PROPERTIES OF PAIRS OF MATRICES AND STATE MODELS FOR TWO-DIMENSIONAL SYSTEMS. PART 2: MODELS STRUCTURE AND REALIZATION PROBLEMS

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1. Introduction

In this chapter we shall focus on some aspects of the construction of two-dimensional state space realizations.

We take into account external descriptions given by rational transfer matrices and autoregressive behaviours represented as kernels of polynomial matrices. Our investigation mainly concentrates on some connections between the structure of state model matrices and the algebraic varieties associated with the polynomials involved in the external representations.

2. Realization and transfer function singularities

Given a two-dimensional rational proper matrix $W(z_1, z_2)$, it is well-known [1–4] that it is always possible to construct state models with structure (I,2,1) whose transfer matrices coincide with $W(z_1, z_2)$.

The problem we shall tackle in this section consists in connecting the structure of $W(z_1, z_2)$ with the properties of the pairs (A_1, A_2) entering in the state models. In particular the dimension of the realizations and the features of the minimal ones constitute and interesting and still non entirely understood aspect of this matter. To avoid cumbersome notations, we shall consider here scalar transfer functions.

Let

$$W(z_1, z_2) = p(z_1, z_2)/q(z_1, z_2) \tag{2.1}$$

be an irreducible proper rational function, with

$$p(z_1,z_2)=\sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij} z_1^i z_2^j,$$
$$q(z_1,z_2)=\sum_{i=0}^{m}\sum_{j=0}^{n} q_{ij} z_1^i z_2^j. \tag{2.2}$$

Properness implies $q_{00} \neq 0$, so we can assume $q_{00} = 1$. Moreover, in solving the realization problem, we can restrict to the case of a strictly proper $W(z_1,z_2)$ (i.e.$p_{00} = 0$). Assume that m and n represent the degrees in z_1 and in z_2 of at least one between $p(z_1,z_2)$ and $q(z_1,z_2)$. Then multiplication of p and q by $z_1^{-m}z_2^{-n}$ provides an irreducible representation

$$W(z_1,z_2)=\tilde{p}(z_1^{-1},z_2^{-1})/\tilde{q}(z_1^{-1},z_2^{-1}), \tag{2.3}$$

with

$$\tilde{p}(z_1^{-1},z_2^{-1})=\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m-i,n-j} z_1^{-i} z_2^{-j},$$
$$\tilde{q}(z_1^{-1},z_2^{-1})=\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m-i,n-j} z_1^{-i} z_2^{-j}. \tag{2.4}$$

Note that $\tilde{q}$ includes the monomial $z_1^{-m}z_2^{-n}$ and that all monomials in $\tilde{p}(z_1^{-1},z_2^{-1})$ and in $\tilde{q}(z_1^{-1},z_2^{-1})$ have degrees in z_1^{-1} and z_2^{-1} less than or equal to m and n, respectively.

In the sequel it will be convient to adopt the change of variables

$$\eta = z_1^{-1}, \qquad \xi = z_1^{-1}z_2 \tag{2.5}$$

and refer to the following representation of the transfer function

$$W'(\eta,\xi)=\frac{f(\eta,\xi)}{g(\eta,\xi)}, \tag{2.6}$$

with

$$\begin{aligned} f(\eta,\xi) &= \xi^n[\tilde{p}(z_1^{-1},z_2^{-1})]_{z_1^{-1}=\eta, z_1^{-1}z_2=\xi} \\ &= \eta^{m+n-1}a_1(\xi)+\eta^{m+n-2}a_2(\xi)+\ldots+a_{m+n}(\xi), \\ g(\eta,\xi) &= \xi^n[\tilde{q}(z_1^{-1},z_2^{-1})]_{z_1^{-1}=\eta, z_1^{-1}z_2=\xi} \\ &= \eta^{m+n-1}b_1(\xi)+\eta^{m+n-2}b_2(\xi)+\ldots+b_{m+n}(\xi). \end{aligned} \tag{2.7}$$

The nonzero monomials of f and g are represented as points of the dashed parallelogram in Figure 1. In general (2.6) needs not be irreducible. An obvious necessary condition for the irreducibility of (2.6) is that $a_{m+n}(\xi)$ or $b_{m+n}(\xi)$ be nonzero. This condition is also sufficient. Actually, we shall prove that if p and q are coprime, powers of η are the only possible common

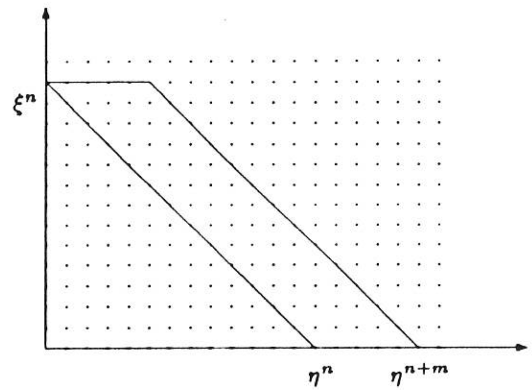

Fig. 1.

factors of f and g. Indeed, suppose $c(\eta,\xi)$ be a common factor of f and g, so that

$$f(\eta,\xi)=c(\eta,\xi)\bar{f}(\eta,\xi),$$
$$g(\eta,\xi)=c(\eta,\xi)\bar{g}(\eta,\xi). \tag{2.8}$$

Using (2.5), we get

$$\tilde{p}(z_1^{-1},z_2^{-1})=z_1^n z_2^{-n} c\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right)\bar{f}\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right),$$
$$\tilde{q}(z_1^{-1},z_2^{-1})=z_1^n z_2^{-n} c\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right)\bar{g}\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right). \tag{2.9}$$

Clearly there exist positive integers r and s with $r+s \geq n$ such that $\bar{f}z_2^{-r}, \bar{g}z_2^{-s}$ and cz_2^{-s} are in $\boldsymbol{R}[z_1^{-1},z_2^{-1}]$. Thus, letting $t=r+s-n$, we obtain the following factorizations in $\boldsymbol{R}[z_1^{-1},z_2^{-1}]$,

$$z_1^{-n}z_2^{-t}\tilde{p}(z_1^{-1},z_2^{-1})=\left[z_2^{-s}c\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right)\right]\left[\bar{f}\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right)z_2^{-r}\right],$$

$$z_1^{-n}z_2^{-t}\tilde{q}(z_1^{-1},z_2^{-1})=\left[z_2^{-s}c\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right)\right]\left[\bar{g}\left(z_1^{-1},\frac{z_1^{-1}}{z_2^{-1}}\right)z_2^{-r}\right].$$

Since $\tilde{p}$ and $\tilde{q}$ are coprime, the irreducible factors of cz_2^{-s} are z_1^{-1} and/or z_2^{-1}, which implies that $c(z_1^{-1},z_1^{-1}/z_2^{-1})$ must be $z_1^{-h}z_2^k$ for some nonnegative h and k. Because $g(\eta,\xi)$ is monic in η, $c(\eta,\xi)$ is necessarily a nonnegative power of η.

Every state space realization $\Sigma=(A_1,A_2,B_1,B_2,C)$ of $W(z_1,z_2)$ satisfies

$$C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)=W(z_1,z_2) \tag{2.10}$$

or, equivalently,

$$C(\eta I - A_1 - A_2\xi)^{-1}(B_1 + B_2\xi) = W'(\eta, \xi). \tag{2.11}$$

When no cancellations arise between f and g in (2.6), the degree of $g(\eta, \xi)$ with regards to η , and [as a consequence of (2.11)], a lower bound for the dimension of Σ, are given by $n + m$. Actually, it is not known whether all transfer functions admit a realization of dimension $m + n$. In the following we shall give some evidence to support this conjecture, by providing a constructive procedure for obtaining a pair of $(m + n) \times (m + n)$ matrices A_1, A_2, such that

$$\det(\eta I - A_1 - A_2\xi) = g(\eta, \xi). \tag{2.12}$$

It is not known if the condition

$$C\operatorname{adj}(\eta I - A_1 - A_2\xi)(B_1 + B_2\xi) = f(\eta, \xi) \tag{2.13}$$

can always be safisfied using the entries of B_1, B_2, C and exploiting the fact that not all entries of A_1 and A_2 are determined by (2.12).

The procedure can be summarized in the following steps [5].

Step 1. Since the case $m = 0$ corresponds to a one- dimensional realization problem and can be solved by standard techniques, we assume in (2.7) $m \geq 1$. Write A_1 and A_2 in the form

$$A_1 = \left[\begin{array}{cccc:c} 0 & & & & \boldsymbol{r}_0 \\ 1 & & & & \boldsymbol{r}_1 \\ & 1 & & & \boldsymbol{r}_2 \\ & & \ddots & & \vdots \\ & & & 1 & \boldsymbol{r}_m \\ \cdots & \cdots & & \cdots & \cdots \\ & \mathbf{0} & & & \mathbf{0} \end{array}\right],$$

$$A_2 = \left[\begin{array}{c:cccc} & & & & \boldsymbol{s}_0 \\ \mathbf{0} & & & & \boldsymbol{s}_1 \\ & & & & \boldsymbol{s}_2 \\ & & & & \boldsymbol{s}_m \\ \cdots & \cdots & & & \cdots \\ & 1 & & & \\ \mathbf{0} & & 1 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{array}\right], \tag{2.14}$$

where $\boldsymbol{r}_0, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_m, \boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_m$ are arbitrary row vectors of dimension n. Then we have

$$\begin{aligned} &\det(\eta I - A_1 - A_2\xi) \\ &= \det\begin{bmatrix} & \boldsymbol{r}_0 & \\ -\xi & \eta & \\ & -\xi & \ddots \\ & & -\xi \quad \eta \end{bmatrix} + \eta \det\begin{bmatrix} & \boldsymbol{r}_1 & \\ -\xi & \eta & \\ & -\xi & \ddots \\ & & -\xi \quad \eta \end{bmatrix} \\ &+\xi \det\begin{bmatrix} & \boldsymbol{s}_0 & \\ -\xi & \eta & \\ & -\xi & \ddots \\ & & -\xi \quad \eta \end{bmatrix} + \eta^2 \det\begin{bmatrix} & \boldsymbol{r}_2 & \\ -\xi & \eta & \\ & -\xi & \ddots \\ & & -\xi \quad \eta \end{bmatrix} \\ &+\eta\xi \det\begin{bmatrix} & \boldsymbol{s}_1 & \\ -\xi & \eta & \\ & -\xi & \ddots \\ & & -\xi \quad \eta \end{bmatrix} + \ldots + \eta^{n+m} \\ &= \sum_{i=0}^{m} (\boldsymbol{r}_i \eta^i + \boldsymbol{s}_i \eta^{i-1}\xi) \begin{bmatrix} \eta^{n-1} \\ \eta^{n-2}\xi \\ \vdots \\ \xi^{n-1} \end{bmatrix}. \end{aligned} \tag{2.15}$$

Step 2. Rewrite $g(\eta,\xi)$ as a sum of homogeneous polynomials in η and ξ

$$g(\eta,\xi)=[g_0]\begin{bmatrix}\eta^n\\ \eta^{n-1}\xi\\ \vdots\\ \xi^n\end{bmatrix}+[g_1]\,\eta\begin{bmatrix}\eta^n\\ \eta^{n-1}\xi\\ \vdots\\ \xi^n\end{bmatrix}+\ldots+[g_m]\,\eta^m\begin{bmatrix}\eta^n\\ \eta^{n-1}\xi\\ \vdots\\ \xi^n\end{bmatrix}, \tag{2.16}$$

so that (2.12) is equivalent to the following set of independent linear equations in the unknowns $\mathbf{r}_i$ and $\boldsymbol{s}_i$,

$$\begin{aligned}
\boldsymbol{r}_0\begin{bmatrix}\eta^{n-1}\\ \eta^{n-2}\xi\\ \vdots\\ \xi^{n-1}\end{bmatrix}&=0,\\
(\boldsymbol{r}_1\eta+\boldsymbol{s}_1\xi)\begin{bmatrix}\eta^{n-1}\\ \eta^{n-2}\xi\\ \vdots\\ \xi^{n-1}\end{bmatrix}&=[g_0]\begin{bmatrix}\eta^n\\ \eta^{n-1}\xi\\ \vdots\\ \xi^n\end{bmatrix},\\
(\boldsymbol{r}_2\eta+\boldsymbol{s}_2\xi)\begin{bmatrix}\eta^{n-1}\\ \eta^{n-2}\xi\\ \vdots\\ \xi^{n-1}\end{bmatrix}&=[g_1]\begin{bmatrix}\eta^n\\ \eta^{n-1}\xi\\ \vdots\\ \xi^n\end{bmatrix},\\
&\cdots\\
(\boldsymbol{r}_m\eta+\boldsymbol{s}_m\xi)\begin{bmatrix}\eta^{n-1}\\ \eta^{n-2}\xi\\ \vdots\\ \xi^{n-1}\end{bmatrix}&=[g_m]\begin{bmatrix}\eta^n\\ \eta^{n-1}\xi\\ \vdots\\ \xi^n\end{bmatrix}.
\end{aligned} \tag{2.17}$$

While $\boldsymbol{r}_0$ is uniquely determined by (2.15), the pairs $\boldsymbol{r}_1, \boldsymbol{s}_i$, $i = 1,2...m$, are not, and each of them contributes $n-1$ degrees of freedom to system (2.15).

Note that, once (2.12) has been satisfied, the number $mn+3n+2m$ of free parameters in $A_1, A_2.B_1, B_2, C$ exceeds the number $mn+n+m$ of coefficients of $f(\eta,\xi)$.

In presence of cancellations between f and g, we can show that there are

cases where realizations with dimension less than $n+m$ can be constructed. So the degree of the denominator in (2.3) needs not constitute a lower bound for the dimension of minimal realizations.

Example. The following transfer function

$$W(z_1,z_2)=(z_1+z_2)^3+2z_2^2+z_2 =\frac{(z_1^{-1}+z_2^{-1})^3+2z_1^{-3}z_2^{-1}+z_1^{-3}z_2^{-2}}{z_1^{-3}z_2^{-3}} \tag{2.18}$$

can be rewritten as

$$W'(\eta,\xi)=\frac{(\xi+1)^3+2\xi^2\eta+\xi\eta^2}{\eta^3} \tag{2.19}$$

and is realized by a fourth-order system (see Section 3). In this case the dimension of the realization is less than $m+n$. As it will be shown, fourth-order realizations are minimal and, consequently, the η-degree of the denominator of $W'(\eta.\xi)$, once common factors have been eliminated, needs not provide the dimension of minimal realizations.

Remark. It is worthwhile to notice that if we adopt the particular structure of the matrices A_1, A_2, B_1, B_2, C that characterizes the Roesser's model, $n+m$ effectively represents a lower bound for the dimension of minimal realizations. Indeed the transfer function of Roesser's model, [Chapter 8, (2.3)], is given by

$$\bar{C}\begin{bmatrix} z_1^{-1}I-\bar{A}_{11} & -\bar{A}_{12}\\ -\bar{A}_{21} & z_2^{-1}I-\bar{A}_{22}\end{bmatrix}^{-1}\begin{bmatrix}\bar{B}_1\\ \bar{B}_2\end{bmatrix} \tag{2.20}$$

and, therefore, has a denominator with z_1^{-1} and z_2^{-1} degrees given by the dimensions of $\bar{A}_{11}$ and $\bar{A}_{22}$, respectively. Consequently Roesser's models that realize the transfer function [Chapter 8, (2.3)] must have a state space with dimension greater than or equal to $m+n$.

So far our discussion of state models mainly concentrated on the connection between the characteristic polynomial of the pair (A_1, A_2) and the rational transfer function of the system.

There are other polynomial structures, besides the characteristic polynomial $\det(I-A_1z_1-A_2z_2)$, that play an important role in analyzing the dynamical properties of two-dimensional systems. These constitute the two- dimensional counterpart of the well-known PBH controllability and reconstructibility matrices [6 8], and will be denoted as

$$\mathcal{R}=[\,I-A_1z_1-A_2z_2 \quad B_1z_1+B_2z_2\,], \tag{2.21}$$

$$\mathcal{O}=\begin{bmatrix} I-A_1z_1-A_2z_2\\ C\end{bmatrix}. \tag{2.22}$$

Through the rest of this section we shall investigate how the structure of the transfer function influences, through the so-called rank singularities, the rank of the matrices $\mathcal{R}$ and $\mathcal{O}$ associated with any realization of W and, consequently, the possibility of synthesing asymptotic observers and stabilizing compensators.

As an aid to discuss the connections between $W(z_1, z_2)$ and its state models, we proceed to introduce the class of coprime realizations. A realization $\Sigma = (A_1, A_2, B_1, B_2, C)$ of (2.1) is called 'coprime' if the characteristic polynomial $\det(I - A_1 z_1 - A_2 z_2)$ coincides with $q(z_1, z_2)$. Since it is always possible to get coprime realizations of $W(z_1, z_2)$ with arbitrarily high dimension, these need not be minimal. On the other hand, as it will be shown in Section 3, the minimality of a realization of $W(z_1, z_2)$ does not imply its coprimeness. Note that since two-dimensional minimal realizations in general are not algebraically equivalent, the existence of a minimal noncoprime realization does not imply that all minimal realizations of W are not coprime. In particular $W(z_1, z_2)$ in Section 3 admits both coprime and noncoprime minimal realizations. However, it is not known whether all two-dimensional transfer functions have minimal coprime realizations.

Let $\mathcal{V}(\mathcal{R})$ and $\mathcal{V}(\mathcal{O})$ denote the varieties of the points where $\mathcal{R}$ and $\mathcal{O}$ are not full rank, and $\mathcal{V}(W)$ the set of rank singularities of W, i.e., the variety of the ideal generated by the polynomials p and q in (2.1). For all realizations of W, we have

$$\mathcal{V}(W) \subseteq \mathcal{V}(\mathcal{R}) \cup \mathcal{V}(\mathcal{O}) \tag{2.23}$$

and the equality sign holds if and only if $\mathcal{R}$ and $\mathcal{O}$ are relative to a coprime realization.

If $P(z_1, z_2)Q^{-1}(z_1, z_2)$ is a right coprime MFD of a $p \times m$ transfer matrix $W(z_1, z_2)$, (2.23) still holds. In this case $\mathcal{V}(W)$ is the variety of the maximal order minors of $[Q^T \quad P^T]$ and coprime realizations are characterized by the condition

$$\det(I - A_1 z_1 - A_2 z_2) = \det Q(z_1, z_2).$$

To illustrate the consequences of (2.23), let $\Sigma = (A_1, A_2, B_1, B_2, C)$ be a coprime realization of a given $p \times m$ strictly proper transfer matrix

$$W(z_1, z_2) = P(z_1, z_2)Q^{-1}(z_1, z_2) \tag{2.24}$$

and $\Sigma_r = (F_1, F_2, G_1, G_2, H, J)$ an arbitrary two- dimensional system with p inputs and m outputs. If $R^{-1}(z_1, z_2)S(z_1, z_2)$ is any MFD of the transfer matrix of Σ_r satisfying $\det(I - F_1 z_1 - F_2 z_2) = \det R(z_1, z_2)$ the closed loop characteristic polynomial of the feedbak connection of Σ and Σ_r is given by

$$\Delta_c(z_1, z_2) = \det(RQ + SP). \tag{2.25}$$

Using the Binet–Chauchy formula, $\Delta_c(z_1, z_2)$ is expressed as the sum of the products of all possible minors of maximal order $c_i, i = 1, 2 \ldots \rho$, of $[R \quad S]$ into the corresponding minors of the same order m_i, $i = 1, 2, \ldots, \rho$, of $[Q^T \quad P^T]$, that is

$$\det(RQ + SP) = \sum_{i=1}^{\rho} c_i m_i.$$

Hence $\det(RQ + SP)$ belongs to the ideal $\mathcal{I}(m_1, m_2, \ldots, m_\rho)$ for any choice of the compensator. Conversely, given any polynomial $p \in \mathcal{I}$, there exists a compensator $R^{-1}S$ such that

$$RQ + SP = pI. \tag{2.26}$$

Hence the characteristic polynomial Δ_c is a power of p and $\mathcal{V}(\bar{\Delta})$ is freely assignable except that it must include $\mathcal{V}(W)$ and does not contain $(0, 0)$. We summarize our conclusions.

Proposition 1. *The system Σ admits a stabilizing compensator if and only if $\mathcal{V}(W)$ does not intersect the closed unit polydisc $\mathcal{P}_1$.*

An interesting issue associated with the synthesis of two- dimensional compensators is that of checking feedback stabilizability. That is, how can a particular two-dimensional system be recognized as being stabilizable without an explicit computation of $\mathcal{V}(W)$?

As we shall see, the stabilizability property can be expressed in terms of the spectral properties of a pair of commutative matrices M_1, M_2 that can be obtained using a Gröbner basis of $\mathcal{I}$.

Let $\mathcal{G} = \{g_1, g_2, \ldots, g_\mu\}$ be a Gröbner basis of $\mathcal{I}$, and let $q_1 = 1, q_2, \ldots, q_\sigma$ be the monic mononials in $\boldsymbol{R}[z_1, z_2]$ which are not multiple of the leading power products of any of the polynomials in $\mathcal{G}$. Thus consider $(q_1 + \mathcal{I}, q_2 + \mathcal{I}, \ldots, q_\sigma + \mathcal{I})$ as a basis for the finite-dimensional vector space $\boldsymbol{R}[z_1, z_2]/\mathcal{I}$ and let M_1, M_2 be a pair of commutative matrices that represents the linear commutative transformations

$$\phi_i : q + \mathcal{I} \mapsto q z_i + \mathcal{I}, \quad i = 1, 2, \tag{2.27}$$

acting on $\boldsymbol{R}[z_1, z_2]/\mathcal{I}$. The matrices M_1, M_2 (and, consequently, the transformations ϕ_1 and ϕ_2) have some interesting properties, we summarize in the following proposition.

Proposition 2. [9–11] *The matrices M_1, M_2 of size $\sigma \times \sigma$ defined above have the following properties:*

(1) *the smallest M_1, M_2-invariant subspace of $\boldsymbol{R}^\sigma$ that includes*

$$e_1^T = [1 \quad 0 \quad \ldots \quad 0]^T$$

is $\mathbf{R}^\sigma$;

(2) $p(z_1, z_2) \in \mathcal{I}$ *if and only if* $p(M_1, M_2) = 0$;

(3) $(\alpha_1, \alpha_2) \in \mathcal{V}(W)$ *if and only if there exists a nonzero vector* $\boldsymbol{v}$ *such that*

$$M_1 \boldsymbol{v} = \alpha_1 \boldsymbol{v}, \qquad M_2 \boldsymbol{v} = \alpha_2 \boldsymbol{v}; \tag{2.31}$$

(4) $(\alpha_1, \alpha_2) \in \mathcal{V}(W)$ *if and only if in the Frobenius upper triangular form of* M_1, M_2 *given by*

$$T_1 = Z^{-1} M_1 Z = \begin{bmatrix} t_{11}^{(1)} & & & * \\ & t_{22}^{(1)} & & \\ & & \ddots & \\ & & & t_{\sigma\sigma}^{(1)} \end{bmatrix}, \qquad T_2 = Z^{-1} M_2 Z = \begin{bmatrix} t_{11}^{(2)} & & & * \\ & t_{22}^{(2)} & & \\ & & \ddots & \\ & & & t_{\sigma\sigma}^{(2)} \end{bmatrix} \tag{2.32}$$

$(\alpha_1, \alpha_2) = (t_{ii}^{(1)}, t_{ii}^{(2)})$ *for some* i.

Proof. Property (1) is immediate, since $\phi_1^i \phi_2^j (1 + \mathcal{I})$, $i, j = 0, 1, \ldots$, generate $\mathbf{R}[z_1, z_2]/\mathcal{I}$ or, equivalently, $M_1^i M_2^j e_1$, $i, j = 0, 1, \ldots$, span $\mathbf{R}^\sigma$.

(2) Let

$$p(z_1, z_2) = \sum_{i,j} p_{ij} z_1^i z_2^j \in \mathcal{I},$$

This implies

$$0 = \sum_{i,j} p_{ij} (z_1^i + \mathcal{I})(z_2^j + \mathcal{I}) = \sum_{i,j} p_{ij} \phi_1^i \phi_2^j (1 + \mathcal{I}) \tag{2.33}$$

and equivalently

$$0 = \sum_{i,j} p_{ij} M_1^i M_2^j. \tag{2.34}$$

Multiplying (2.31) on the left by $M_1^r M_2^s e_1$ and recalling the matrix commutativity we have

$$0 = \left(\sum p_{ij} M_1^i M_2^j\right)(M_1^r M_2^s e_1), \quad r, s = 0, 1, \ldots.$$

This proves that $p(M_1, M_2) = 0$.

The viceversa is easily obtained by following backward the lines of the proof above.

(3) Assume that (2.28) holds and consider any polynomial $p(z_1, z_2) = \sum p_{ij} z_1^i z_2^j \in \mathcal{I}$. By property (1),

$$0 = p(M_1, M_2) = \sum p_{ij} M_1^i M_2^j,$$
$$0 = \sum p_{ij} M_1^i M_2^j \boldsymbol{v} = \sum p_{ij} \alpha_1^i \alpha_2^j \boldsymbol{v},$$
$$0 = \sum p_{ij} \alpha_1^i \alpha_2^j = p(\alpha_1, \alpha_2).$$

Since $p(z_1, z_2)$ is arbitrary in $\mathcal{I}$, $(\alpha_1, \alpha_2) \in \mathcal{V}(W)$. Viceversa, assume that (α_1, α_2) belongs to $\mathcal{V}(W)$. Let $\psi_i(\xi)$ be the minimum polynomial of M_i, $i = 1, 2$, and denote by k_1 and k_2 the algebraic multiplicities of $z_1 - \alpha_1$ and $z_2 - \alpha_2$, in $\psi_1(z_1)$ and $\psi_2(z_2)$ respectively:

$$\psi_1(z_1) = h_1(z_1)(z_1 - \alpha_1)^{k_1}, \quad h_1(\alpha_1) \neq 0,$$
$$\psi_2(z_2) = h_2(z_2)(z_2 - \alpha_2)^{k_2}, \quad h_2(\alpha_2) \neq 0.$$

Note that $h_1(z_1) h_2(z_2) \in \mathcal{I}$, since $h_1(\alpha_1) h_2(\alpha_2) \neq 0$. Let t, $0 \le t \le k_1$, be the largest integer such that

$$h_1(z_1) h_2(z_2)(z_1 - \alpha_1)^t \notin \mathcal{I}$$

and let r, $0 \le r < k_2$, be the largest integer such that

$$s(z_1, z_2) = h_1(z_1) h_2(z_2)(z_1 - \alpha_1)^t (z_2 - \alpha_2)^r \notin \mathcal{I}.$$

We then have that

$$s(z_1, z_2) \notin \mathcal{I},$$
$$s(z_1, z_2)(z_1 - \alpha_1) \in \mathcal{I},$$
$$s(z_1, z_2)(z_2 - \alpha_2) \in \mathcal{I}.$$

Hence

$$\boldsymbol{v} := s(M_1, M_2) e_1 \neq 0 \tag{2.35}$$

and

$$(M_1 - \alpha_1 I)\boldsymbol{v} = 0, \qquad (M_2 - \alpha_2 I)\boldsymbol{v} = 0.$$

The last two equations show that the vector $\boldsymbol{v}$ defined in (2.32) is a common eigenvector.

(4) Since T_1 and M_1 as well as T_2 and M_2 are connected by a common similarity transformation, property (2) holds for matrices T_1 and T_2 too.

Therefore $p(z_1,z_2)$ belongs to $\mathcal{I}$ if and only if $p(T_1,T_2)=0$. Let $p(z_1,z_2)\in\mathcal{I}$. Then

$$0=p(T_1,T_2)=\begin{bmatrix} p(t_{11}^{(1)},t_{11}^{(2)}) & & & & \\ & p(t_{22}^{(1)},t_{22}^{(2)}) & & * & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & p(t_{\sigma\sigma}^{(1)},t_{\sigma\sigma}^{(2)}) \end{bmatrix}.$$

Since $p(z_1,z_2)$ is arbitrary in $\mathcal{I}$, $p(t_{ii}^{(1)},t_{ii}^{(2)})=0$ implies $(t_{ii}^{(1)},t_{ii}^{(2)})\in\mathcal{V}(W)$. Viceversa, let $(\alpha_1,\alpha_2)\in\mathcal{V}(W)$ and suppose, by contradiction,

$$(\alpha_1,\alpha_2)\neq(t_{ii}^{(1)},t_{ii}^{(2)}),\quad i=1,2,\ldots,\nu.$$

Then, there exists a polynomial $p(z_1,z_2)$ vanishing in $(t_{ii}^{(1)},t_{ii}^{(2)})$, $1=1,2,\ldots\nu$, and different from zero in (α_1,α_2). We therefore have

$$p(T_1,T_2)=\begin{bmatrix} 0 & & & \\ & 0 & * & \\ & & \cdots & \\ & & \cdots & \\ & & & 0 \end{bmatrix}$$

so that

$$p(T_1,T_2)^\nu=0 \quad\text{and}\quad p(z_1,z_2)^\nu\in\mathcal{I}.$$

Since $p(\alpha_1,\alpha_2)^\nu$ is different from zero, $(\alpha_1,\alpha_2)\notin\mathcal{V}(W)$, contrary to the assumption.

Corollary. The following facts are equivalent:

(1) *Σ is output feedback stabilizable;*
(2) *any common eigenvector of M_1 and M_2 refers to a pair of eigenvalues (α_1,α_2) such that $|\alpha_1|>1$ and/or $|\alpha_2|>1$;*
(3) *any pair $(t_{ii}^{(1)},t_{ii}^{(2)})$ in the triangular form of M_1 and M_2 satisfies $|t_{ii}^{(1)}|>1$ and/or $|t_{ii}^{(2)}|>1$.*

3. Realization with structural constraints on A_1 and A_2

In this section we reconsider some structural constraints on the matrix pairs (A_1,A_2), with the aim of investigating what are the implications in the framework of the realization theory. As we shall see, the assumptions on A_1 and A_2 not only influence the characteristic polynomial of the realization and, consequently, the denominators of the transfer functions, but also the numerators. We shall consider three cases:

(1) A_1 and A_2 commute;
(2) A_1 and A_2 have property P;
(3) A_1 and A_2 are finite memory.

The following proposition provides a precise characterization of the scalar transfer functions that admit a realization with $[A_1,A_2]=0$.

Proposition 3. *A strictly proper two-dimensional transfer function can be realized by a state model with $[A_1,A_2]=0$ if and only if $W(z_1,z_2)$ admits the following partial fractions expansion*

$$W(z_1,z_2)=\sum_{i=1}^{t}\frac{p_i(z_1,z_2)}{(1-\lambda_i z_1-\mu_i z_2)^{\nu_i}},\tag{3.1}$$

with $p_i(0,0)=0$, $i=1,2,\ldots t$, and $\deg p_i\le\nu_i$ for all i such that $(\lambda_i,\mu_i)\neq(0,0)$.

Proof. Assume that A_1 and A_2 commute. Referring to the basis adopted in Lemma 4 of Section 5 of Chapter 8 for obtaining the block diagonal form [Chapter 8, (5.23)] of A_1 and A_2, partition B_1, B_2 and C conformably. Thus we have

$$W(z_1,z_2)=\sum_{i=1}^{t}C_i(I-A_{ii}^{(1)}z_1-A_{ii}^{(2)}z_2)^{-1}(B_i^{(1)}z_1+B_i^{(2)}z_2)\tag{3.2}$$

and letting

$$\begin{aligned}&\det(I-A_{ii}^{(1)}z_1-A_{ii}^{(2)}z_2)=(1-\lambda_i z_1-\mu_i z_2)^{\nu_i},\\ &C_i\operatorname{adj}(I-A_{ii}^{(1)}z_1-A_{ii}^{(2)}z_2)(B_i^{(1)}z_1+B_i^{(2)}z_2)=p_i(z_1,z_2),\end{aligned}\tag{3.3}$$

we get (3.1).
To prove the converse, express each term in (3.1) in the form

$$\frac{p_i(z_1,z_2)}{(1-\lambda_i z_1-\mu_i z_2)^{\nu_i}}=z_1\frac{p_i'(z_1,z_2)}{(1-\lambda_i z_1-\mu_i z_2)^{\nu_i}}+z_2\frac{p_i''(z_1,z_2)}{(1-\lambda_i z_1-\mu_i z_2)^{\nu_i}}\tag{3.4}$$

with $\deg p_i'<\nu_i$, $\deg p_i''<\nu_i$, and use the techniques in [12] to get

$$\begin{aligned}z_1\frac{p_i'(z_1,z_2)}{(1-\lambda_i z_1-\mu_i z_2)^{\nu_i}}&=\bar C_i(I-\bar A_{ii}^{(1)}z_1-\bar A_{ii}^{(2)}z_2)^{-1}\bar B_i^{(1)}z_1,\\ z_2\frac{p_i''(z_1,z_2)}{(1-\lambda_i z_1-\mu_i z_2)^{\nu_i}}&=\hat C_i(I-\hat A_{ii}^{(1)}z_1-\hat A_{ii}^{(2)}z_2)^{-1}\hat B_i^{(2)}z_2,\end{aligned}\tag{3.5}$$

with $[\bar A_{ii}^{(1)},\bar A_{ii}^{(2)}]=0$ and $[\hat A_{ii}^{(1)},\hat A_{ii}^{(2)}]=0$. So, a commutative realization of

(3.4) is given by

$$A_{ii}^{(1)} = \begin{bmatrix} \bar{A}_{ii}^{(1)} & 0 \\ 0 & \hat{A}_{ii}^{(1)} \end{bmatrix}, \qquad A_{ii}^{(2)} = \begin{bmatrix} \bar{A}_{ii}^{(2)} & 0 \\ 0 & \hat{A}_{ii}^{(2)} \end{bmatrix},$$
$$B_i^{(1)} = \begin{bmatrix} \bar{B}_i \\ 0 \end{bmatrix}, \qquad B_i^{(2)} = \begin{bmatrix} 0 \\ \hat{B}_i \end{bmatrix}, \qquad C_i = [\bar{C}_i \quad \hat{C}_i]. \tag{3.6}$$

Finally, (3.1) is realized by the parallel connection of systems (3.6).

As shown by Proposition 3, the factorizability of the denominator of $W(z_1, z_2)$ into linear factors is not sufficient to guarantee the existence of a realization with $[A_1, A_2] = 0$. In other words, the class of commutative pairs of matrices is not large enough for realizing all transfer functions with denominators of the form $\prod_j (1 - \lambda_j z_1 - \mu_j z_2)$.

The larger class, constituted by the pairs of matrices having property P, fits the requirement, since it allows to realize exactly the set of these transfer functions. A pair of $n \times n$ matrices (A_1, A_2) has the property P if there is an ordering of the eigenvalues λ_i of A_1 and μ_j of A_2 such that for any noncommutative polynomial $\pi(\xi_1, \xi_2)$, the eigenvalues of the matrix $\pi(A_1, A_2)$ are $\pi(\lambda_i, \mu_i)$, $i = 1, 2, \ldots, n$.

This property is equivalent [13] to any one of the following conditions,

(1) there is an invertible matrix T such that $T^{-1}A_1T$ and $T^{-1}A_2T$ are upper (lower) triangular;

(2) the Lie algebra $\mathcal{L}$ generated by A_1 and A_2 is solvable;

(3) for every noncommutative polynomial $\pi(\xi_1, \xi_2)$ the matrix $\pi(A_1, A_2)[A_1, A_2]$ is nilpotent.

Proposition 4. [14] *Let $W(z_1, z_2) = p(z_1, z_2)/q(z_1, z_2)$ be a proper transfer function, with p and q coprime polynomials. Then $W(z_1, z_2)$ is realizable by a two-dimensional system with A_1 and A_2 having property P if and only if $q(z_1, z_2)$ factors completely in the complex field into linear factors.*

Proof. Assume that A_1 and A_2 have property P. Since $q(z_1, z_2)$ divides $\det(I - A_1 z_1 - A_2 z_2)$, it factors into linear elements. Conversely, note that, starting from two-dimensional systems with A_1 and A_2 lower triangular and connecting them in series and parallel, the A_1 and A_2 matrices of the resulting systems still preserve the lower triangular structure. So we need only to take into account the realization of the following transfer functions,

$$W_1(z_1, z_2) = z_1, \qquad W_2(z_1, z_2) = z_2,$$
$$W_3(z_1, z_2) = \frac{1}{1 - \lambda z_1 - \mu z_2}. \tag{3.7}$$

These have realizations $(A_1, A_2, B_1, B_2, C, D)$ of dimension 1 given by

$$\Sigma_1 = (0, 0, 1, 0, 1.0),$$
$$\Sigma_2 = (0, 0, 0, 1, 1.0),$$
$$\Sigma_3 = (\lambda, \mu, \lambda, \mu, 1, 1), \tag{3.8}$$

respectively.

Finally, consider pairs of matrices (A_1, A_2) which are finite memory, i.e. $\det(I - A_1 z_1 - A_2 z_2) = 1$. Clearly every finite memory system is the realization of a polynomial transfer function in z_1 and z_2. On the other hand, every polynomial transfer function $W(z_1, z_2)$ can be realized by a finite memory system. In fact, it is always possible [7] to construct coprime realizations of $W(z_1, z_2)$, and these are finite memory.

However, if we look for minimal realizations of polynomial transfer functions, in general it is not true that these are necessarily finite memory. This phenomenon holds independently of the (real or complex) field where the matrix elements take their values and implies that minimal realizations of polynomial transfer functions need not be internally stable. To prove our statement, we shall go through the following steps:

1. We show that there exists a two-dimensional polynomial $n(z_1, z_2)$ of degree 3 that cannot be realized in dimension 3.

2. We construct an (infinite memory) realization of dimension 3 for the transfer function $n(z_1, z_2)/(1 + 2z_2)$ and a series connection of such a realization with a realization of dimension 1 for $1 + 2z_2$. The realization of $n(z_1, z_2)$ obtained in this way is minimal, internally unstable and exhibits pole/zero cancellations.

Step 1. Assume that A_1 and A_2 belong to $C^{3\times 3}$ and are the state updating matrices of a finite memory two-dimensional system.

For all matrices

$$C = [c_1 \quad c_2 \quad c_3]$$

in $C^{1\times 3}$ and $B_1, B_2 \in C^{3\times 1}$, the system transfer function is a polynomial,

$$W(z_1, z_2) = C\,\mathrm{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2), \tag{3.9}$$

of degree not greater than 3, that we shall rewrite as

$$W(z_1, z_2) = p_1(z_1, z_2) + p_2(z_1, z_2) + p_3(z_1, z_2), \tag{3.10}$$

where p_i are homogeneous forms of degree i, $i = 1, 2, 3$.

Referring to the cases considered in Section 3 of Chapter 8, we see that, independently of the choice of B_1, B_2 and C, the polynomials p_i must satisfy the following constraints:

Case 1: p_3 belongs either to the principal ideal $(z_1)^2$ or to the principal ideal $(z_2)^2$;

Case 2: p_3 belongs to the principal ideal $(z_1 z_2)$;

Case 3: if $p_3 = q^3$, where q is a first-order form, then either $q = z_1$ or $q = z_2$;

Case 4: if $p_3 = q^3$, where q is a first-order form, then q is also a factor of p_2.

The above constraints show that the polynomial

$$n(z_1, z_2) = (z_1 + z_2)^3 + 2z_2^2 + z_2 \tag{3.11}$$

cannot be realized by a finite memory third-order two- dimensional system. Therefore, if we look for a third-order two-dimensional realization of $n(z_1, z_2)$, a system with

$$\det(I - A_1 z_1 - A_2 z_2) = 1$$

would be needed. However, in that case pole/zero cancellations between $\det(I - A_1 z_1 - A_2 z_2)$ and $C\mathrm{adj}(I - A_1 z_1 - A_2 z_2)(B_1 z_1 + B_2 z_2)$ must occur, and therefore the degree of the latter must be greater than or equal to 4, which is impossible for third-order systems.

Consequently, the dimension of any state space realization of $n(z_1, z_2)$ is greater than 3.

Step 2. Consider the following two-dimensional systems:

$$\Sigma_1 = (A_1, A_2, B_1, B_2, C)$$

with

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad C = [1 \quad 0 \quad 0]$$

and

$$\Sigma_2 = (F_1, F_2, G_1, G_2, H, D)$$

with

$$F_1 = F_2 = 0, \qquad G_1 = 0, \qquad G_2 = 2, \qquad H_1 = 0, \qquad D = 1.$$

They realize $n(z_1, z_2)/(1 + 2z_2)$ and $1 + 2z_2$ respectively. Then the series connection of Σ_1 and Σ_2 is a strictly proper two-dimensional system

$$\hat{\Sigma} = (\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2, \hat{C})$$

with

$$\hat{A}_1 = \begin{bmatrix} F_1 & 0 \\ B_1 H & A_1 \end{bmatrix}, \qquad \hat{A}_2 = \begin{bmatrix} F_2 & 0 \\ B_2 H & A_2 \end{bmatrix},$$
$$\hat{B}_1 = \begin{bmatrix} G_1 \\ B_1 D \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} G_2 \\ B_2 D \end{bmatrix}, \quad \hat{C} = [0 \quad C] \tag{3.12}$$

which provides a fourth-order, and hence a minimal realization of $n(z_1, z_2)$. Since the characteristic polynomials of $\hat{\Sigma}$ is

$$\det(I - \hat{A}_1 z_1 - \hat{A}_2 z_2) = 1 + 2z_2 \tag{3.13}$$

the realization $\hat{\Sigma}$ above is not finite memory.

The above example allows to point out some interesting consequences:

(1) Minimal realizations of polynomial transfer functions need not be finite memory.

(2) Pole/zero cancellations are allowed in minimal realizations. Actually in $\hat{\Sigma}$ we have

$$\hat{C}\mathrm{adj}(I - \hat{A}_1 z_1 - \hat{A}_2 z_2)(\hat{B}_1 z_1 + \hat{B}_2 z_2) = n(z_1, z_2)(1 + 2z_2),$$
$$\det(I - \hat{A}_1 z_1 - \hat{A}_2 z_2) = 1 + 2z_2$$

(3) Minimal realizations of polynomial (and hence BIBO stable) transfer functions may be unstable.

In fact in the above example the variety of $1 + 2z_2$ intersects the unit closed polydisc $\mathcal{P}_1$.

4. Two-dimensional behaviours

The state models and the realization problems considered so far make explicit reference to input-output maps with quarter plane causality and, consequently, to local state updating equations that preserve this causality. Actually, it is of central importance to realize that there is often no natural, intrinsic direction of the evolution for systems defined over a two-dimensional domain. In this case any choice of a preferred direction is somewhat artificial, and there are various possible definitions of past and future. The behavioural approach to dynamical systems [14–16] allows to cope with general causality structures in the two-dimensional context and avoids any a priori assumption concerning the input-output representation of the external data.

Although in a more general context, some algebraic problems which are typical of the state space realizations discussed in the previous sections, arise also when we look for state models of two-dimensional behaviours.

Let $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ be a $p \times q$ polynomial matrix and consider the autoregressive two-dimensional system $\mathcal{S} = (\boldsymbol{Z}^2, \boldsymbol{R}^q, \mathcal{B})$, where the 'behaviour'

$$\mathcal{B} := \{\boldsymbol{w} : \boldsymbol{Z}^2 \to \boldsymbol{R}^q : R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})\boldsymbol{w} = 0\} \tag{4.1}$$

represents the set of all admissible trajectories of $\mathcal{S}$.

As usual, σ_1 and σ_2 denote the shift operators, defined by

$$\begin{aligned} \sigma_1 \boldsymbol{w}(h,k) &= \boldsymbol{w}(h+1,k), \\ \sigma_2 \boldsymbol{w}(h,k) &= \boldsymbol{w}(h,k+1). \end{aligned} \tag{4.2}$$

In general it is not specified which components $w_1, w_2, \dots w_q$ are inputs (i.e. free variables) and which are outputs. It has been shown [14] that any behaviour can be considered as the sum of a controllable part, involving the free variables and an autonomous part. The construction of state models for controllable behaviours is essentially based on the procedures used for realizing two-dimensional transfer matrices [16] and, therefore, involves the same minimality and structure arguments mentioned in Sections 2–3.

Intuitively speaking, a system is autonomous if there exists a portion T of $\boldsymbol{Z}^2$ with 'large' complement, such that the knowledge of the trajectory in T completely specifies the trajectory in $\boldsymbol{Z}^2 \setminus T$. Here 'large' means that $\boldsymbol{Z}^2 \setminus T$ must include the intersection of $\boldsymbol{Z}^2$ with a sector of $\boldsymbol{R}^2$.

The following proposition shows how autonomy, existence of free variables and structure of R are related each other.

Proposition 5. [16] *The following facts are equivalent:*

(1) $\mathcal{S} = (\boldsymbol{Z}^2, \boldsymbol{R}^q, \mathcal{B})$ *is autonomous;*

(2) $\mathcal{S}$ *has no free variables;*

(3) $\mathcal{B} = \ker R$ *implies that* R *has full column rank.*

Moreover, $\mathcal{B}$ *is finite-dimensional if and only if* $\mathcal{S}$ *is autonomous and* R *is right factor prime.*

We shall now discuss in some detail the state representation of an autonomous two-dimensional system $\mathcal{S} = (\mathbf{Z}^2, \boldsymbol{R}^q, \mathcal{B})$, under the assumption that $\mathcal{B}$ has finite dimension.

The general result we aim to prove is that any such a system can be represented by a model with the following equations.

$$\begin{aligned} \boldsymbol{x}(h,k) &= A_1^h A_2^k \boldsymbol{x}(0,0), \\ \boldsymbol{w}(h,k) &= C\boldsymbol{x}(h,k), \end{aligned} \tag{4.3}$$

where A_1 and A_2 constitute a pair of commuting invertible matrices. Moreover, when minimal realizations are considered, A_1 and A_2 are uniquely determined up to similarity transformations, and their dimension coincides with that of a suitable quotient over the ring of Laurent polynomials in two variables.

To keep the notation simpler, we consider only the scalar case ($q = 1$), assuming that the right factor prime matrix R which characterizes the autoregressive structure of $\mathcal{B}$ via

$$\mathcal{B} = \{\boldsymbol{w} : R\boldsymbol{w} = 0\} \tag{4.4}$$

is a row vector

$$R(z_1, z_2) = [\, r_1(z_1, z_2) \quad r_2(z_1, z_2) \quad \dots \quad r_t(z_1, z_2) \,]. \tag{4.5}$$

Before embarking into the derivation of a state model, we shall discuss various connections between the ideals in $A_\pm := \boldsymbol{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]$ and the ideals in $A_+ := \boldsymbol{R}[z_1, z_2]$ as well as an abstract characterization of $\mathcal{B}$ based on some algebraic properties of dual spaces. The relevant facts are summarized below

4.1. *Ideals in $A_\pm$ and in A_+*

Consider the following map

$$|\cdot| : A_\pm \to A_+ : p \mapsto |p| := z_1^{-i} z_2^{-j} p,$$

where i and j are the minimum degrees of the monomials that appear in the nonzero Laurent polynomial p with regards to the variables z_1 and z_2, respectively. More precisely, if

$$p = \sum_{h,k \in \boldsymbol{Z}} p_{hk} z_1^h z_2^k,$$

then

$$\begin{aligned} i &:= \min\{h \in \boldsymbol{Z} \mid \exists k \in \boldsymbol{Z}, p_{hk} \neq 0\}, \\ j &:= \min\{k \in \boldsymbol{Z} \mid \exists h \in \boldsymbol{Z}, p_{hk} \neq 0\}. \end{aligned}$$

In case $p = 0$, we define $|p| = 0$. Clearly, for every nonzero Laurent polynomial p, $|p|$ includes a monomial in z_1 and a monomial in z_2 with nonzero coefficients.

The operation just described, of shifting the support of a Laurent polynomial into the positive orthant of $\boldsymbol{Z} \times \boldsymbol{Z}$, associates with the ideal $\mathcal{I}_\pm := (r_1, r_2, \dots, r_t)_\pm$ generated in $A_\pm$ by the elements of the matrix R an ideal $\mathcal{I}_+ := (|r_1|, |r_2|, \dots, |r_t|)_+$ generated in A_+ by $|r_1|, |r_2|, \dots, |r_t|$.

The ideals in $\mathcal{I}_\pm$ and $\mathcal{I}_+$ are connected as follows [17]:

(1) $p \in \mathcal{I}_\pm$ if and only if there exists a pair of nonnegative integers (i, j) such that $z_1^i z_2^j p \in \mathcal{I}_+$;

(2) the quotient space $A_\pm/\mathcal{I}_\pm$ is finite-dimensional if and only if the same holds for $A_+/\mathcal{I}_+$.

4.2. *Duality properties*

Introduce a nondegenerate bilinear function

$$\langle \cdot,\cdot \rangle : A_\pm \times \boldsymbol{R}^{\boldsymbol{Z}\times \boldsymbol{Z}} \to \mathbf{R},$$

by assuming

$$\langle p, w\rangle = \sum_{ij} p_{ij} w(i,j)$$

for all polynomials $p = \sum p_{ij} z_1^i z_2^j$ in $A_\pm$ and all signals w in $\boldsymbol{R}^{\boldsymbol{Z}\times \boldsymbol{Z}}$. In this way

(1) the universe $\boldsymbol{R}^{\boldsymbol{Z}\times \boldsymbol{Z}}$ of all signals with support in $\boldsymbol{Z}\times\boldsymbol{Z}$ is isomorphic to the algebraic dual of $A_\pm$, i.e., to the space of the linear functionals on $A_\pm$;

(2) the behaviour $\mathcal{B}$ can be identified with the orthogonal complement of $\mathcal{I}_\pm$ with regards to such bilinear function

$$\mathcal{B} = \mathcal{I}_\pm^\perp, \tag{4.6}$$

and, by duality,

$$\mathcal{B}^\perp = \mathcal{I}_\pm^{\perp\perp} = \mathcal{I}_\pm .$$

The proof of (4.6) is an easy consequence of the following identity

$$p(\sigma_1,\sigma_2,\sigma_1^{-1},\sigma_2^{-1})w(h,k) = \langle p z_1^h z_2^k, w\rangle,$$

in fact, $w \in \mathcal{B}$ implies $p(\sigma_1,\sigma_2,\sigma_1^{-1},\sigma_2^{-1})w = 0$ and, therefore, $\langle p, w\rangle = 0$, $\forall p \in \mathcal{I}_\pm$, and viceversa, given $w \in \mathcal{I}_\pm^\perp$ and $p\in\mathcal{I}_\pm$, we have $\langle p z_1^h z_2^k, w\rangle = 0$, $\forall h,k\in\mathbf{Z}$, which implies $p(\sigma_1,\sigma_2,\sigma_1^{-1},\sigma_2^{-1})w = 0$;

(3) $A_\pm/\mathcal{B}^\perp = A_\pm/\mathcal{I}_\pm$ and $\mathcal{B}$ constitute a dual pair with regards to the bilinear function

$$\langle [p], w\rangle := \langle p, w\rangle, \quad [p] := p + \mathcal{I}_\pm;$$

(4) if the matrix R is right factor prime, then $\mathcal{B}$ and $A_\pm/\mathcal{I}_\pm$ are finite-dimensional isomorphic vector spaces.

If

$$([p_1],[p_2],\dots,[p_n])$$

is a basis of $A_\pm/\mathcal{I}_\pm$, the linear map

$$\psi : \mathcal{B} \to A_\pm/\mathcal{I}_\pm : w \mapsto \sum_{i=1}^n \langle p_i, w\rangle [p_i]$$

provides explicitely an isomorphism between $\mathcal{B}$ and $A_\pm/\mathcal{I}_\pm$.

From now on, we suppose that a basis $([p_1],[p_2],\dots,[p_n])$ has been chosen in $A_\pm/\mathcal{I}_\pm$, and consider the corresponding dual basis $(w_1,w_2,\dots,w_n)$ in $\mathcal{B}$.

The relations $\langle [p_i], w_j\rangle = \delta_{ij}$, $i,j = 1,2,\dots n$, imply

$$[p] = \sum_{i=1}^n \langle [p], w_i\rangle [p_i], \quad \forall [p] \in A_\pm/\mathcal{I}_\pm, \tag{4.7}$$

and, on the other hand,

$$w = \sum_{i=1}^n \langle [p_i], w\rangle w_i, \quad \forall w \in \mathcal{B}. \tag{4.8}$$

Introduce the following invertible linear maps

$$\begin{aligned} \mathcal{Z}_1 &: A_\pm/\mathcal{I}_\pm \to A_\pm/\mathcal{I}_\pm : [p] \mapsto [z_1 p], \\ \mathcal{Z}_2 &: A_\pm/\mathcal{I}_\pm \to A_\pm/\mathcal{I}_\pm : [p] \mapsto [z_2 p]. \end{aligned} \tag{4.9}$$

Clearly $\mathcal{Z}_1\mathcal{Z}_2 = \mathcal{Z}_2\mathcal{Z}_1$ and the adjoint maps of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ in $\mathcal{B}$ are σ_1 and σ_2, respectively.

The matrices $N_i = [n_{hk}^{(i)}]$, $i = 1,2$, representing the linear transformations $\mathcal{Z}_i$, $i=1,2$, with respect to the basis $([p_1],[p_2],\dots,[p_n])$ are given by $n_{hk}^{(i)} = \langle [z_i p_k], w_h\rangle$. Hence, the matrices representing σ_1 and σ_2 with respect to the dual basis are N_1^T and N_2^T, respectively.

We are now in a position for providing a state variable realization of the autonomous finite-dimensional system $\mathcal{S}$ characterized by the polynomial matrix (4.5).

For any $\boldsymbol{w} \in \mathcal{B}$, we introduce the following signal

$$\boldsymbol{x} : \boldsymbol{Z}\times\boldsymbol{Z} \to \boldsymbol{R}^n : (h,k) \mapsto \begin{bmatrix} \langle [p_1], \sigma_1^h\sigma_2^k w\rangle \\ \langle [p_2], \sigma_1^h\sigma_2^k w\rangle \\ \dots \\ \langle [p_n], \sigma_1^h\sigma_2^k w\rangle \end{bmatrix}.$$

The value of $\boldsymbol{x}$ at (h,k) provides the components of $\sigma_1^h\sigma_2^k w$ with respect to the basis $(w_1,w_2,\dots w_n)$. Since the one step updated value of $\boldsymbol{x}$ is given by

$$\boldsymbol{x}(h+1,k) = \begin{bmatrix} \langle [p_1], \sigma_1^{h+1}\sigma_2^k w\rangle \\ \langle [p_2], \sigma_1^{h+1}\sigma_2^k w\rangle \\ \dots \\ \langle [p_n], \sigma_1^{h+1}\sigma_2^k w\rangle \end{bmatrix} = N_1^T \boldsymbol{x}(h,k),$$

it is clear that, once $\boldsymbol{x}(0,0)$ is known, $\boldsymbol{x}(h,k)$ is easily computed for all $(h,k)\in\boldsymbol{Z}\times\boldsymbol{Z}$,

$$\boldsymbol{x}(h,k) = (N_1^T)^h (N_2^T)^k \boldsymbol{x}(0,0).$$

Moreover, the value of w at (h,k) can be recovered from $\boldsymbol{x}(h,k)$ as

$$w(h,k) = (\sigma_1^h\sigma_2^k w)(0,0) = \langle [1], \sigma_1^h \sigma_2^k w\rangle$$
$$= \langle c_1[p_1] + c_2[p_2] + \ldots + c_n[p_n], \sigma_1^h\sigma_2^k w\rangle = C\boldsymbol{x}(h,k).$$

Here $C := [\,c_1 \quad c_2 \quad \ldots \quad c_n\,]$ denotes the row vector of the components of $[1]$ with regards to the basis $([p_1],[p_2],\ldots,[p_n])$ in $A_\pm/\mathcal{I}_\pm$.

Letting $N_1^T = A_1$ and $N_2^T = A_2$ the above results can be summarized in model (4.3) or, equivalently, in the following model

$$\begin{cases} \sigma_1\boldsymbol{x} = A_1\boldsymbol{x}, \\ \sigma_2\boldsymbol{x} = A_2\boldsymbol{x}, \\ w = C\boldsymbol{x}, \end{cases} \tag{4.10}$$

which provides a recursive version of (4.3) Every signal of the autonomous behaviour $\mathcal{B}$ is uniquely determined by the corresponding value of the state $\boldsymbol{x}$ at any point (h,k) and, conversely, different states at (h,k) induce different signals in the autonomous system.

When p_j, $j = 1,2,\ldots n$, are monic monomials, i.e., $p_j = z_1^{\mu_j} z_2^{\nu_j}$, $j = 1,2,\ldots,n$, the structure of the corresponding dual basis is very appealing. In fact the element w_j is the unique element of $\mathcal{B}$ taking the values 1 at (μ_j,ν_j) and 0 at (μ_i,ν_i), $i = 1,2,\ldots,j-1,j+1,\ldots,n$. Moreover, for every $(h,k) \in \boldsymbol{Z}\times\boldsymbol{Z}$, the components of the state vector $\boldsymbol{x}(h,k)$ are the values of w at $\{(\mu_1+h,\nu_1+k),(\mu_2+h,\nu_2+k),\ldots,(\mu_n+h,\nu_n+k)\}$.

To conclude this section, we present an algorithm for computing the matrices A_1, A_2 and C which realize a finite-dimensional autonomous behaviour $\mathcal{B}$. As we shall see, the interest of the procedure exceeds, to some extent, that of obtaining A_1, A_2 and C starting from the polynomials $r_1, r_2, \ldots, r_t$. Indeed it provides also some connections between the ideals $\mathcal{I}_\pm$ and $\mathcal{I}_+$ generated in $A_\pm$ and A_+ by $r_1, r_2, \ldots, r_t$ and relates the shift maps ζ_i, introduced in (4.9), acting on $A_\pm/\mathcal{I}_\pm$, with the shift maps ϕ_i, introduced in Section 5 of Chapter 8, acting on $A_+/\mathcal{I}_+$.

Let $\mathcal{G} = \{g_1, g_2, \ldots, g_h\}$, $g_i \in A_+$, be a Gröbner basis of $\mathcal{I}_+ = (|r_1|, |r_2|, \ldots, |r_t|) \subseteq A_+$, and denote by $\{q_1 = 1, q_2, \ldots, q_m\}$ the set of monic monomials that are not multiple of the leading power products of any of the polynomials in $\mathcal{G}$. As in Section 5 of Chapter 8, the linear transformations

$$\phi_i : \ A_+/\mathcal{I}_+ \to A_+/\mathcal{I}_+ : \ q + \mathcal{I}_+ \mapsto z_i q + \mathcal{I}_+, \quad i = 1,2, \tag{4.11}$$

will be represented with respect to the basis

$$(q_1 + \mathcal{I}_+, q_2 + \mathcal{I}_+, \ldots, q_m + \mathcal{I}_+) \tag{4.12}$$

by a pair of commuting matrices M_1 and M_2.

Some relevant properties of the maps ϕ_i and, consequently, of the matrices M_i, $i = 1,2$, are summarized in the following proposition.

Proposition 6. *Let ϕ_i, $i = 1,2$, be a pair of commuting linear transformations acting on a finite-dimensional space X Then*

(1) *for all nonnegative integers r and s, $\mathrm{Im}\phi_1^r\phi_2^s$ is a $\{\phi_1,\phi_2\}$-invariant subspace of X;*

(2)

$$\mathrm{Im}\phi_1^r\phi_2^{s+1} \subseteq \mathrm{Im}\phi_1^r\phi_2^s, \tag{4.13}$$
$$\mathrm{Im}\phi_1^{r+1}\phi_2^s \subseteq \mathrm{Im}\phi_1^r\phi_2^s; \tag{4.14}$$

(3) *there exists an integer $\nu > 0$ such that*

$$\mathcal{L} := \mathrm{Im}\phi_1^\nu\phi_2^\nu = \mathrm{Im}\phi_1^r\phi_2^s \tag{4.15}$$

for all pairs (r,s) with $r \geq \nu$ and $s \geq \nu$;

(4) *$\phi_1|\mathcal{L}$ and $\phi_2|\mathcal{L}$ are invertible linear transformations.*

Proof. (1) Assume $\boldsymbol{x}$ be in $\mathrm{Im}\phi_1^r\phi_2^s$, i.e., $\boldsymbol{x} = \phi_1^r\phi_2^s\mathbf{y}$, for some $\boldsymbol{y}$. Then $\phi_i\boldsymbol{x} = \phi_1^r\phi_2^s(\phi_i\boldsymbol{y}), i = 1,2$, is in $\mathrm{Im}\phi_1^r\phi_2^s$ too.

(2) Let $\boldsymbol{x} \in \mathrm{Im}\phi_1^r\phi_2^{s+1}$. Then $\boldsymbol{x} = \phi_1^r\phi_2^{s+1}\boldsymbol{y}$, which proves (4.13). The proof of (4.14) is analogous.

(3) Clearly the descending chain

$$\mathrm{Im}\phi_1\phi_2 \supseteq \mathrm{Im}\phi_1^2\phi_2^2 \supseteq \ldots \tag{4.16}$$

becomes stationary for some $\nu \leq \dim X$. Consider any pair of integers (r,s), with $r \geq \nu$ and $s \geq \nu$, and let $\mu := \max\{r,s\}$. The stationarity of the chain and (2) imply

$$\mathrm{Im}\phi_1^\nu\phi_2^\nu \supseteq \mathrm{Im}\phi_1^r\phi_2^s \supseteq \mathrm{Im}\phi_1^\mu\phi_2^\mu = \mathrm{Im}\phi_1^\nu\phi_2^\nu.$$

(4) Suppose $\phi_1|\mathcal{L}$ be noninvertible. This would imply

$$\mathrm{Im}\phi_1^{\nu+1}\phi_2^{\nu+1} \subseteq \phi_1\mathrm{Im}\phi_1^\nu\phi_2^\nu \subset \mathrm{Im}\phi_1^\nu\phi_2^\nu,$$

which contradicts the stationarity of the chain (4.16).

Introduce now the following maps:

$$\begin{aligned} \eta &= A_+/\mathcal{I}_+ \to \mathcal{L} : p + \mathcal{I}_+ \mapsto z_1^\nu z_2^\nu + \mathcal{I}_+, \\ \chi &= \mathcal{L} \to A_\pm/\mathcal{I}_\pm : z_1^\nu z_2^\nu p + \mathcal{I}_+ \mapsto [z_1^\nu z_2^\nu p]. \end{aligned} \tag{4.17}$$

Both of them are well-defined linear maps. Indeed this is obvious for η. As far as χ is concerned,

$$z_1^\nu z_2^\nu p \equiv z_1^\nu z_2^\nu q \mod \mathcal{I}_+$$

implies that $z_1^\nu z_2^\nu(p-q)$ is in $\mathcal{I}_+$ and hence in $\mathcal{I}_\pm$. So χ maps equivalent polynomials modulo $\mathcal{I}_+$ into equivalent polynomials modulo $\mathcal{I}_\pm$ and is therefore a well-defined linear map.

Proposition 7. *The linear maps η and χ defined in (4.17) have the following properties:*

(1) *η is onto;*
(2) *χ is a bijection;*
(3) *the diagram*

$$\begin{array}{ccccc} A_+/\mathcal{I}_+ & \xrightarrow{\eta} & \mathcal{L} & \xrightarrow{\chi} & A_\pm/\mathcal{I}_\pm \\ \downarrow \phi_i & & \downarrow \phi_i|\mathcal{L} & & \downarrow \zeta_i \\ A_+/\mathcal{I}_+ & \xrightarrow{\eta} & \mathcal{L} & \xrightarrow{\chi} & A_\pm/\mathcal{I}_\pm \end{array}$$

commutes.

Proof. (1) is obvious. (2) Let $\chi(z_1^\nu z_2^\nu p+\mathcal{I})=0$. Since $z_1^\nu z_2^\nu p$ is in $\mathcal{I}_\pm$, there exists a nonnegative integer ρ such that $z_1^{\nu+\rho} z_2^{\nu+\rho} p$ is in $\mathcal{I}_+$. Therefore, both $z_1^{\nu+\rho} z_2^{\nu+\rho} p+\mathcal{I}_+$ and its image under $(\phi_1|\mathcal{L}\circ\phi_2|\mathcal{L})^{-\rho}$ are the zero element of $\mathcal{L}$, which implies that $z_1^\nu z_2^\nu p$ is in $\mathcal{I}_+$ too. So χ is injective. To prove that χ is onto, it will be sufficient to check that every monomial $m\in A_\pm$ is equivalent, modulo $\mathcal{I}_\pm$, to some polynomial in the ideal of A_+ generated by $z_1^\nu z_2^\nu$. If so, every element in $A_\pm/\mathcal{I}_\pm$ would be expressible as a linear combination of terms with structure $[z_1^\nu z_2^\nu p]$, which by definition are in the image of χ .

Let $t=\dim A_\pm/\mathcal{I}_\pm$ and choose t polynomials $p_1,p_2,\dots,p_t$ in A_+, so that $[p_1],[p_2],\dots,[p_t]$ is a basis of $A_\pm/\mathcal{I}_\pm$. Clearly $[z_1^k z_2^k p_1],[z_1^k z_2^k p_2],\dots,[z_1^k z_2^k p_t]$ is still a basis of $A_\pm/\mathcal{I}_\pm$, for any integer k and, on the other hand, $z_1^k z_2^k p_i$, $i=1,2,..,t$, are elements of the ideal generated by $z_1^\nu z_2^\nu$ in A_+, provided that k is chosen larger enough. So every monomial $m\in A_\pm$ is a linear combination of $z_1^k z_2^k p_i$, $i=1,2,...,t$, modulo $\mathcal{I}_\pm$, and, therefore, $[m]$ is the image, under χ of some element of $\mathcal{L}$.

The proof of (3) is straightforward and requires to check that the maps obtained by following the diagram from one initial point to a terminal point along each displayed route are the same.

Basing on the proposition above, the following steps provide an explicit realization algorithm.

Step 1. Since η is onto, it maps the basis (4.12) of $A_+/\mathcal{I}_+$ into a set of generators of $\mathcal{L}$. Within this set we can choose a basis of $\mathcal{L}$, which in turn is mapped by χ into a basis of $A_\pm/\mathcal{I}_\pm$. The columns of $M_1^\nu M_2^\nu$ represent the components, with regards to the basis (4.12), of the η images of the basis vectors. Therefore, the selection of a basis in $\mathcal{L}$ reduces to choose a maximal set of linearly independent columns in $M_1^\nu M_2^\nu$

Step 2. Once a basis in $\mathcal{L}$ has been selected, the linear maps $\phi_i|\mathcal{L}$, $i=1,2$, are represented by a pair of matrices N_1,N_2. Assuming that in the previous step the indices of the columns we selected in $M_1^\nu M_2^\nu$ are $i_1,i_2,\dots i_\nu$, the hth column of N_1 (of N_2) is constituted by the components of the i_hth column of $M_1^{\nu+1}M_2^\nu$ (of $M_1^\nu M_2^{\nu+1}$) with regards to the $i_1,i_2,\dots i_\nu$ columns of $M_1^\nu M_2^\nu$. The matrices N_1 and N_2 represent also ζ_1 and ζ_2 with regards to the basis in $A_\pm/\mathcal{I}_\pm$.

Step 3. The vector $1+\mathcal{I}_+$ is mapped by η into $z_1^\nu z_2^\nu+\mathcal{I}_+\in\mathcal{L}$. Its representation, with regards to the basis of $\mathcal{L}$, is given by a vector $\boldsymbol{v}\in\boldsymbol{R}^n$, whose elements are the components of $M_1^\nu M_2^\nu e_1$ with regards to the $i_1,i_2,\dots,i_\nu$ columns of $M_1^\nu M_2^\nu$. The components of [1] with regards to the basis of $A_\pm/\mathcal{I}_\pm$ are the elements of the vector $C^T=N_1^{-\nu}N_2^{-\nu}\boldsymbol{v}$.

The connections between the matrices M_i and N_i, $i=1,2$, become more evident when, by choosing a suitable basis in $A_+/\mathcal{I}_+$, M_i are put in the form

$$M_1=\begin{bmatrix} X_1 & & \\ & Y_1 & \\ & & Z_1\end{bmatrix},\qquad M_2=\begin{bmatrix} X_2 & & \\ & Y_2 & \\ & & Z_2\end{bmatrix},\tag{4.18}$$

with X_1,X_2 invertible and $[X_1,X_2]=0$; Y_1 nilpotent and $[Y_1,Y_2]=0$; and Z_2 nilpotent and $[Z_1,Z_2]=0$.

Since, for large k, we have

$$M_1^k M_2^k=\begin{bmatrix} X_1^k X_2^k & & \\ & 0 & \\ & & 0\end{bmatrix},$$

$\mathcal{L}$ is generated by the first $t=\dim X_i$ vectors of the basis of $A_+/\mathcal{I}_+$ that leads to (4.18) and X_1,X_2 represent $\phi_1|\mathcal{L}$ and $\phi_2|\mathcal{L}$.

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