FINITE MEMORY REALIZATION OF 2D FIR FILTERS

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Abstract- Some properties of state space realizations of 2D FIR filters are investigated. It is shown that hidden modes are allowed in minimal realizations and, consequently, there exist minimal realizations of FIR filters which are not finite memory.

I. INTRODUCTION

When one designs a 2D filter, the performance requirements are usually specified by an input/output description. On the other hand, the design is completed when the specifications are met by a suitable state space model that "realizes" the input/output description, by displaying the structure and the connections of the physical components that are needed in the synthesis procedure.

The translation of one description into another is an important problem which, while discussed and solved in the early 70's in the 1D context [1,2], is still far from a complete solution in the 2D case.

It is well known that any linear stationary recursive 2D filter, represented by a proper rational transfer function $W(z_1, z_2)$, can be converted to an equivalent first order state space model with the following structure [3],

$$\begin{aligned} x(h,k) &= A_1x(h,k+1) + A_2x(h+1,k) \\ &\quad + B_1u(h,k+1) + B_2u(h+1,k) \\ y(h,k) &= Cx(h,k) + Du(h,k) \end{aligned} \tag{1}$$

System (1) is usually denoted as $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ and called a (state space) realization of $W(z_1, z_2)$.

The research efforts made in the last few years enlightened many connections between the input-output description given by $W(z_1, z_2)$ and the internal one, given by Σ and opened several avenues for subsequent investigations.

However if we look for minimal realizations, their structure still constitutes the bottleneck of the 2D theory. At the moment several counterexamples are available, that provide negative answers to questions we could naively hope to solve by just extending 1D results. In particular, we know that minimal realizations of a 2D filter in general are not unique (modulo algebraic equivalence) and their dimension depends on the (complex or real) field where system matrices take their values. Moreover reachability and observability properties (whatever may be their definition) are not suitable for characterizing minimal realizations. An even more stricking difference w.r. to the classical case is that hidden modes are allowed in minimal 2D realizations. This result, that will be proved in Section III, encompasses many interesting consequences, ranging from the "internal" stability of minimal realizations of "externally stable" filters to the existence of minimal realizations that are not modally controllable and reconstructible [4,5].

In this paper we shall be concerned with two properties of 2D state models and 2D filters, that are strongly connected each other. The first one is the so called "finite memory" property [6] of the state space model or, equivalently, of the pair (A_1, A_2). A 2D system is finite memory if, for any arbitrary initial set of local states,

$$\mathcal{X}_0 = \{x(i,-i), i \in \mathbf{Z}\} \tag{2}$$

the state free evolution goes to zero in a finite number of steps. This property depends only on the structure of the pair (A_1, A_2) and requires that the 2D characteristic polynomial of the system satisfies

$$\det(I - A_1z_1 - A_2z_2) = 1 \tag{3}$$

The second property is the finite impulse response (FIR) of the filter or, equivalently, the polynomial character of $W(z_1, z_2)$.

The connections between these properties are very well understood in the 1D situation, where

i) finite memory reduces to the nilpotency of the matrix A,

ii) finite memory implies that the transfer function is FIR,

iii) minimal realizations of FIR transfer functions are finite memory.

Only proposition ii) has an immediate extension to 2D systems, as can be easily seen from the expression

$$W(z_1, z_2) = \frac{C\mathrm{adj}(I - A_1z_1 - A_2z_2)(B_1z_1 + B_2z_2)}{\det(I - A_1z_1 - A_2z_2)} \tag{4}$$

The objective pursued in section 2 is to investigate some connections between the finite memory property of a 2D state space model and the structure of the pair (A_1, A_2), while in section 3 we shall cope with the minimal realization problem of FIR transfer functions in two variables. One characteristic feature of 2D minimal realizations is that the numerator and the denominator of (4) need not be coprime polynomials. Therefore we expect that the characteristic polynomial of a minimal 2D state model may be a multiple of the transfer function denominator.

In that case the FIR property of a transfer function does not imply the internal stability of its minimal realizations.

II. FINITE MEMORY 2D STATE MODELS

When analyzing the structure of 1D state space models, canonical forms with respect to similarity transformations provide an extremely useful tool. Actually the interest of canonical forms

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relies in reducing the computation in the analysis and design of control systems and, also, in evidentiating characteristic properties which are invariant for the orbits generated under matrix similarity equivalence.

In the 2D framework, given a finite memory pair (A_1, A_2) of dimension n, all pairs in the orbit $\{(T^{-1}A_1T, T^{-1}A_2T),\ T \text{ nonsing.}\}$ characterize 2D systems with the same property. Therefore, in order to obtain a complete classification of the set $\mathcal{F}$ of finite memory pairs, all we need is to give a subset $\mathcal{C} \subset \mathcal{F}$ having nonempty intersection with each orbit. In the sequel sets of canonical forms will be directly computed for $n \leq 3$ using techniques which cannot be extended to higher dimensional systems. As far as we know, there are no general procedures for computing sets of canonical forms $\mathcal{C}$ with arbitrary dimension of the state space.

What we can easily see is that all pairs in $\mathcal{F}$ consist of nilpotent matrices, but the converse is not true, since the nilpotency of A_1 and A_2 does not imply finite memory. To show this, just take

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

So, for instance, $\mathcal{C}$ cannot be the set of alla pairs (A_1, A_2) where A_1 is any Jordan nilpotent matrix and A_2 is a generic nilpotent matrix.

On the other hand, $\mathcal{F}$ includes the set $\mathcal{T}$ of triangular nilpotent pairs. The orbits generated by the elements of $\mathcal{T}$, however, do not cover the set $\mathcal{F}$. In fact, consider the following finite memory pair

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{5}$$

If the matrices in (5) were triangularizable, every element of the multiplicative semigroup generated by A_1 and A_2 would be nilpotent. Since

$$A_1A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is not nilpotent, this is not true and shows that $\mathcal{T}$ does not intersect all orbits of $\mathcal{F}$.

If we confine ourselves to pairs of matrices with dimension less than or equal to 3, a finite set of canonical forms (w.r. to similarity transformations) can be computed directly. Since these will be needed in the next section, we shall sketch here their construction.

If the dimension of A_1 and A_2 is 2, then the finite memory condition (3) implies the simultaneous triangularizability of A_1 and A_2. Actually, the case $A_1 = 0$ is trivial; so there is no restriction in assuming that A_1 is a Jordan block,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{6}$$

and A_2 any 2×2 nilpotent matrix

$$A_2 = \begin{bmatrix} u \\ v \end{bmatrix} [\alpha \quad \beta], \quad u\alpha + v\beta = 0 \tag{7}$$

We therefore have, $\det(I - A_1z_1 - A_2z_2) = 1 - \alpha v z_1 z_2$. Since finite memory gives $\alpha v = 0$, $u\alpha + v\beta = 0$ implies $\beta v = \alpha u = 0$. We conclude that the structure of A_2 is as follows

$$A_2 = \begin{bmatrix} 0 & u\beta \\ 0 & 0 \end{bmatrix}.$$

For $n = 3$, referring to the ranks of A_1 and A_2, we classify all possible cases as follows:

Case 1: $(\text{rank}A_1)(\text{rank}A_2) = 0$

One of the matrices is 0 and the other one can be reduced to a nilpotent matrix in Jordan form. No further constraints arise when considering equation (3).

Case 2: $(\text{rank}A_1)(\text{rank}A_2) = 1$

Both matrices have rank 1. There is no restriction in assuming that A_1 is in Jordan form

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} := J^{(1)}$$

and A_2 is a generic 3×3 nilpotent matrix of rank 1

$$A_2 = [u \quad v \quad w]^T [\alpha \quad \beta \quad \gamma], \quad \alpha u + \beta v + \gamma w = 0 \tag{8}$$

Similarity transformations induced by

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{11} & 0 \\ 0 & t_{32} & t_{33} \end{bmatrix}, \quad t_{11}t_{33} \neq 0$$

preserve the structure of $J^{(1)}$ and reduce A_2 to one of the following structures

$$A_2' = \begin{bmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ w & 0 & 0 \end{bmatrix} \neq 0, \quad A_2'' = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \neq 0,$$
$$A_2''' = \begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & w & 0 \end{bmatrix} \neq 0 \tag{9}$$

Finally, equation (3) introduces a further constraint on A_2', so that the admissible pairs are

$$A_1 = J^{(1)}, \quad A_2' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ w & 0 & 0 \end{bmatrix} \neq 0$$
$$A_1 = J^{(1)}, \quad A_2'' = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \neq 0 \tag{10}$$
$$A_1 = J^{(1)}, \quad A_2''' = \begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & w & 0 \end{bmatrix} \neq 0$$

Case 3: $(\text{rank}A_1)(\text{rank}A_2) = 2$

Suppose A_1 of rank 2 reduced to its Jordan form

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} := J^{(2)}$$

and A_2 be a generic 3×3 nilpotent matrix of rank 1 (the case $\text{rank}A_2 = 2$ can be dealt with by simmetry). Similarity transformations induced by

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{11} & t_{12} \\ 0 & 0 & t_{11} \end{bmatrix}, \quad t_{11} \neq 0, \tag{11}$$

preserve the structure of $J^{(2)}$, while reducing A_2 to one of the structures given in (9). The finite memory condition implies then that the admissible pairs are only two, namely,

$$A_1 = J^{(2)}, \quad A_2 = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$
$$A_1 = J^{(2)}, \quad A_2 = \begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0 \tag{12}$$

Case 4: $(\mathrm{rank}A_1)(\mathrm{rank}A_2) = 4$

Assume $A_1 = J^{(2)}$ and let A_2 be a generic 3x3 nilpotent matrix of rank 2

$$A_2 = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \quad r_i \in \mathbf{R}^{1\times 3}$$

If r_1 belongs to the row span of r_2 and r_3, then there is no restriction in assuming $r_1 = 0$. In fact in this case a similarity transformation exists, as induced by a suitable matrix (11), that annihilates the first row of A_2.

If r_1 does not belong to the row span of r_2 and r_3, then, using a similarity transformation as above, we may assume that either $r_2 = 0$ (and $r_3 \neq 0$) or $r_3 = 0$ (and $r_2 \neq 0$).

In case $r_1 = 0$, the finite memory condition induces further constraints on A_2 , that reduces to

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -u & 0 & 0 \\ 0 & u & 0 \end{bmatrix} \neq 0 \tag{13}$$

In case $r_2 = 0$, there is no way of satisfying the finite memory condition, while in case $r_3 = 0$ the matrix A_2 reduces to

$$A_2 = \begin{bmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{bmatrix}, \quad u, w \neq 0 \tag{14}$$

which concludes the computation of canonical forms for $n = 3$.

Although the set $\mathcal{F}$ has not yet been characterized by canonical forms for matrices of arbitrary dimension, the finite memory property of a pair (A_1, A_2) can be always expressed in terms of equivalent conditions which involve the linear space and the additive semigroup generated by A_1 and A_2, or the family of matrices $A_1{}^i\sqcup^j A_2$ recursively defined by

$$A_1{}^i\sqcup^0 A_2 = A_1^i, \quad A_1{}^0\sqcup^j A_2 = A_2^j$$
$$A_1{}^i\sqcup^j A_2 = A_1(A_1{}^{i-1}\sqcup^j A_2) + A_2(A_1{}^i\sqcup^{j-1} A_2), \text{ if } i,j > 0$$

The picture is illustrated by the following proposition [5,7]

PROPOSITION The following properties are equivalent:

(i) $\det(I - A_1 z_1 - A_2 z_2) = 1$ (finite memory property)

(ii) the additive semigroup generated by A_1 and A_2, i.e. the set of matrices $aA_1 + bA_2$, a, b non negative integers, is constituted by nilpotent matrices

(iii) the linear space generated by A_1 and A_2, i.e. the set of matrices $\alpha A_1 + \beta A_2$, $\alpha, \beta \in \mathbf{R}$, is constituted by nilpotent matrices

(iv) $A_1{}^i\sqcup^j A_2 = 0, \quad i + j \geq n$

(v) $\mathrm{tr}(A_1{}^i\sqcup^j A_2) = 0, \quad \forall i, j \geq 0$

III. STATE SPACE MODELS OF FIR FILTERS

Obtaining a finite memory realization of a polynomial transfer function is quite easy if we are not required to obtain a low dimensional system [5].

However if we look for *minimal* realizations, in general it is not true that these are necessarily finite memory (which makes a remarkable difference with the 1D case). In this section we aim to show that this phenomenon holds independently of the (real or complex) field where the matrix elements take their values and implies that minimal realizations of FIR transfer functions need not be internally stable. To prove our statement, we shall go through the following steps:

1. we show that there exists a 2D polynomial $n(z_1, z_2)$ of degree 3, that cannot be realized in dimension 3;
2. we construct an (infinite memory) realization of dimension 3 for the transfer function $n(z_1, z_2)/(1+z_2)$ and a series connection of such a realization with a realization of dimension 1 for $(1 + z_2)$. The realization of $n(z_1, z_2)$ obtained in this way is minimal, internally unstable and exhibits pole/zero cancellations.

Step 1 Assume that A_1 and A_2 belong to $\mathbf{C}^{3\times 3}$ and are the state updating matrices of a finite memory 2D state model.

For all matrices

$$C = [\, c_1 \quad c_2 \quad c_3 \,]$$

in $\mathbf{C}^{3\times 3}$ and B_1, B_2 in $\mathbf{C}^{3\times 1}$, the system transfer function is a polynomial

$$W(z_1, z_2) = C\mathrm{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2) \tag{15}$$

of degree not greater than 3, that we shall rewrite as

$$W(z_1, z_2) = p_1(z_1, z_2) + p_2(z_1, z_2) + p_3(z_1, z_2),$$

where p_i are homogenous forms of degree i, $i = 1, 2, 3$.

Referring to the cases considered in section 2, we see that, independently of the choice of B_1, B_2 and C, the polynomials p_i must satisfy the following constraints:

case 1) p_3 belongs either to the principal ideal (z_1^2) or to the principal ideal (z_2^2).
case 2) p_3 belongs to the principal ideal $(z_1 z_2)$.
case 3) if $p_3 = q^3$, where q is a first order form, then either $q = z_1$, or $q = z_2$.
case 4) if $p_3 = q^3$, where q is a first order form, then q is also a factor of p_2.

The above constraints show that the polynomial

$$n(z_1, z_2) = (z_1 + z_2)^3 + z_2^2 + z_2 \tag{16}$$

cannot be realized by a finite memory third order 2D system. Therefore, if we look for a third order 2D realization of $n(z_1, z_2)$, a system with

$$\det(I - A_1 z_1 - A_2 z_2) \neq 1$$

would be needed. However in that case pole/zero cancellations between $\det(I - A_1 z_1 - A_2 z_2)$ and $C\mathrm{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2)$ must occur, and therefore the degree of $C\mathrm{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2)$ must be greater than or equal to 4, which is impossible for third order systems.

Consequently, the dimension of any state space realization of $n(z_1, z_2)$ is greater than 3.

Step 2 The following 2D systems: $\Sigma_1 = (A_1, A_2, B_1, B_2, C)$, with

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0]$$

and $\Sigma_2 = (F_1, F_2, G_1, G_2, H, D)$ with

$$F_1 = F_2 = 0, G_1 = 0, G_2 = H = 1, D = 1.$$

realize $n(z_1, z_2)/(1 + z_2)$ and $1 + z_2$ respectively. Then the series connection of Σ_1 and Σ_2 is a strictly proper 2D system $\hat{\Sigma} = (\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2, \hat{C})$ with

$$\hat{A}_1 = \begin{bmatrix} F_1 & 0 \\ B_1 H & A_1 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} F_2 & 0 \\ B_2 H & A_2 \end{bmatrix},$$
$$\hat{B}_1 = \begin{bmatrix} G_1 \\ B_1 D \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} G_2 \\ B_2 D \end{bmatrix}, \quad \hat{C} = [0 \quad C],$$

which provides a fourth order, and hence a minimal realization of $n(z_1, z_2)$.

Since the characteristic polynomials of $\hat{\Sigma}$ is

$$\det(I - \hat{A}_1 z_1 - \hat{A}_2 z_2) = 1 + z_2$$

the realization $\hat{\Sigma}$ above is not finite memory.

The example allows to point out some interesting consequences:

i) minimal realizations of FIR transfer functions need not be finite memory

ii) pole/zero cancellations are allowed in minimal realizations., Actually in $\hat{\Sigma}$ we have

$$C\mathrm{adj}(I - \hat{A}_1 z_1 - \hat{A}_2 z_2)(\hat{B}_1 z_1 + \hat{B}_2 z_2) = n(z_1, z_2)(1 + z_2)$$
$$\det(I - \hat{A}_1 z_1 - \hat{A}_2 z_2) = 1 + z_2$$

iii) minimal realizations of FIR (and hence BIBO stable) transfer functions may be unstable. In fact in the example the variety of $1 + z_2$ intersects the unit closed polydisc $P_1 = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$.

Remark The above example shows that FIR filters can have infinite memory (even unstable) minimal realizations. This does not necessarily imply the existence of FIR filters whose minimal realizations are all infinite memory. As a matter of fact $n(z_1, z_2)$ admits also the following finite memory realization $\bar{\Sigma} = (\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C})$ with

$$\bar{A}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\bar{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \bar{C} = [1 \quad 0 \quad 0 \quad 0]$$

The systems $\bar{\Sigma}$ and $\hat{\Sigma}$ provide two nonequivalent minimal realizations of the same transfer function $n(z_1, z_2)$. This is not surprising in the 2D case, since minimal realizations need not be algebraically equivalent.

Whether an FIR filter exists whose minimal realizations are all infinite memory systems is still an open question.

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