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Feedback Strategies in Two-Dimensional Control Theory

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1 INTRODUCTION

In this chapter we illustrate the main features of feedback control strategies in the field of two-dimensional (2-D) systems and their use in solving the problems of pole placement, LQ optimal control, and input–output decoupling. As is well known, in one-dimension (1-D) theory there are essentially two approaches to tackle feedback problems. The first consists of constructing the input value at some instant t as a static linear function of the state or output values at the same instant. In the second approach the input value is generated by a dynamic feedback compensator, whose output is obtained by causally processing the state or the output of the system.

Both schemes also apply to 2-D systems. Within this context, however, the partial ordering structure that characterizes 2-D dynamics (quarter-plane causality) allows us to consider a class of feedback schemes where the state or output values at the instant (h, k) influence the input values at instants that are not causally related to (h, k), leading to a closed-loop system where in general the original quarter-plane causality is lost. Of course, this is not acceptable if we look for a solution that maintains the original 2-D causality.

The state equation [1] of a 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ having m inputs and p outputs is given by

$$\begin{aligned} \mathbf{x}(h+1,k+1) &= A_1\mathbf{x}(h+1,k) + A_2\mathbf{x}(h,k+1) \\ &\quad + B_1\mathbf{u}(h+1,k) + B_2\mathbf{u}(h,k+1) \\ \mathbf{y}(h,k) &= C\mathbf{x}(h,k) + D\mathbf{u}(h,k) \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the m-dimensional vector of input values, $\mathbf{y}$ the p-dimensional vector of output values, $\mathbf{x}$ the n-dimensional local state vector, and A_1, A_2, B_1, B_2, C, and D are matrices of appropriate dimensions.

The transfer matrix of Σ,

$$W(z_1, z_2) := C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D \tag{2}$$

is a $p \times m$ matrix whose entries are proper rational functions in two variables. The system (1) is called *strictly proper* if $D = 0$ and *bicausal* if D is an invertible matrix.

Having in mind the dynamical structure of (1), the control laws that preserve the quarter-plane causality of the closed-loop system are provided [2–4] by the static state feedback

$$\mathbf{u}(h,k) = K\mathbf{x}(h,k), \qquad K \in \mathbf{R}^{m\times n} \tag{3}$$

and by the dynamic state feedback (2-D state feedback compensator) represented by the following recursive equation:

$$\begin{aligned} \mathbf{u}(h,k) &= \sum_{i,j} H_{i,j}\mathbf{u}(h-i,k-j) + \sum_{i,j} \mathbf{x}(h-i,k-j), \\ &\qquad H_{i,j} \in \mathbf{R}^{m\times m}, \quad K_{i,j} \in \mathbf{R}^{m\times n} \end{aligned} \tag{4}$$

Denoting by

$$\mathcal{X}_t = \sum_{i=-\infty}^{+\infty} \mathbf{x}(i+t,-i)z^i$$

the "global state" on the separation set $\mathcal{C}_t = \{(i+t,-i), i \in \mathbf{Z}\}$ and by

$$\mathcal{U}_t = \sum_{i=-\infty}^{+\infty} \mathbf{u}(i+t,-i)z^i$$

the restriction of the input function to $\mathcal{C}_t$, the state updating equation (1) can be rewritten in the form

$$\mathcal{X}_{t+1} = (A_1 + A_2 z)\mathcal{X}_t + (B_1 + B_2 z)\mathcal{U}_t \tag{5}$$

Since (5) can be viewed as a linear 1-D system, evolving on the vector space of global states, the feedback control law

$$\mathcal{U}_t(z) = \left(\sum_{i=-N}^{N} K_i z^i \right) \mathcal{X}_t(z) \tag{6}$$

preserves the 1-D causality of the system (5). From the 2-D point of view, however, by expressing (6) in the form

$$\mathbf{u}(h,k) = \sum_{i=-N}^{N} K_i \mathbf{x}(h-i, k+i), \qquad K_i \in \mathbf{R}^{m \times n} \tag{7}$$

we can see that the resulting 2-D feedback system does not preserve the original quarter-plane causality. In fact, because of (1), $\mathbf{x}(h,k)$ depends on the input values at $(h-1,k)$ and $(h,k-1)$, which in turn depend on the states at $(h-1-N, k+N)$, $(h-N, k+N-1), \ldots, (h+N, k-1-N)$. The closed-loop system that results by applying (7) belongs to the class of weakly causal 2-D systems [5,6]. Assuming that $K = \bar{K}H$, $K_i = \bar{K}_i H$, and $K_{i,j} = \bar{K}_{i,j}H$, in (3), (4), and (7), one obtains the explicit expression for the corresponding output feedback laws.

As mentioned above, 2-D feedback control strategies can be used to tackle the following problems:

P1: Stabilization
P2: Noninteracting control
P3: Minimization of a quadratic cost functional

In solving problems P1 and P2, dynamic feedback laws preserving quarter-plane causality are allowed, while these are no longer suitable for solving problem P3. As we shall see, the input that minimizes the cost functional is generated through a feedback scheme where all values of the local states on the separation set $\mathcal{C}_{h+k} = \{(h+i, k-i), i \in \mathbf{Z}\}$ enter in the expression of the input value at (h,k).

2 STABILIZATION

This section is devoted to illustrating the synthesis of output feedback compensators that (internally) stabilize the plants represented by strictly proper 2-D systems. By definition, a 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ is internally stable if for any global state $\mathcal{X}_0 = \{\mathbf{x}(i,-i), i \in \mathbf{Z}\}$ with

$$\sup_i \|\mathbf{x}(i,-i)\| < \infty$$

the free evolution of the local states satisfies

$$\lim_{h+k \to +\infty} \mathbf{x}(h,k) = 0 \tag{8}$$

According to the previous definition, a stabilizing output feedback compensator is a p-input, m-output 2-D system $\bar{\Sigma} = (\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C}, \bar{D})$ that makes internally stable the closed-loop system resulting from the feedback interconnection of Σ and $\bar{\Sigma}$.

Both stability and stabilizability properties can be checked by analyzing the intersection of suitable polynomial varieties with the closed unit bidisk

$$\mathcal{P}_1 = \{(z_1, z_2) : |z_1| \le 1, |z_2| \le 1\}$$

The following results have been obtained in [7,8]:

1. Σ is internally stable if and only if the characteristic polynomial
$$\det(I - A_1 z_1 - A_2 z_2) \tag{9}$$
associated with the pair (A_1, A_2) is devoid of zeros in $\mathcal{P}_1$.
2. Σ admits a stabilizing output feedback compensator if and only if
$$[\mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O})] \cap \mathcal{P}_1 = \emptyset \tag{10}$$
Here $\mathcal{V}(\mathcal{R})$ and $\mathcal{V}(\mathcal{O})$ denote the complex varieties of the ideals generated by the maximal order minors of the following polynomial matrices in two variables:
$$\mathcal{R} = [I - A_1 z_1 - A_2 z_2 \quad B_1 z_1 + B_2 z_2] \tag{11}$$
and
$$\mathcal{O} = \begin{bmatrix} I - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix} \tag{12}$$

The meaning of condition (10) can be fully understood by examining the analogous condition in the 1-D framework. It is well known that a discrete-time 1-D system (A, B, C) is stabilizable via dynamic output feedback if and only if

$$[\mathcal{V}(\mathcal{R}_1) \cup \mathcal{V}(\mathcal{O}_1)] \cap \mathcal{D}_1 = \emptyset$$

where $\mathcal{D}_1$ denotes the closed unit disk $\{z : |z| \le 1\}$, and $\mathcal{R}_1$ and $\mathcal{O}_1$ are the PBH controllability and reconstructibility matrices:

$$\mathcal{R}_1 = [I - Az \quad Bz], \qquad \mathcal{O}_1 = \begin{bmatrix} I - Az \\ C \end{bmatrix}$$

If $N_R(z) D_R^{-1}(z)$ is a right coprime MFD of the transfer matrix of (A, B, C), so that $\det D_R$ divides $\det(I - Az)$, then $[\mathcal{V}(\mathcal{R}_1) \cup \mathcal{V}(\mathcal{O}_1)]$ is the zero set of $\det(I - Az)/\det D_R$:

$$\mathcal{V}(\mathcal{R}_1) \cup \mathcal{V}(\mathcal{O}_1) = \mathcal{V}\left(\frac{\det(I - Az)}{\det D_R}\right) \tag{13}$$

This implies that the points of $[\mathcal{V}(\mathcal{R}_1) \cup \mathcal{V}(\mathcal{O}_1)]$ are naturally associated with the so-called "hidden modes" of the system and 1-D stabilizability reduces to have all hidden modes converging to zero.

The situation is quite different in the 2-D case. Given any factor right coprime MFD $N_R(z_1,z_2)D_R^{-1}(z_1,z_2)$ of the transfer matrix $W(z_1,z_2)$ of Σ, the ideal of the maximal order minors of $[N_R \quad D_R]$ does not depend on the particular right coprime MFD and coincides with the analogous ideal of any left coprime MFD of $W(z_1,z_2)$ (see [9]). This ideal will be unambiguously denoted by $\mathcal{I}(W)$ and the points of the corresponding finite, possibly nonempty, variety $\mathcal{V}(W)$ are called "rank singularities" of Σ [or $W(z_1,z_2)$].

The variety $\mathcal{V}(W)$ turns out to be of great importance in investigating the relationship between the set of points $[\mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O})]$ and the variety of the characteristic polynomial $\mathcal{V}(\det(I - A_1z_1 - A_2z_2))$. For the 2-D case this relationship is more complex than (13), essentially because $\mathcal{V}(W)$ is an invariant subset of $[\mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O})]$ with respect to the specific realization taken into account.

In [9,10] the following result has been obtained:

THEOREM 1 Let $W(z_1,z_2)$ be the transfer matrix of a 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C)$ and $N_R(z_1,z_2)D_R^{-1}(z_1,z_2)$ be a right coprime MFD of W. Then the variety of the maximal order minors of (11) and (12) is given by

$$\mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O}) = \mathcal{V}(W) \cup \mathcal{V}\left(\frac{\det(I - A_1z_1 - A_2z_2)}{\det D_R(z_1,z_2)}\right) \tag{14}$$

Equation (14) shows that the set of critical points for 2-D stabilizability includes both the hidden modes variety [i.e., the algebraic curve associated with the polynomial $h(z_1,z_2) := \det(I - A_1z_1 - A_2z_2)/\det D_R(z_1,z_2)$, and the variety of rank singularities $\mathcal{V}(W)$]. The difference between the structures of (13) and (14) has important consequences on the stabilizability of the state-space realizations of input–output maps given by transfer matrices.

Clearly, minimal realizations in the 1-D case are stabilizable, since no hidden modes are left in the system, and therefore the polynomial matrices $\mathcal{R}_1$ and $\mathcal{O}_1$ are full rank everywhere. On the other hand, as a consequence of Theorem 1, the 2-D stabilizability condition (10) requires that both $\mathcal{V}(h)$ and $\mathcal{V}(W)$ do not intersect $\mathcal{P}_1$. For a given transfer matrix $W(z_1,z_2)$ it is always possible to compute a realization which fulfills the requirement that $\mathcal{V}(h) \cap \mathcal{P}_1 = \emptyset$; nevertheless, if $\mathcal{V}(W) \cap \mathcal{P}_1 \neq \emptyset$, there are no stabilizable realizations of W, since the set of the rank singularities $\mathcal{V}(W)$ is independent of the realization.

Theorem 1 contains a first result about the constraints that have to be fulfilled by the closed-loop polynomial variety, but it does not specify to what extent it can be modified using output feedback. The assignability of the variety has been further investigated and the key result is given by the following theorem.

THEOREM 2 [9] Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a realization of a strictly proper transfer matrix $W(z_1, z_2)$. For any output feedback compensator $\bar{\Sigma}$, the variety of the closed-loop polynomial $\bar{\Delta}(z_1, z_2)$ satisfies the inclusion

$$\mathcal{V}(\bar{\Delta}) \supseteq \mathcal{V}(h) \cup \mathcal{V}(W)$$

Vice versa, given any algebraic curve $\mathcal{C}$ that includes $\mathcal{V}(h) \cup \mathcal{V}(W)$ and excludes the origin, there exists a compensator $\bar{\Sigma}$ such that $\mathcal{V}(\bar{\Delta}) = \mathcal{C}$.

At this point two problems naturally arise:

1. Given a polynomial $c(z_1, z_2)$ in $\mathbf{R}[z_1, z_2]$, decide about the assignability of the variety $\mathcal{V}(c) := \mathcal{C}$.
2. If $\mathcal{V}(c)$ is assignable, find algorithms for realizing a compensator $\bar{\Sigma}$.

The solution of the first problem consists in verifying if

$$(0,0) \notin \mathcal{C} \tag{15}$$

$$\mathcal{V}(h) \subseteq \mathcal{C} \tag{16}$$

$$\mathcal{V}(W) \subseteq \mathcal{C} \tag{17}$$

Checking (15) is trivial. Moreover, once the polynomial $h(z_1, z_2)$ has been computed, we can easily verify (16) using a linear algorithm to see if h divides $c^{\deg h}$. Condition (17) can be checked by first computing a set of generators of $\mathcal{I}(W)$ and successively exploiting them for constructing a pair of commuting matrices M_1 and M_2, with the property

$$p(z_1, z_2) \in \mathcal{I}(W) \Leftrightarrow p(M_1, M_2) = 0 \tag{18}$$

Thus $\mathcal{V}(W) \subseteq \mathcal{C}$ if and only if $c(M_1, M_2)$ is a nilpotent matrix. For the construction of M_1 and M_2 the reader is referred to [11].

It remains to show how to compute the polynomial h and a set of generators for $\mathcal{I}(W)$ starting from the system matrices A_1, A_2, B_1, B_2, C. For this, let

$$[C \operatorname{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2)][I_m \det(I - A_1 z_1 - A_2 z_2)]^{-1} = \bar{N}\bar{D}^{-1}$$

be a MFD of the transfer matrix $W(z_1, z_2)$. The generators set can be obtained by evaluating the maximal order minors $m_1, m_2, \ldots, m_t$ in $[\bar{N}^T \quad \bar{D}^T]$ and then by eliminating their greatest common denominator (g.c.d.) $d(z_1, z_2)$. Thus h is given by

$$h = \frac{\det(I - A_1 z_1 - A_2 z_2)}{\det D_R} = \frac{\det(I - A_1 z_1 - A_2 z_2) d(z_1, z_2)}{\det \bar{D}}$$

As far as the second problem is concerned, suppose that a variety $\mathcal{C} = \mathcal{V}(c)$ that fulfills conditions (15)–(17) has been given, and suppose that we want to synthesize a compensator $\bar{\Sigma}$ that produces a closed-loop polynomial $\bar{\Delta}$ whose variety is $\mathcal{C}$. The procedure can be summarized as follows:

1. Evaluate a right coprime MFD $N_R D_R^{-1}$ of W. This can be performed by using the primitive factorization algorithm [12] or other algorithms that do not require primitive factorizations [13].
2. Compute the maximal order minors $m_1, m_2, \ldots, m_t$.
3. Compute an integer r and a Gröbner basis $g_1, g_2, \ldots, g_w$ such that $c^r = \sum_i m_i g_i$. A technique for performing this step has been presented in [11].
4. Solve the Bézout equation $c^r I_m = XD_R + YN_R$.
5. Use the realization algorithm given in [9] for computing a coprime realization of $X^{-1}Y$.

3 NONINTERACTING CONTROL

The noninteracting control problem consists in designing a 2-D dynamic state feedback compensator and a static precompensator which guarantee that the transfer matrix of the resulting closed-loop system is diagonal. As is well known [14,15], in the 1-D environment the problem above can be solved if and only if it is possible to find a bicausal precompensator that decouples the original system. Moreover, if we are not interested in stabilizing the system and we only look at the realization of a noninteracting control, it has been proved that a static state feedback compensator can be substituted for the dynamic, with no augmentation of the order of the system. The properties above do not hold for 2-D systems. In particular, it has been shown [16] that a 2-D bicausal decoupling precompensator cannot be replaced by a feedback scheme unless some assumptions are introduced on the structure of system (1). The simplest of these consists in assuming that the matrix

$$[B_1 \quad B_2] \tag{19}$$

is injective. Confining ourselves to systems that fulfill this restriction, we can prove the following theorem, which provides necessary and sufficient conditions for solving the noninteracting control problem.

THEOREM 3 [14] Let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be an m-input, m-output 2-D system with dimension n and consider the polynomial matrix

$$M_0 = \begin{bmatrix} C_1(A_1z_1 + A_2z_2)^{d_1}(B_1z_1 + B_2z_2) \\ C_2(A_1z_1 + A_2z_2)^{d_2}(B_1z_1 + B_2z_2) \\ \vdots \\ C_m(A_1z_1 + A_2z_2)^{d_m}(B_1z_1 + B_2z_2) \end{bmatrix} \tag{20}$$

where C_i is the ith row of C and

$$d_i = \min\{j : C_i(A_1z_1 + A_2z_2)^j(B_1z_1 + B_2z_2) \neq 0\}$$

Assume also that $[B_1 \quad B_2]$ is injective. Then Σ can be decoupled by a dynamic state feedback compensator plus a static precompensator if and only if:

(a) There exists a constant nonsingular matrix Q_0 such that M_0Q_0 is diagonal:

$$M_0Q_0 = \text{diag}\,\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$$

where ϵ_i, $i = 1, 2, \ldots, m$ are homogeneous polynomials in $\mathbb{R}[z_1, z_2]$ of degree $d_i + 1$.

(b) $M_0^{-1}C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)$ is proper rational.

Interestingly enough, the proof of the theorem is constructive and provides a synthesis procedure for a decoupling compensator. Here we shall only outline the sufficiency part, which will be needed in the subsequent discussion on stable decoupling.

First, recalling assumption (a), we have

$$\begin{aligned} &C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) \\ &\qquad \times [M_0^{-1}(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)]^{-1}Q_0 \\ &\quad = M_0Q_0 = \text{diag}\,\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\} \end{aligned} \tag{21}$$

Now, using assumption (b), it can be shown that

$$M_0^{-1}C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)$$

is a bicausal transfer matrix, which in turn implies that

$$P(z_1, z_2) := [M_0^{-1}C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)]^{-1}Q_0 \tag{22}$$

is the transfer matrix of a bicausal decoupling precompensator.

So the sufficiency part of the proof reduces to show that $W(z_1, z_2)P(z_1, z_2)$ can be expressed as the transfer matrix of an interconnected system that includes the static precompensator Q_0 and a suitable causal feedback compensator $K(z_1, z_2)$ that solves the following equation:

$$\begin{aligned} &W(z_1, z_2)P(z_1, z_2) \\ &\quad = C[I - A_1z_1 - A_2z_2 - (B_1z_1 + B_2z_2)K(z_1, z_2)]^{-1}(B_1z_1 + B_2z_2)Q_0 \end{aligned} \tag{23}$$

Note that the right-hand side of (23) can be rewritten as

$$\begin{aligned} &C(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2) \\ &\quad \cdot [I - K(z_1, z_2)(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)]^{-1}Q_0 \end{aligned}$$

Hence, expressing $P(z_1, z_2)$ as $P_S + Q_0$, where P_S denotes the strictly proper part of P and taking the inverse on both sides of (23), one gets

$$[I + P_S(z_1, z_2)Q_0^{-1}]^{-1} = I - K(z_1, z_2)(I - A_1z_1 - A_2z_2)^{-1}(B_1z_1 + B_2z_2)$$

Now introduce a 2-D realization $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}, I)$ of $[I + P_s(z_1, z_2)Q_0^{-1}]^{-1}$ and let

$$\tilde{K}(z_1, z_2) := K(z_1, z_2)(I - A_1 z_1 - A_2 z_2)^{-1}$$

Then the search for a causal $K(z_1, z_2)$ that solves (23) reduces to finding a causal rational matrix $\tilde{K}(z_1, z_2)$ that solves the following equation:

$$\tilde{C}(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2)^{-1}(\tilde{B}_1 z_1 + \tilde{B}_2 z_2) = -\tilde{K}(z_1, z_2)(B_1 z_1 + B_2 z_2)$$

By the injectivity of $[B_1 \quad B_2]$, there exists a constant matrix F such that

$$F(B_1 z_1 + B_2 z_2) = \tilde{B}_1 z_1 + \tilde{B}_2 z_2$$

We therefore have that a solution of (23) is provided by the proper rational matrix

$$K(z_1, z_2) = -\tilde{C}(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2)^{-1} F(I - A_1 z_1 - A_2 z_2) \tag{24}$$

which constitutes the transfer matrix of a causal decoupling compensator.

If the original system Σ satisfies both the decouplability conditions of Theorem 3 and the state feedback stabilizability condition,

$$\text{rank}[I - A_1 z_1 - A_2 z_2 \quad B_1 z_1 + B_2 z_2] = n \quad \forall (z_1, z_2) \in \mathcal{P}_1$$

we can solve simultaneously the stabilization and decoupling problems. In other words, it is possible to synthesize a decoupling compensator that stabilizes the closed-loop system. This property is an immediate consequence of the following facts:

1. State feedback stabilization preserves decouplability.
2. Decoupling can be performed by a state feedback compensator that preserves internal stability.

These enable us to solve the stable decoupling problem using a two-step procedure: first, state feedback stabilization is performed on the plant; and second, the resulting system is decoupled using an appropriate state feedback compensator that does not affect internal stability.

The proof of the first fact is immediate since the matrix M_0 in the original system coincides with the corresponding matrix in the closed-loop system. So both the original and the closed-loop systems satisfy condition 1 of Theorem 3. Moreover, the transfer matrix W_F of the feedback system differs from W in a bicausal multiplicative factor. Therefore, $M_0^{-1} W_F$ is proper rational.

To prove the second fact, note that if the plant Σ is internally stable, $\det(I - A_1 z_1 - A_2 z_2)$ does not vanish in $\mathcal{P}_1$ and consequently, in (22) all singularities of $P(z_1, z_2)^{-1}$ are external to $\mathcal{P}_1$. Assuming that $\tilde{\Sigma}$ is a coprime realization of $(I + P_S Q_0^{-1})^{-1}$, the characteristic polynomial $\det(I - \tilde{A}_1 z_1 - \tilde{A}_2 z_2)$ does not vanish in $\mathcal{P}_1$ and therefore $K(z_1, z_2)$ in (24) admits an internally stable realization $\tilde{\Sigma}$. Moreover, $M_0 Q_0$ is the transfer matrix of the series connection of the bicausal precompensator $P(z_1, z_2)$ and the plant Σ. Therefore, the feedback connection of Σ and $\bar{\Sigma}$ is internally stable.

4 OPTIMAL CONTROL

The compensator synthesis procedure illustrated in Section 2 enables us to obtain any preassigned variety of the closed-loop polynomial provided that the constraints specified by Theorem 2 are satisfied. However, the pole placement design essentially affects the asymptotic behavior and exhibits a poor control of the short-term system response.

In this section we tackle the problem from a different point of view and outline a state feedback synthesis procedure based on the minimization of a quadratic cost functional J. The system that constitutes the end result of the optimal design is not merely internally stable but satisfies additional requirements on the state and input evolutions that are summarized by J.

Assume that the initial global state (8) is an l_2 sequence and consider the cost functional

$$J = \sum_{h+k\geq 0} \mathbf{x}^T(h,k)Q\mathbf{x}(h,k) + \mathbf{u}^T(h,k)R\mathbf{u}(h,k) \tag{25}$$

with $Q \geq 0$ and $R > 0$. Theorems 4 and 5 provide a complete solution to the following optimal control problems:

1. Given $\mathcal{X}_0$, derive conditions for the existence and the uniqueness of an input function $\mathbf{u}(\cdot,\cdot)$ that minimizes J.
2. Whenever these conditions are satisfied, explicitly compute the optimal input function and the corresponding value of J.

The following theorem shows that the existence and uniqueness of a stabilizing optimal control reduce to rank conditions on polynomial matrices in two variables.

THEOREM 4 [17] For any $\mathcal{X}_0 \in l_2$, there exists an l_2-solution of the optimal control problem [i.e., an input $\mathbf{u}(\cdot,\cdot)$ in l_2 such that the corresponding state evolution $\mathbf{x}(\cdot,\cdot)$ is in l_2 and the value of J is minimized] if and only if the polynomial matrix (10) has full rank on the set

$$\mathcal{M} = \{(z_1, z_2) \in \mathbf{C} \times \mathbf{C} : |z_2| = |z_2| \leq 1\}$$

and the polynomial matrix

$$\begin{bmatrix} I - A_1 z_1 - A_2 z_2 \\ Q \end{bmatrix} \tag{26}$$

has full rank on the unit torus

$$\mathcal{T}_1 = \{(z_1, z_2) \in \mathbf{C} \times \mathbf{C} : |z_1| = |z_2| = 1\}$$

When the conditions of Theorem 4 are fulfilled, the optimal control law is obtained via an algebraic Riccati equation whose coefficients are polynomial matrices in one variable.

THEOREM 5 [17] Assume that (10) is full rank on $\mathcal{M}$ and (26) is full rank on $\mathcal{T}_1$. Then the following algebraic Riccati equation (ARE-z)

$$\begin{aligned} P(z) = Q + (A_1^T + A_2^T z^{-1})P(z)(A_1 + A_2 z) - (A_1^T + A_2^T z^{-1})P(z)(B_1 + B_2 z) \\ \cdot [R + (B_1^T + B_2^T z^{-1})P(z)(B_1 + B_2 z)]^{-1}(B_1^T + B_2^T z^{-1})P(z)(A_1 + A_2 z) \end{aligned}$$

in the unknown matrix $P(z)$ admits a unique solution in an open annulus that includes the unit circle γ_1, with the following properties:

(a) $P(e^{j\omega}) = P^*(e^{j\omega}) \geq 0, \ \forall \omega \in [0, 2\pi]$

(b) The matrix

$$K(z) := -[R + (B_1^T + B_2^T z^{-1})P(z)(B_1 + B_2 z)]^{-1}(B_1^T + B_2^T z^{-1})P(z)(A_1 + A_2 z)$$

is analytic in an open annulus that includes γ_1.

The coefficients of its Laurent series expansion

$$K(z) = \sum_{i=-\infty}^{+\infty} K_i z^i \tag{27}$$

provide a stabilizing feedback law

$$u(h, k) = \sum_{i=-\infty}^{+\infty} K_i x(h + i, k - i) \tag{28}$$

and the minimum value of J is given by

$$J_{\min} = \frac{1}{2\pi} \int_0^{2\pi} \hat{\mathcal{X}}_0^*(\omega) P(e^{j\omega}) \hat{\mathcal{X}}_0(\omega)\, d\omega$$

where $\hat{\mathcal{X}}_0(\omega)$ denotes the Fourier transform of the l_2-sequence $\mathcal{X}_0$.

The proofs of Theorems 4 and 5 are quite long and the interested reader is referred to [17]. We shall give here two examples. The first one shows how the solvability conditions based on the rank of (10) and (26) reflect into the analytic structure of $P(e^{j\omega})$. The second one gives an idea of some difficulties involved in the computation of the feedback matrices K_i, even in dimension 1.

EXAMPLE 1 Assume in (1)

$$m = n = 1, \qquad A_1 = B_1 = B_2 = 1, \qquad A_2 = -1$$

and in (25)

$$R = Q = 1$$

In this case the solution of (ARE-z) can be obtained in closed form as

$$P(z) = \frac{-1 \mp \sqrt{5 + 2z + 2z^{-1}}}{-2(2 + z + z^{-1})}$$

Letting $z = e^{j\omega}$, the first solution is negative and the second is given by

$$P(e^{j\omega}) = \frac{1}{-1 + \sqrt{5 + 4\cos\omega}}$$

Since we are looking for nonnegative solutions, we consider only the second one, which is positive for $\omega \in [0, 2\pi]$, except at $\omega = \pi$, where $P(e^{j\omega})$ diverges. Actually, this is not surprising because (10) is not full rank at $(\frac{1}{2}, -\frac{1}{2}) \in \mathcal{M}$. Hence for some initial global state in l_2 a stabilizing optimal feedback law does not exist.

EXAMPLE 2 Let's change only the sign of A_2 in Example 1. In this case the unique positive definite solution of (ARE-z) along γ_1 is given by

$$P(e^{j\omega}) = \frac{2}{\sqrt{1 + 16(1 + \cos\omega)^2} - (3 + 4\cos\omega)}$$

and the corresponding feedback matrix is

$$K(e^{j\omega}) = \frac{4(1 + \cos\omega)}{1 + \sqrt{1 + 16(1 + \cos\omega)^2}}$$

A plot of $P(e^{j\omega})$ is given in Figure 1. Since $P(e^{j\omega})$ attains its minimum value at $\omega = \pi$, we have

$$J_{\min}(\mathcal{X}_0) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{j\omega}) \|\hat{\mathcal{X}}_0(\omega)\|^2 \, d\omega \geq \|\mathcal{X}_0\|^2 P(e^{j\pi})$$

and $J_{\min}(\mathcal{X}_0)$ can be made arbitrarily close to the lower bound if we consider initial global states whose spectral content is concentrated in a narrow neighborhood of $e^{j\pi}$.

The computation of the values of K_h depends on evaluation of the following integrals:

$$K_h = \frac{1}{2\pi} \int_0^{2\pi} \frac{4(1 + \cos\omega)\cos(\omega h)}{1 + \sqrt{1 + (1 + \cos\omega)^2}} \, d\omega, \qquad h = 0, 1, \ldots$$

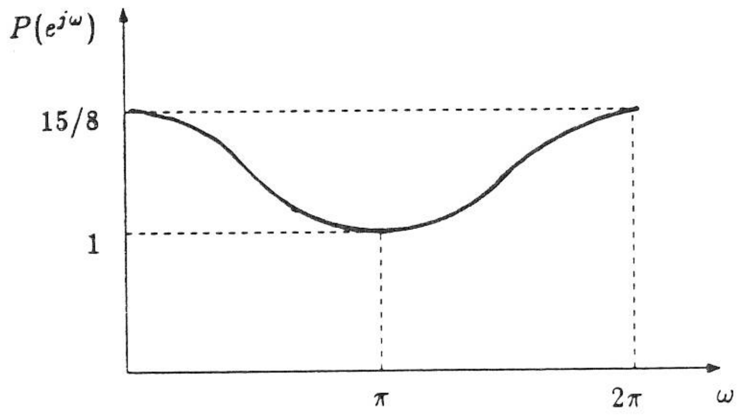

Figure 1. Plot of $P(e^{j\omega})$.

When infinitely many K_h's are different from zero, the optimal feedback law (28) cannot be implemented by a finite-dimensional device and the resulting closed-loop system is a half-plane causal 2-D system, whose updating equation, in principle, requires coping with an infinite-dimensional state vector.

To overcome storage and computation problems, it seems natural to investigate whether, when (1) satisfies the rank condition of Theorem 4, the stabilizing feedback matrix could be constrained to have all elements in the bilateral polynomials ring $\mathbf{R}[z, z^{-1}]$. An obvious advantage of this control law is that $\mathbf{u}(h, k)$ would depend only on a finite number of local states, which makes the closed-loop system weakly causal [5,6]. The question above can be answered positively. Actually, the bilateral polynomial matrix

$$K_N(z) = \sum_{i=-N}^{N} K_i z^i$$

obtained by truncation of the Laurent series (27) gives a stabilizing state feedback, provided that N is large enough. Even more, when N diverges and l_2 initial states are considered, the corresponding cost functional J_N converges asymptotically to the minimum value $J_{\min}$.

5 CONCLUSIONS

The classical problems of stabilization, noninteracting control, and minimization of a quadratic cost functional, which are solved in 1-D theory using control strategies based on static-state feedback, require dynamic feedback in the 2-D context.

It has been shown that stabilization and decoupling can be achieved under fairly general assumptions using 2-D dynamic compensators. In both problems, static feedback laws are quite inefficient and one has to resort to methods based on dynamic compensation. However, if the conditions given in Theorems 2 and 3 are fulfilled, the solutions one obtains lead to closed-loop systems that still preserve the quarter-plane causality.

This is no longer true when we solve the optimal control problem. In this case the feedback solution gives rise to half-plane causal systems and to weakly causal systems when suboptimal strategies are adopted.

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