), $k(z)$ can be directly computed as a solution of the following equation

$$\begin{aligned} &k^2(z)\,a(z^{-1})\,b(z)\,r + k(z)\,[r\,a(z)\,a(z^{-1}) - q\,b(z)\,b(z^{-1}) - r] - \\ &- q\,a(z)\,b(z^{-1}) = 0 \end{aligned} \tag{3.13}$$

So we reduced to prove that, if conditions 1, 2 and 3 do not hold, equation (3.13) isn't solvable in $\mathbb{R}[z, z^{-1}]$.

Assume by contradiction that $k(z)$ belongs to $\mathbb{R}[z, z^{-1}]$ and satisfies (3.13). It is clear from (3.12) that $p(z)$ is a rational function. Consequently, there exists a pair of coprime bilateral polynomials $\phi(z, z^{-1})$ and $\gamma(z, z^{-1})$ such that

$$p(z) = \phi(z, z^{-1})/\gamma(z, z^{-1}) \tag{3.14}$$

On the other hand $p(z) = p(z^{-1})$ and $\mathbb{R}[z, z^{-1}]$ is an UFD. Hence

$$\begin{aligned} \gamma(z, z^{-1}) &= cz^i\,\gamma(z^{-1}, z) \\ \phi(z, z^{-1}) &= cz^i\,\phi(z^{-1}, z) \end{aligned} \tag{3.15}$$

and, evaluating (3.15) at $z = \pm 1$, it follows that $c = 1$ and $i \equiv 0 \pmod 2$. Consequently, there is no restriction in assuming that ϕ and γ are reciprocal polynomials and in writing ϕ and γ in factored form, namely

$$p(z) = \frac{f(z)\,f(z^{-1})}{g(z)\,g(z^{-1})} \tag{3.16}$$

with $f(z)\,f(z^{-1})$ and $g(z)\,g(z^{-1})$ coprime.

Note that the fact that $p(e^{i\omega})$ is a nonnegative real-valued function for every ω in $[0, 2\pi]$ reflects into the fact that f and g have real coefficients. Next, substitute (3.16) into (3.4), to get

$$k(z)\,\sigma(z, z^{-1}) = -b(z^{-1})\,a(z)\,f(z)\,f(z^{-1}) \tag{3.17}$$

where

$$\sigma(z, z^{-1}) := r\,g(z)\,g(z^{-1}) + b(z)\,b(z^{-1})\,f(z)\,f(z^{-1}) \tag{3.18}$$

is a reciprocal polynomial. To complete the proof, we show that both hypothesis $\deg(\sigma) > 0$ and $\deg(\sigma) \leq 0$ lead to a contradiction.

Lemma 3.2. Assume that 1, 2 and 3 do not hold and let $\deg(\sigma) > 0$. Then $k(z) \notin \mathbb{R}[z, z^{-1}]$.

Proof. Since $f(z)\,f(z^{-1})$ and $g(z)\,g(z^{-1})$ are coprime, we have that $f(z)\,f(z^{-1})$ is a divisor or $k(z)$. Then $\sigma(z, z^{-1})$ divides $b(z^{-1})\,a(z)$ and since $\deg[b(z^{-1})\,a(z)] \leq 2$, as a reciprocal polynomial σ has degree 2.

This easily implies that, modulo nonzero real constants, $\sigma(z, z^{-1})$, $b(z)\,b(z^{-1})$ and $a(z)\,a(z^{-1})$ coincide, so that

$$b(z) = \lambda\, a(z) \tag{3.19}$$

for some nonzero $\lambda \in \mathbb{R}$. Now equation (3.13) reduces to

$$k^2(z)\,a(z)\,a(z^{-1})\,r\lambda = -a(z)\,a(z^{-1})\,\lambda q + k(z)\left[r - a(z)\,a(z^{-1})\,(r - \lambda^2 q)\right] \tag{3.20}$$

If $k(z) \notin \mathbb{R}$, some elementary considerations on the support of the polynomials show that the left and right hand sides of (3.20) cannot be equal. On the other hand, if $k(z) := k \in \mathbb{R}$, (3.20) can be rewritten as

$$a(z)\,a(z^{-1})\left[q\lambda + k(r - \lambda^2 q) + k^2 r\lambda\right] = rk \tag{3.21}$$

This implies $k = 0$ and $q = 0$, which contradicts the conclusions of Lemma 3.1. □

Lemma 3.3. Assume that 1, 2 and 3 do not hold and let $\deg \sigma(z, z^{-1}) \leqq 0$. Then $k(z) \notin \mathbb{R}[z, z^{-1}]$.

Proof. Since $\sigma(z, z^{-1})$ is a reciprocal polynomial, and $\deg(\sigma) < 0$, σ is a real constant.

First, using (3.16) and (3.18), we have

$$\sigma/[g(z)\,g(z^{-1})] = r + b(z)\,b(z^{-1})\,p(z) \tag{3.22}$$

so that

$$p(z) = \frac{1}{b(z)\,b(z^{-1})}\left[\frac{\sigma}{g(z)\,g(z^{-1})} - r\right] \tag{3.23}$$

Since $p(z)$ must be nonnegative, it is clear that σ is different from zero. Substituting (3.23) into (3.3) gives

$$\begin{aligned}\sigma = {} & g(z)\,g(z^{-1})\left[r + r\,a(z)\,a(z^{-1}) + q\,b(z)\,b(z^{-1}) - \right.\\ & \left. - r\sigma^{-1}\,g(z)\,g(z^{-1})\,a(z)\,a(z^{-1})\right]\end{aligned} \tag{3.24}$$

which implies $\deg(\sigma) > 0$, unless $g(z)$ is unit. Now there is no loss of generality in assuming $g(z) = 1$, so that (3.23) and (3.16) imply

$$p(z) = \frac{\sigma - r}{b(z)\,b(z^{-1})} = f(z)\,f(z^{-1}) \tag{3.25}$$

Thus $b(z)$ and $f(z)$ are both units, $p(z)$ is a real constant and, by (3.4),

$$k(z) = \mu\, b(z^{-1})\,a(z) \tag{3.26}$$

where μ is a real constant, different from zero by Lemma 3.1.

Next, since $k(z)$ must satisfy equation (3.13), we obtain the following equation

$$\mu r\,a(z)\,a(z^{-1})\left[1 + \mu\,b(z)\,b(z^{-1})\right] = q\left[1 + \mu\,b(z)\,b(z^{-1})\right] + \mu r \tag{3.27}$$

where $a(z)$ cannot be a unit of $\mathbb{R}[z, z^{-1}]$. The right hand side of (3.27) is a real constant. So on the left hand side we must have $1 + \mu\, b(z)\, b(z^{-1}) = 0$, which in turn implies $\mu r = 0$.

This contradicts the fact that μ is different from zero. $\square$

The classification of Proposition 3.1 shows that the optimal control law of a one dimensional 2D system involves a finite number of local states only in very particular cases. Indeed they reduce to:

1. autonomous 2D systems ($b_1 = b_2 = 0$), for which the control problem does not make sense.
2. "dead beat" 2D systems ($a_1 = a_2 = 0$), for which the zero input is obviously the optimal control.
3. asymptotically stable 2D systems ($|a_1| + |a_2| < 1$) with state weighting matrix $q = 0$, for which the cost of the free evolution is zero.
4. 2D systems "isomorphic" to 1D systems ($a_1 = b_1 = 0$ or $a_2 = b_2 = 0$), that exhibit only an horizontal or a vertical dynamics. In this case the control law is a static one

$$u(h, k) = k_0\, x(h, k)$$

5. 2D systems that reduce to the previous case by a one-step time-shift of the input function ($a_1 = b_2 = 0$ or $a_2 = b_1 = 0$). In these cases the control laws become

$$u(h, k) = k_1\, x(h + 1, k - 1)$$

or

$$u(h, k) = k_{-1}\, x(h - 1, k + 1)$$

The extension of the previous result to higher dimensional cases constitutes a topic of current investigation.

As a final remark, we only mention the fact that in cases when $K(z)$ given in (2.14) has an infinite support, it is possible to use weakly causal feedback laws that approximate the optimal control law. These can be obtained by a suitable truncation of the Laurent power series expansion (2.14), namely

$$K_N(z) = \sum_{i=-N}^{N} K_i z^i \qquad (3.28)$$

Truncations (3.28) constitute the "right" approximation to the optimal control law. Actually

i) provided that N is large enough, (3.28) is a stabilizing state feedback matrix

ii) even more important, when N diverges the corresponding cost function

$$J_N(\mathscr{X}_0) = \sum_{t=0}^{\infty} (2\pi)^{-1} \int_0^{2\pi} [\,{}^{\wedge}\mathscr{X}_t^*(\omega)\, Q\, {}^{\wedge}\mathscr{X}_t(\omega) + $$
$$+ {}^{\wedge}\mathscr{X}_t^*(\omega)\, \hat{K}_N^*(e^{i\omega})\, R\, \hat{K}_N(e^{i\omega})\, {}^{\wedge}\mathscr{X}_t(\omega)]\, d\omega \qquad (3.29)$$

asymptotically converges to the minimum value (2.10).

The discussion of these topics, however, is beyond the scope of our paper. It can be found in [2].

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