State realization of 2D finite dimensional autonomous systems

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Abstract The paper deals with the state space realization of autonomous autoregressive 2D systems in the context of the behavioural approach. An arbitrary autoregressive 2D system Σ can be viewed as the sum of an externally controllable subsystem Σ^c with an autonomous one Σ^a, so that a state space realization of Σ can be obtained by separately realizing Σ^c and Σ^a. Since a procedure for realizing externally controllable systems in state/driving-variable form is already available in the literature, the general realization problem is easily reduced to the autonomous case. Here some properties of finite dimensional autonomous systems are discussed allowing for a realization procedure that uses the Gröbner bases theory.

Keywords : autonomous systems; (externally) controllable systems; Gröbner basis; state/driving-variable realization.

1 Introduction

In this paper we will present some results and algorithms involved in the construction of 2D state space models on the basis of external data. Following the behavioural approach to dynamical systems introduced in [1]- [3], external data are characterized by means of a family of laws telling us that certain signals can occur and others cannot. Moreover all the components of the external data play completely symmetric roles, so that no input/output structure is a priori assumed.

Realization theory has been developed for the most part in the 1D environment, where state space models have shown to be a very convenient framework for the mathematical analysis and synthesis of real time data processors and controllers. In this context the state is very naturally viewed as a set of latent variables which parametrize the content of system memory and the realization problems are inextricably connected with the definitions of past and future, that underlie the notion of memory.

When trying to formulate state concepts for 2D systems, it is of central importance to realize that there is often no natural, intrinsic direction of the evolution for systems defined over a two dimensional domain. In this case any choice of a preferred direction is artificial, and there are various possible definitions of past and future. Some of them are discussed in [3], where their connections with 2D state representations available in the literature are illustrated in detail. Obviously, the most classical example is provided by the quarter plane causality structure. It underlies the so called SW-state representation, which is the behavioural counterpart of the state space model of quarter plane impulse responses, introduced by Attasi [4], Roesser [5] and Fornasini and Marchesini [6] within the classical input/output framework.

Singular state space models have been analyzed to cope with more general causality structures in e.g. Lévy [7], Kaczorek [8] and Lewis [9]. In this paper an alternative approach to non causal structures is considered, basing on the introduction of a set of auxiliary free variables in SW state representation. These act in some sense like an input driving the system dynamics and are therefore called the "driving-variables". State /driving-variable models allow to compute recursively joint input-output trajectories from the values of the auxiliary free variables via the state. Although only the realization of 2D AR equations is considered, we remark however that transfer functions can also be handled in this context, since they can be identified with (externally) controllable AR systems, see [1].

It is shown in [1] that every controllable AR 2D system (and hence every 2D rational transfer function) can be realized in SW state/driving-variable form, independently of the existence of quarter plane causal structure between the system variables. This results from the possibility of generating the joint input-output trajectories $w = \mathrm{col}(u, y)$ of a controllable AR system from (auxiliary) driving-variables trajectories v such that the relationship between w and v is quarter plane causal, even if u and y are not causally related. Viewing v as the new "input" and w as the new "output", a 2D state space model (for instance, of the FM type [6]) can then be obtained by the classical realization procedures, yielding the desired state/driving-variable realization.

The realizability of arbitrary, i.e. not necessarily controllable, AR systems is still an open problem. Reducing this problem to the realization of autonomous systems is the first goal of our paper. Indeed, we show that every AR system Σ can be viewed as the sum of an externally controllable system Σ^c and an autonomous one Σ^a and, therefore, the realization of Σ can be obtained by realizing separately Σ^c and Σ^a. Since a procedure for realizing externally controllable systems is given in [1], it will be enough to derive a realization procedure for autonomous systems.

This contribution focuses on finite dimensional autonomous AR systems with q (real-valued) variables defined over $\mathbf{Z}^2$. It turns out that, in this case, the admissible system trajectories constitute a finite dimensional vector space $\mathcal{B}$, given by $\mathcal{B} = \{w : \mathbf{Z}^2 \rightarrow \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}$, with σ_1 and σ_2 the two dimensional shifts and $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ a factor right prime 2D polynomial matrix.

The second goal we pursue in this paper is that of representing the autonomous behaviour via a state model, characterized by a pair of commuting matrices that describe the state evolution in the two directions of the grid. The realization algorithm exploits the algebraic duality between $\mathcal{B}$ and a suitable quotient module over the space of 2D polynomial rows, as well as the correspondence betweeen the shift operators in $\mathcal{B}$ and a pair of adjoint operators in the quotient module. These operators are represented by a pair of commutative invertible matrices and can be obtained by computer algebra techniques and linear manipulations.

2 Autonomous 2D systems

Following the behavioural approach to dynamical systems introduced in [2] and [3], we characterize a 2D system by means of its behaviour, which consists of the set of all the signals which are compatible with the system laws. Moreover, we do not start with a given input/output structure, i.e., the system signals are stacked ogether in a (multivariate) signal w instead of being divided into inputs u and outputs y. A 2D system Σ with q real valued variables defined over $\mathbf{Z}^2$ and with behaviour $\mathcal{B} \subseteq \{w : \mathbf{Z}^2 \rightarrow \mathbf{R}^q\}$ will be denoted by $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$.

In the sequel we will be interested in the class of autoregressive 2D systems. $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is said to be an autoregressive (AR) system if there exists a 2D Laurent polynomial matrix $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\} =: \ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}),$$

with σ_1 and σ_2 the 2D shift operators. These are respectively defined by

$$\sigma_1 w(t_1, t_2) = w(t_1 + 1, t_2)$$

$$\sigma_2 w(t_1, t_2) = w(t_1, t_2 + 1)$$

for all $w : \mathbf{Z}^2 \to \mathbf{R}^q$ and all $(t_1, t_2) \in \mathbf{Z}^2$.

Clearly, the set $\{w : \mathbf{Z}^2 \to \mathbf{R}^q\}$ of $\mathbf{R}^q$-valued functions defined on $\mathbf{Z}^2$ is a real vector space with respect to functions (pointwise) addition and (pointwise) scalar multiplication. A 2D system $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is said to be linear if $\mathcal{B}$ is a linear subspace of $\{w : \mathbf{Z}^2 \to \mathbf{R}^q\}$. A linear 2D system is said to be finite dimensional if its behaviour $\mathcal{B}$ is a finite dimensional vector space, otherwise Σ is said to be infinite dimensional.

In order to define the notion of autonomy we introduce the following nomenclature. A subset of $\mathbf{R}^2$ is said to be 2D-unbounded if it contains a plane sector $\mathcal{S}(v, v_1, v_2) := \{v + \alpha v_1 + \beta v_2 \mid \alpha, \beta \geq 0\}$ with $v, v_1, v_2 \in \mathbf{R}^2$ and v_1, v_2 linearly independent. $U \subseteq \mathbf{Z}^2$ is a 2D-unbounded set if $U = \mathcal{U} \cap \mathbf{Z}^2$ for some 2D-unbounded set $\mathcal{U}$ in $\mathbf{R}^2$.

Definition 1 $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is an autonomous 2D system if there exists a subset $T \subseteq \mathbf{Z}^2$ such that $\mathbf{Z}^2 \setminus T$ is 2D unbounded and satisfies the following condition:

$$\{w_1, w_2 \in \mathcal{B} \text{ and } w_1|_T = w_2|_T\} \;\Rightarrow\; \{w_1 = w_2\}.$$

So, intuitively, a system is autonomous if the evolution of its traiectories in a sufficently large portion of the discrete plane is completely specified by what occurs in remaining portion of the domain $\mathbf{Z}^2$. As stated in proposition 2.1, for autoregressive systems the autonomy is equivalent to the absence of free variables.

Notation Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be a system in the variables $(w_1, \ldots, w_q)^T := w$. The variable w_i, $i \in \{1, \ldots, q\}$, is a free variable if, for every $\alpha : \mathbf{Z}^2 \to \mathbf{R}$, there exists some $w \in \mathcal{B}$ such that $w_i = \alpha$. Similarly, a vector $(w_{i_1}, \ldots, w_{i_l})^T$, with $i_j \in J \subseteq \{1, \ldots, q\}$ and $i_j \neq i_k$ if $j \neq k$, is a vector of free variables if for every $\beta : \mathbf{Z}^2 \to \mathbf{R}^l$ there exists $w \in \mathcal{B}$ such that $(w_{i_1}, \ldots, w_{i_l})^T = \beta$. The number of free variables in a system Σ is defined as the maximum dimension of a vector of free variables in Σ.

Lemma 2.1 Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be an autoregressive 2D system such that $\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}$. Then:

1. The number l of free variables in Σ is $q -$ rank R.
2. If $l > 0$, there exists a nonzero $q \times l$ polynomial matrix $M(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that $\mathcal{B} \supseteq \text{im} M(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$, where $M(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ is viewed as an operator from $\{v : \mathbf{Z}^2 \to \mathbf{R}^l\}$ into $\{w : \mathbf{Z}^2 \to \mathbf{R}^q\}$. Moreover, im M has l free variables.

PROOF: In order to prove the statement 1 we first consider the case where R has full row rank. In this case, there is a column permutation Π such that $R\Pi = [\ P \mid Q\ ]$ with P square $r \times r$ and nonsingular. This means that the equation $Rw = 0$ is equivalent to

$$Pw_1 = -Qw_2, \tag{2.1}$$

with $\text{col}(w_1, w_2) = \Pi w$. Since P is full row rank, it can be shown that $P(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ is a surjective operator, and w_2 is a $q - r$ dimensional vector of free variables. So $l \geq q - r$. Now, it remains to see that none of the components of w_1 is free. Let $P^*(z_1, z_2, z_1^{-1}, z_2^{-1})$ be such that $P^*P = \text{diag}(p) =: D$, with $p :=$ det P, and define $E := -P^*Q$. Premultiplying (2.1) by P^* yields

$$Dw_1 = Ew_2 \tag{2.2}$$

In particular, if $w_2 \equiv 0$, (2.2) implies that the components w_{1i} of w_1 must satisfy:

$$p(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w_{1i} = 0 \quad i = 1, \ldots, r \tag{2.3}$$

and are hence not free. This shows that the number l of free variables in $\mathcal{B} :=$ ker R is exactly $l = q - r$.
If R has not full row rank, there exists a factorization

$$R = F\overline{R} \tag{2.4}$$

such that F has full column rank, $\overline{R}$ has full row rank and rank $F =$ rank $\overline{R} =$ rank R. Let $\overline{F}$ be an $r \times r$ submatrix of F obtained by selecting r linearly independent rows. It is not difficult to see that

$$\overline{R}w = 0 \Rightarrow F\ \overline{R}w = 0 \Rightarrow \overline{F}\ \overline{R}w = 0,$$

or, equivalently,

$$\mathcal{B}_1 := \ker\ \overline{R} \subseteq \mathcal{B} := \ker\ R \subseteq \ker\ \overline{F}\ \overline{R} =: \mathcal{B}_2.$$

Since $\overline{R}$ and $\overline{F}\ \overline{R}$ are both matrices with full row rank r, it follows from the previous reasoning that both $\mathcal{B}_1$ and $\mathcal{B}_2$ have $l = q - r$ free variables proving the first statement of the lemma.
As for 2, if $l = q - r$, without loss of generality the matrix $\overline{R}$ in the factorization (2.4) can be taken to be a full rank factor left prime 2D polynomial matrix of size $r \times q$ (note that if $\overline{R}$ is not left prime its nontrivial left factors can be extracted and included in F). In this case it follows from [1, Theorem 1] that there exists a $q \times l$ matrix $M(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that $\ker\ \overline{R}(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) = \text{im}\ M(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$. So, $\mathcal{B} \supseteq \ker\ \overline{R} = \text{im}\ M$. Finally, we note that the number of free variables in $\ker \overline{R}$ is still l, and hence $\text{im}\ M = \ker\ \overline{R}$ has l free variables.

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Proposition 2.1 An autoregressive system $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is autonomous if and only if it has no free variables.

PROOF: (i) Suppose that Σ is a system without free variables. This means that $\mathcal{B} = \ker\ R$, for some full rank matrix $R(z_1, z_2, z_1^{-1}, z_2^{-1})$. Let P be a $q \times q$ matrix obtained by taking q rows of R, and define $P^* := DP^{-1}$ with $D := \text{diag}(p)$ and $p := \det\ P$ (note that P^* is a polynomial matrix). Clearly,

$$Rw = 0 \Rightarrow Pw = 0 \Rightarrow Dw = 0 \Leftrightarrow p(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w_i = 0 \quad i = 1, \ldots, q,$$

where w_i denotes the i-th component of w.
We next show that $\ker\ D$ is an autonomous behaviour. Since $\mathcal{B} \subseteq \ker\ D$, this implies that also $\mathcal{B}$ (and hence Σ) is autonomous. Since the support of p if finite, then there exist integers l_1, L_1, l_2, L_2 such that such support is included in the set $T := \{(h_1, h_2) \in \mathbf{Z}^2 \mid l_1 \le h_1 \le L_1 \text{ or } l_2 \le h_2 \le L_2\}$. It can be shown that the solutions of the equation are completely determined by their values on T, i.e., if w_1, w_2 are elements of $\ker\ D$ and $w_1|_T = w_2|_T$, then $w_1 = w_2$. Since $\mathbf{Z}^2 \setminus T$ is 2D-unbounded, this means that $\ker\ D$ is autonomous.

(ii) Assume that Σ has $l > 0$ free variables and let M be as in lemma 2.1. Denote respectively by $\overline{m}_i$ and $\underline{m}_i$ the maximum and the minimum of the exponents of z_i in the entries of M, and define the extent of M as $e(M) :=$

$\sqrt{2}$ max $\{\overline{m}_1 - \underline{m}_1, \overline{m}_2 - \underline{m}_2\}$. Further, denote the Euclidean distance by $d(\cdot,\cdot)$. Given any subset $T \subseteq \mathbf{Z}^2$ such that $\mathbf{Z}^2 \setminus T$ is 2D-unbounded, define two trajectories v' and $v" \in \{v : \mathbf{Z}^2 \to \mathbf{R}^q\}$ in the following way. The trajectory v' is simply the zero trajectory. As for $v"$, we require that $v"(t_1, t_2) = 0$ if $d((t_1, t_2), T) \leq e(M)$; for (t_1, t_2) with $d((t_1, t_2), T) > e(M)$, we define $v"(t_1, t_2)$ in such a way that $Mv" \neq 0$. Note that this is possible since the value of $Mv"$ at a point $(t_1^*, t_2^*) \in \mathbf{Z}$ depends only on the values of $v"$ at the points (t_1, t_2) such that $d((t_1, t_2), (t_1^*, t_2^*)) \leq e(M)$. Let now $w' := Mv' = 0$ and $w" := Mv"$. Clearly, $w', w" \in \mathcal{B}$. Moreover $w'|_T = w"|_T = 0$, and $w" \neq 0 = w'$. This shows that Σ is not autonomous.

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Contrary to the one dimensional case, where autonomous linear systems are necessarily finite dimensional [2], autonomous 2D systems may be infinite dimensional. The following proposition characterizes the autonomy and the finite dimensionality properties of AR 2D systems.

Proposition 2.2 Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be an AR 2D system with behaviour $\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}$, where $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ is a 2D polynomial matrix. Then Σ is autonomous if and only if R has full column rank. Moreover Σ is finite dimensional if and only if R is right factor prime.

PROOF The first part of the result is an immediate consequence of lemma 2.1 and proposition 2.1. The second statement follows from [7, Theorem 3.8].

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An extreme example of non-autonomous systems is the class of (externally) controllable systems. For these systems, the evolution of the trajectories outside a restricted part T of the domain eventually becomes independent of what occurs in T. Formally, we define (external) controllability as follows.

Definition 2 A 2D system $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ is (externally) controllable if the following condition holds. There exists a positive real number ρ such that, for all $T_1, T_2 \subseteq \mathbf{Z}^2$ with $d(T_1, T_2) > \rho$ and for all $w_1, w_2 \in \mathcal{B}$, there exists $w \in \mathcal{B}$ such that $w|_{T_i} = w_i|_{T_i}, i = 1, 2$. Here $d(T_1, T_2)$ denotes the euclidean distance between the sets T_1 and T_2.

Remark We refer to the above notion of controllability as to external controllability in order to make a distinction from the classical notion, which applies to state space realization. Our definition is given at an external level, as it only refers to the (external) system variables $w \in \mathcal{B}$. However, when no possibility of confusion arises, we will simply refer to it as controllability.

The next result provides a characterization of controllability for autoregressive systems.

Proposition 2.3 [1] Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ be an AR 2D system. Then Σ is controllable if and only if there exists a factor left-prime polynomial matrix $P(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid P(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}. \qquad |||||$$

An interesting feature of controllability and autonomy is the fact that these are complementary properties, in the sense that an arbitrary AR system can be viewed as the sum of a controllable system with an autonomous one.

Notation Given two systems $\Sigma_i = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}_i), i = 1, 2$, the sum $\Sigma_1 + \Sigma_2$ of Σ_1 and Σ_2 is defined as $\Sigma_1 + \Sigma_2 := (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$, where $\mathcal{B} := \mathcal{B}_1 + \mathcal{B}_2$

Proposition 2.4 Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B})$ a 2D system. Then there exist AR systems $\Sigma^c = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^c)$ and $\Sigma^a = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^a)$ such that:

1. Σ^c is controllable,
2. Σ^a is autonomous, and
3. $\Sigma = \Sigma^c + \Sigma^a$.

PROOF: Let $R(z_1, z_2, z_1^{-1}, z_2^{-1})$ such that

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}.$$

Then there exist polynomial matrices $F(z_1, z_2, z_1^{-1}, z_2^{-1})$, with full column rank, and $P(z_1, z_2, z_1^{-1}, z_2^{-1})$, with full row rank and factor left prime, such that $R = FP$. Without loss of generality we can assume that $P = [P_1 \;\; P_2]$, with P_1 a square and full rank matrix. Define

$$\mathcal{B}^c = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid Pw = 0\},$$

$$\mathcal{B}^a = \{w : \mathbf{Z}^2 \to \mathbf{R}^q \mid w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, w_2 = 0 \text{ and } FP_1 w_1 = 0\},$$

$\Sigma^c = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^c)$ and $\Sigma^a = (\mathbf{Z}^2, \mathbf{R}^q, \mathcal{B}^a)$. Clearly, by proposition 2.3, Σ^c is controllable and, by proposition 2.2, Σ^a is autonomous. We will show that $\Sigma = \Sigma^c + \Sigma^a$. Since both $\mathcal{B}^c$ and $\mathcal{B}^a$ are subspaces of $\mathcal{B}$,

$$\mathcal{B}^c + \mathcal{B}^a \subseteq \mathcal{B}.$$

In order to prove the reciprocal inclusion, assume that $w \in \mathcal{B}$ is given. Let $w^a := \begin{bmatrix} w_1^a \\ w_2^a \end{bmatrix}$, with $w_2^a := 0$ and w_1^a such that $P_1 w_1^a = Pw$ (note that $P_1(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ is a surjective operator, as P_1 has full row rank). Further define $w^c := w - w^a$. Now, since $FP_1 w_1^a = FPw = Rw = 0$, it follows that $w^a \in \mathcal{B}^a$. Moreover,

$$Pw^c = Pw - Pw^a = Pw - [P_1 \;\; P_2] \begin{bmatrix} w_1^a \\ w_2^a \end{bmatrix} = Pw - P_1 w_1^a = 0$$

and hence $w^c \in \mathcal{B}^c$. So, $w = w^c + w^a$ with $w^c \in \mathcal{B}^c$ and $w^a \in \mathcal{B}^a$, proving that $\mathcal{B} \subseteq \mathcal{B}^c + \mathcal{B}^a$. This yields the desired result.

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Remark Note that the above sum $\mathcal{B} = \mathcal{B}^c + \mathcal{B}^a$ (and hence $\Sigma = \Sigma^c + \Sigma^a$) is not necessarily a direct sum, i.e. we may have $\mathcal{B}^c \cap \mathcal{B}^a \neq \{0\}$. However, it can be shown that $\mathcal{B} = \mathcal{B}^c \oplus \mathcal{B}^a$ if and only if in the decomposition $R = FP$ the matrix P is zero-left-prime. Moreover, it is not difficult to prove that $\Sigma = \Sigma_1^c + \Sigma_1^a = \Sigma_2^c + \Sigma_2^a$ imply $\Sigma_1^c = \Sigma_2^c$, i.e. in the decomposition $\Sigma = \Sigma^c + \Sigma^a$ the system Σ^c is unique. In fact, it turns out that Σ^c is the largest controllable subsystem of Σ. Curiously, this uniqueness does not necessarily hold for Σ^a. This is illustrated in the example below.

Example Let $\Sigma = (\mathbf{Z}^2, \mathbf{R}^3, \mathcal{B})$ with

$$\mathcal{B} = \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0\}.$$

where

$$R(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} z_1 - 1 & 0 & (z_1 - 1)(z_1 + z_2) \\ 0 & z_2 - 1 & (z_2 - 1)(z_2 - z_1) \end{bmatrix}.$$

Then, clearly, R can be decomposed as $R = FP$ with

$$F(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} z_1 - 1 & 0 \\ 0 & z_2 - 1 \end{bmatrix}$$

and

$$P(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} 1 & 0 & z_1 + z_2 \\ 0 & 1 & z_2 - z_1 \end{bmatrix}.$$

Constructing Σ^c and Σ^a as in the proof of the proposition 2.4 yields $\Sigma^c = (\mathbf{Z}^2, \mathbf{R}^3, \mathcal{B}^c)$ with

$$\mathcal{B}^c := \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid Pw = 0\},$$

and $\Sigma^a = (\mathbf{Z}^2, \mathbf{R}^3, \mathcal{B}^a)$ with

$$\mathcal{B}^a := \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, w_3 = 0, F \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0\}.$$

So, $\Sigma = \Sigma^a + \Sigma^c$. Let now $\bar{\Sigma}^a = (\mathbf{Z}^2, \mathbf{R}^3, \bar{\mathcal{B}}^a)$, with

$$\bar{\mathcal{B}}^a := \{w : \mathbf{Z}^2 \to \mathbf{R}^3 \mid w_2 = 0, F\bar{P} \begin{bmatrix} w_1 \\ w_3 \end{bmatrix} = 0\}$$

and

$$\bar{P}(z_1, z_2, z_1^{-1}, z_2^{-1}) := \begin{bmatrix} 1 & z_1 + z_2 \\ 0 & z_1 - z_2 \end{bmatrix}.$$

Applying the same reasoning as in the proof of proposition 2.4, it is easily shown that also $\bar{\Sigma}^a + \Sigma^c = \Sigma$. So, in the decomposition $\Sigma = \Sigma^a + \Sigma^c$ the autonomous subsystem is not unique.

The decomposition of an arbitrary AR 2D system Σ into the sum of a controllable part Σ^c and an autonomous part Σ^a can be used to obtain state space realizations of Σ by separately realizing Σ^c and Σ^a.

As shown in [1], every controllable AR system admits a state/driving variable realization of the form

$$\begin{cases} S(\sigma)\mathbf{x} &= 0 & (2.5) \\ \sigma_1 \mathbf{x} &= (A_1\sigma + A_0)\mathbf{x} + (B_1\sigma + B_0)v & (2.6) \\ w &= C\mathbf{x} + Dv & (2.7) \end{cases}$$

with $\sigma := \sigma_2^{-1}\sigma_1$ the diagonal shift, $\mathbf{x}$ the state and v an auxiliary free driving variable. Moreover, the matrices $S(z)$, $A(z) := A_1 z + A_0$ and $B(z) := B_1 z + B_0$ are such that $A(\sigma)\ker\ S(\sigma) \subseteq \ker\ S(\sigma)$ and $\operatorname{im} B(\sigma) \subseteq \ker\ S(\sigma)$.

This model can be interpreted as follows. On each diagonal line $\mathcal{L}_k := \{(i, k-i) | i \in \mathbf{Z}\}$, $k \in \mathbf{Z}$, the state trajectories must satisfy the constraint of (2.5). Equation (2.6) yields the state on $\mathcal{L}_{k+1}$ once the state and the driving variable on $\mathcal{L}_k$ are given. We remark that, due to the special structure of $S(z)$, $A(z)$ and $B(z)$, if $\mathbf{x}|_{\mathcal{L}_k}$ satisfies (2.5), then the corrisponding state $\mathbf{x}|_{\mathcal{L}_{k+1}}$ computed from (2.6) also satisfies this restriction. Therefore we can view the equation (2.5) as a constraint on the admissible initial states along the diagonal line, say $\mathcal{L}_0$, and use (2.6) and (2.7) to propagate the $(\mathbf{x}, w)$-trajectories on the half-plane $\mathcal{H}_0 := \bigcup_{k\geq 0} \mathcal{L}_k$. By means of (2.6) the state $\mathbf{x}(i+1,j)$ is computed from the values of $\mathbf{x}$ and v on the nearest neighbours (i,j) and $(i+1,j-1)$ of $(i+1,j)$ (see figure 1).

This updating structure is the same as for the 2D input-state-output (i/s/o) model introduced in [6] known in the literature as the FM model. However, here the system dynamics is driven by the auxiliary variable v instead of being driven by the system inputs, and the model output is the system variable w,which includes both inputs and outputs. Another important distinction is that the FM i/s/o model does not include an explicit restriction on the admissible initial states as in (2.5). As it will later become clear, such restriction is essential for the realization of autonomous systems in state/driving variable form.

In view of foregoing considerations, it turns out that in order to study the realizability of an arbitrary 2D system $\Sigma = \Sigma^c + \Sigma^a$ by state/driving variable model as (2.5), (2.6) and (2.7) it is enough to focus on the realizability of the autonomous part Σ^a. This problem will be considered in the next section for the autonomous finite dimensional case.

3 Autonomous finite dimensional systems

We shall assume throughout that R is a full column rank, factor right prime matrix, with elements in the Laurent polynomial ring $\mathbf{R}[z_1, z_2, z_1^{-1}, z_2^{-1}] := A_\pm$. Moreover, for sake of simplicity, we shall first restrict to scalar behaviours, by assuming that $R = \begin{bmatrix} r_1 & r_2 & \cdots & r_t \end{bmatrix}^T$ is a column vector, and successively extend the results to the general case.

3.1 Scalar case

In the subsequent discussion a significant role will be played by some connections between the ideals in $A_\pm$ and the ideals in $A_+ := \mathbf{R}[z_1, z_2]$ and by an abstract characterization of the system behaviour based on the algebraic properties of dual spaces. Let us first consider the following map

$$|\cdot| : A_\pm \to A_+ : p \mapsto |p| := z_1^{-i} z_2^{-j} p$$

where i and j are the minimum degrees of the monomials that appear in the nonzero Laurent polynomial p w.r.t. to the variables z_1 and z_2 respectively. More precisely, if

$$p = \sum_{h,k \in \mathbf{Z}} p_{hk} z_1^h z_2^k$$

then

$$i := \min\{h \in \mathbf{Z} \mid \exists k \in \mathbf{Z}, p_{hk} \neq 0\}$$
$$j := \min\{k \in \mathbf{Z} \mid \exists h \in \mathbf{Z}, p_{hk} \neq 0\}.$$

In case $p = 0$, we define $|p| = 0$. Clearly, for every nonzero Laurent polynomial p, $|p|$ includes a monomial in z_1 and a monomial in z_2 with nonzero coefficients.

The operation just described, of shifting the support of a Laurent polynomial into the positive orthant of $\mathbf{Z} \times \mathbf{Z}$, associates with the ideal $\mathcal{I}_\pm := (r_1, r_2, \ldots, r_t)_\pm$ generated in $A_\pm$ by the elements of the matrix R an ideal $\mathcal{I}_+ := (|r_1|, |r_2|, \ldots, |r_t|)_+$ generated in A_+ by $|r_1|, |r_2|, \ldots, |r_t|$. Some relevant connections between $\mathcal{I}_\pm$ and $\mathcal{I}_+$ are summarized in the following Lemma.

Lemma 3.1 i) $p \in \mathcal{I}_\pm$ if and only if there exists a pair of integers (i,j) such that $z_1^i z_2^j p \in \mathcal{I}_+$.
ii) The quotient space $A_\pm / \mathcal{I}_\pm$ is finite dimensional if and only if the same holds for $A_+ / \mathcal{I}_+$.

PROOF i) is obvious. As far as ii) is concerned, suppose first that $A_+/\mathcal{I}_+$ is a nonzero finite dimensional space. This implies that $\mathcal{I}_+$, and hence $\mathcal{I}_\pm$, include two nonzero polynomials $f(z_1)$ and $g(z_2)$, with deg $f > 0$ and deg $g > 0$. It is easily seen that the cosets

$$[z_1^i z_2^j] := z_1^i z_2^j + \mathcal{I}_\pm, \quad 0 \leq i < \deg f,\ 0 \leq j < \deg g$$

constitute a finite set of generators for the quotient space $A_\pm / \mathcal{I}_\pm$.
Conversely suppose that $A_\pm / \mathcal{I}_\pm$ is finite dimensional and let d be any common factor of $|r_1|, |r_2|, \ldots, |r_t|$
It is clear that $\mathcal{I}_\pm \subseteq (d)_\pm$, where $(d)_\pm$ is the principal ideal of $A_\pm$ generated by d. We therefore have

$$\dim A_\pm / \mathcal{I}_\pm \geq \dim A_\pm / (d)_\pm$$

and $A_\pm / (d)_\pm$ is finite dimensional. This implies that the polynomial sets $\{z_1^i, i \in \mathbf{Z}\}$ and $\{z_2^j, j \in \mathbf{Z}\}$ are linearly dependent modulo $(d)_\pm$, and hence

$(d)_\pm$ includes two nonzero polynomials $f(z_1)$ and $g(z_2)$. Since d must be the constant polynomial, $r_1, \ldots, r_t$ are coprime and A_+/I_+ is finite dimensional.

|||||

We introduce next a special nondegenerate bilinear function

$$\langle \cdot, \cdot \rangle : A_\pm \times \mathbf{R}^{\mathbf{Z}\times\mathbf{Z}} \to \mathbf{R},$$

by assuming

$$\langle p, w \rangle = \sum_{ij} p_{ij} w(i,j)$$

for all polynomials $p = \sum p_{ij} z_1^i z_2^j$ in $A_\pm$ and all signals w in $\mathbf{R}^{\mathbf{Z}\times\mathbf{Z}}$.

For instance, if p is the Laurent polynomial $z_1 + z_2^2 - z_1^{-1} z_2 + 3 - z_2^{-3}$ and $w(i,j) = e^{i+j}$, then $\langle p, w \rangle = e^2 + e + 2 - e^{-3}$.

In this way, the "universe" $\mathbf{R}^{\mathbf{Z}\times\mathbf{Z}}$ of all signals with support in $\mathbf{Z} \times \mathbf{Z}$ is isomorphic to the algebraic dual of $A_\pm$, i.e. to the space of the linear functionals on $A_\pm$. Moreover the behaviour $\mathcal{B}$ can be identified with the orthogonal complement of $I_\pm$ w.r. to such bilinear function

$$\mathcal{B} = I_\pm^\perp , \tag{3.1}$$

and, by duality,

$$\mathcal{B}^\perp = I_\pm^{\perp\perp} = I_\pm.$$

The proof of (3.1) is an easy consequence of the following identity

$$p(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) w(h,k) = \langle p z_1^h z_2^k, w \rangle$$

In fact $w \in \mathcal{B}$ implies $p(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) w = 0$ and, therefore, $\langle p, w \rangle = 0$, $\forall p \in I_\pm$. Viceversa, given $w \in I_\pm^\perp$ and $p \in I_\pm$, we have $\langle p z_1^h z_2^k, w \rangle = 0$, $\forall h, k \in \mathbf{Z}$, which implies $p(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) w = 0$.

We now restrict our attention on the space $\mathcal{B}$ and on the quotient space $A_\pm / I_\pm$. Using standard techniques of linear algebra [13], it can be shown that $A_\pm / \mathcal{B}^\perp = A_\pm / I_\pm$ and $\mathcal{B}$ constitute a dual pair w.r. to the bilinear function

$$\langle [p], w \rangle := \langle p, w \rangle.$$

Moreover, the canonical injection

$$i : \mathcal{B} \to \mathbf{R}^{\mathbf{Z}\times\mathbf{Z}}$$

is dual w.r.to the canonical projection π of $A_\pm$ onto $A_\pm/\mathcal{I}_\pm$

$$\pi : A_\pm/\mathcal{I}_\pm \leftarrow A_\pm.$$

For reasons that will be clear later on, we then wish to exhibit explicitely an isomorphism (for the vector space structure) of $\mathcal{B}$ onto $A_\pm/\mathcal{I}_\pm$.

Proposition 3.1 If the matrix R is right factor prime, then $\mathcal{B}$ and $A_\pm/\mathcal{I}_\pm$ are finite dimensional isomorphic vector spaces.

PROOF Since R is right factor prime, $A_+/\mathcal{I}_+$ is finite dimensional. Therefore, by lemma 3.1, $A_\pm/\mathcal{I}_\pm$ is finite dimensional too. Let now

$$([p_1], [p_2], \ldots, [p_n])$$

be a basis of $A_\pm/\mathcal{I}_\pm$ and consider the linear map

$$\psi : \mathcal{B} \to A_\pm/\mathcal{I}_\pm : w \mapsto \sum_{i=1}^{n} \langle p_i, w\rangle [p_i].$$

If $[\alpha_1\ \alpha_2\ \ldots\ \alpha_n]^T \in \mathbf{R}^n$ is orthogonal to $[\langle p_1, w\rangle\ \langle p_2, w\rangle \ldots \langle p_n, w\rangle]^T$ for all $w \in \mathcal{B}$, then $\langle \sum \alpha_i p_i, w\rangle = 0$, $\forall w \in \mathcal{B}$ and

$$\sum_i \alpha_i p_i \in \mathcal{B}^\perp = \mathcal{I}_\pm$$

This implies $\alpha_i = 0$, $i = 1, 2, \ldots, n$ and, consequently, $[\langle p_1, w\rangle\ \langle p_2, w\rangle \ldots \langle p_n, w\rangle]^T$ span $\mathbf{R}^n$ as w varies over $\mathcal{B}$. Therefore ψ is surjective.
Suppose now that $w \in \mathcal{B}$ satisfies $\psi(w) = 0$ or, equivalently, $\langle w, p_i\rangle = 0$, $i = 1, 2, \ldots n$. Since every p in $A_\pm$ can be expressed as $p = \sum_{i=1}^{n} \alpha_i p_i + r$, $r \in \mathcal{I}_\pm$, for all $p \in A_\pm$ we have $\langle w, p\rangle = \langle w, r\rangle = 0$, which implies $w = 0$. Therefore ψ is injective.

|||||

From now on , we suppose that a basis $([p_1], [p_2], \ldots, [p_n])$ has been chosen in $A_\pm/\mathcal{I}_\pm$, and consider the corresponding dual basis $(w_1, w_2, \ldots, w_n)$ in $\mathcal{B}$.
The relations $\langle [p_i], w_j\rangle = \delta_{ij}$, $i, j = 1, 2, \ldots n$ imply

$$[p] = \sum_{i=1}^{n} \langle [p], w_i\rangle [p_i], \quad \forall [p] \in A_\pm/\mathcal{I}_\pm$$

and, on the other hand,

$$w = \sum_{i=1}^{n} \langle [p_i], w \rangle w_i, \quad \forall w \in \mathcal{B}.$$

Introduce the following invertible linear maps

$$Z_1 : A_\pm / I_\pm \to A_\pm / I_\pm : [p] \mapsto [z_1 p]$$

$$Z_2 : A_\pm / I_\pm \to A_\pm / I_\pm : [p] \mapsto [z_2 p].$$

Clearly $Z_1 Z_2 = Z_2 Z_1$ and the adjoint maps of Z_1 and Z_2 in $\mathcal{B}$ are σ_1 and σ_2 respectively.

The matrices $N_i = [n_{hk}^{(i)}]$, $i = 1, 2$ representing the linear transformations Z_i, $i = 1, 2$ w.r. to the basis $([p_1], [p_2], \dots, [p_n])$ are given by $n_{hk}^{(i)} = \langle [z_i p_k], w_h \rangle$. Hence, the matrices representing σ_1 and σ_2 w.r. to the dual basis are N_1^T and N_2^T respectively. In fact, letting $\sigma_i w_j = \sum_h t_{hj}^{(i)} w_h$, $i = 1, 2$, we have

$$\langle [p_k], \sigma_i w_j \rangle = \sum_h t_{hj}^{(i)} \langle [p_k], w_h \rangle = t_{kj}^{(i)} \tag{3.2}$$

and, using the duality,

$$\langle [p_k], \sigma_i w_j \rangle = \langle [z_i p_k], w_j \rangle = \sum_r n_{rk}^{(i)} \langle p_r, w_j \rangle = n_{jk}^{(i)} \tag{3.3}$$

Comparing (3.2) and (3.3) gives the result.

We are now in a position for providing a state driving variable realization of an autonomous finite dimensional system Σ^a.
For any $w \in \mathcal{B}$, we introduce the following signal

$$\mathbf{x} : \mathbf{Z} \times \mathbf{Z} \to \mathbf{R}^n : (h, k) \mapsto \begin{bmatrix} \langle [p_1], \sigma_1^h \sigma_2^k w \rangle \\ \langle [p_2], \sigma_1^h \sigma_2^k w \rangle \\ \dots \\ \langle [p_n], \sigma_1^h \sigma_2^k w \rangle \end{bmatrix}$$

The value of $\mathbf{x}$ at (h, k) provides the components of $\sigma_1^h \sigma_2^k w$ with respect to the basis $(w_1, w_2, \dots w_n)$. It is clear that, once $\mathbf{x}(0,0)$ is known, $\mathbf{x}(h,k)$ is easily computed for all $(h,k) \in \mathbf{Z} \times \mathbf{Z}$

$$\mathbf{x}(h,k) = (N_1^T)^h (N_2^T)^k \mathbf{x}(0,0)$$

Moreover, the value of w at (h,k) can be recovered from $\mathbf{x}(h,k)$ as follows

$$\begin{aligned} w(h,k) &= (\sigma_1^h \sigma_2^k w)(0,0) = \langle [1], \sigma_1^h \sigma_2^k w \rangle \\ &= \langle c_1[p_1] + c_2[p_2] + \ldots + c_n[p_n], \sigma_1^h \sigma_2^k w \rangle = C\mathbf{x}(h,k) \end{aligned}$$

Here $C := [\, c_1 \quad c_2 \quad \ldots \quad c_n \,]$ denotes the row vector of the components of $[1]$ w.r.to the basis $([p_1], [p_2], \ldots, [p_n])$ in $A_\pm / I_\pm$.

The above results are summarized in the following recursive model

$$\begin{cases} \sigma_1 \mathbf{x} = N_1^T \mathbf{x} \\ \sigma_2 \mathbf{x} = N_2^T \mathbf{x} \\ w = C\mathbf{x} \end{cases}$$

Every signal of the autonomous behaviour $\mathcal{B}$ is uniquely determined by the corresponding value of the state $\mathbf{x}$ at any point (h,k) and, conversely, different states at (h,k) induce different signals in the autonomous system. Finally, letting $\sigma := \sigma_1 \sigma_2^{-1}$, $S(\sigma) := \sigma I - N_1^T (N_2^T)^{-1}$ and $A(\sigma) = N_1^T$, we end up with a state driving variable realization of $\mathcal{B}$, as follows

$$\begin{cases} S(\sigma) w = 0 \\ \sigma_1 \mathbf{x} = A(\sigma)\mathbf{x} \\ w = C\mathbf{x} \end{cases}$$

When p_j, $j = 1, 2, \ldots n$ are monic monomials, i.e. $p_j = z_1^{\mu_j} z_2^{\nu_j}$, $j = 1, 2, \ldots, n$, the structure of the corresponding dual basis is very appealing. In fact the element w_j is the unique element of $\mathcal{B}$ taking the values 1 at (μ_j, ν_j) and 0 at (μ_i, ν_i), $i = 1, 2, \ldots, j-1, j+1, \ldots, n$. Moreover, for every $(h,k) \in \mathbf{Z} \times \mathbf{Z}$, the components of the state vector $\mathbf{x}(h,k)$ are the values of w at $\{(\mu_1 + h, \nu_1 + k), (\mu_2 + h, \nu_2 + k), \ldots, (\mu_n + h, \nu_n + k)\}$.

A further reason for using a monomial basis in $A_\pm / I_\pm$ will be made apparent in the following subsection, where an algorithm for the computation of the matrices N_1^T and N_2^T is outlined. Some concepts on computer algebra, and in particular on Gröbner basis theory are required; for details see [10].

3.2 Computational Methods

Let $\mathcal{G} = \{g_1, g_2, \ldots, g_h\}$, $g_i \in A_+$, be a Gröbner basis of the ideal I_+, and denote by $\{q_1 = 1, q_2, \ldots, q_m\}$ the set of monic monomials that are not multiple of the leading power products of any of the polynomials in $\mathcal{G}$. Then

$$(q_1 + I_+, q_2 + I_+, \ldots, q_m + I_+)$$

is a basis of A_+/I_+ and the linear transformations

$$\phi_i : A_+/I_+ \rightarrow A_+/I_+ : q + I_+ \mapsto z_i q + I_+, \quad i = 1, 2$$

are represented by a pair of commuting matrices M_1 and M_2 .

Our purpose here is to supplement the algorithm discussed in [11] for obtaining M_1 and M_2, so as to provide a constructive technique for obtaining the invertible matrices N_1 and N_2 introduced in the previous subsection. The procedure we are going to describe will shed also some light on the connections between the ideals $I_\pm$ and I_+.

Let μ be a positive integer with the property that the subspace of A_+/I_+ spanned by $\{z_1^{\mu+h} z_2^{\mu+k} q_i + I_+\}_{i=1,2,\ldots m}$ is independent of h and k, for all h and $k \geq 0$. Therefore

$$\mathcal{L} := \operatorname{span}_i\{z_1^\mu z_2^\mu q_i + I_+\}$$

is a ϕ_1-ϕ_2-invariant subspace, satisfying

$$\phi_1 \mathcal{L} = \phi_2 \mathcal{L} = \mathcal{L}$$

and the restrictions of ϕ_1 and ϕ_2 to $\mathcal{L}$ constitute a couple of invertible commutative linear transformations.

Upon assuming in $\mathcal{L}$ a basis given by

$$(z_1^\mu z_2^\mu q_{i_1} + I_+, z_1^\mu z_2^\mu q_{i_2} + I_+, \ldots, z_1^\mu z_2^\mu q_{i_\nu} + I_+) \tag{3.4}$$

the restriction of ϕ_1 to $\mathcal{L}$ is associated with a $\nu \times \nu$ invertible matrix N_1 as follows

$$\begin{aligned}&(z_1^{\mu+1} z_2^\mu q_{i_1} + I_+, z_1^{\mu+1} z_2^\mu q_{i_2} + I_+, \ldots, z_1^{\mu+1} z_2^\mu q_{i_\nu} + I_+)\\ &\quad = (z_1^\mu z_2^\mu q_{i_1} + I_+, \; z_1^\mu z_2^\mu q_{i_2} + I_+, \ldots, \; z_1^\mu z_2^\mu q_{i_\nu} + I_+) N_1\end{aligned}$$

Let S_1 be the boolean matrix that selects the basis (3.4) out of the ordered array $(z_1^\mu z_2^\mu q_1 + I_+, z_1^\mu z_2^\mu q_2 + I_+, \ldots, z_1^\mu z_2^\mu q_m + I_+)$

$$\begin{aligned}&(z_1^\mu z_2^\mu q_1 + I_+, z_1^\mu z_2^\mu q_2 + I_+, \ldots, z_1^\mu z_2^\mu q_m + I_+) S_1\\ &\quad = (z_1^\mu z_2^\mu q_{i_1} + I_+, z_1^\mu z_2^\mu q_{i_2} + I_+, \ldots, z_1^\mu z_2^\mu q_{i_\nu} + I_+)\end{aligned}$$

Then, recalling that M_1 and M_2 represent the linear transformations ϕ_1 and ϕ_2 with respect to the basis $(q_1 + I_+, q_2 + I_+, \ldots, q_m + I_+)$, we have

$$\begin{aligned}&(z_1^{\mu+1} z_2^\mu q_{i_1} + I_+, z_1^{\mu+1} z_2^\mu q_{i_2} + I_+, \ldots, z_1^{\mu+1} z_2^\mu q_{i_\nu} + I_+)\\ =&(z_1^\mu z_2^\mu q_1 + I_+, z_1^\mu z_2^\mu q_2 + I_+, \ldots, z_1^\mu z_2^\mu q_m + I_+) S_1 N_1 \\ =&(q_1 + I_+, q_2 + I_+, \ldots, q_m + I_+) M_1^\mu M_2^\mu S_1 N_1\end{aligned} \tag{3.5}$$

On the other hand, the definitions of M_1, M_2 and S also imply

$$\begin{aligned}&(z_1^{\mu+1}z_2^{\mu}q_{i_1}+I_+, z_1^{\mu+1}z_2^{\mu}q_{i_2}+I_+,\ldots,z_1^{\mu+1}z_2^{\mu}q_{i_\nu}+I_+)\\ =&(z_1^{\mu+1}z_2^{\mu}q_1+I_+, z_1^{\mu+1}z_2^{\mu}q_2+I_+,\ldots,z_1^{\mu+1}z_2^{\mu}q_m+I_+)S_1 \qquad (3.6)\\ =&(q_1+I_+,q_2+I_+,\ldots,q_m+I_+)M_1^{\mu+1}M_2^{\mu}S_1\end{aligned}$$

Comparing (3.5) and (3.6) gives $M_1^{\mu+1}M_2^{\mu}S_1 = M_1^{\mu}M_2^{\mu}S_1N_1$. Since $M_1^{\mu}M_2^{\mu}S_1$ has full column rank, letting

$$H := (M_1^T)^{\mu}(M_2^T)^{\mu}M_1^{\mu}M_2^{\mu}$$

we obtain

$$N_1 = (S_1^T H S_1)^{-1}(S_1^T H M_1 S_1) \qquad (3.7)$$

and, similarly,

$$N_2 = (S_2^T H S_2)^{-1}(S_2^T H M_2 S_2) \qquad (3.8)$$

The next proposition shows that the monomials $q_{i_1}, q_{i_2}, \ldots q_{1_\nu}$ resulting from the previous procedure and associated, as shown, to a basis of the subspace $\mathcal{L} \subseteq A_+/I_+$, also provide the basis of the quotient space $A_\pm/I_\pm$ we are looking for.

Proposition 3.2 The monomials $q_{i_1}, q_{i_2}, \ldots, q_{i_\nu}$ constitute a basis of $A_\pm$, modulo $I_\pm$.

PROOF Suppose that $\sum_{h=1}^{n} \alpha_h q_{i_h}$ is in $I_\pm$.
By Lemma 3.1 there exists a positive integer ℓ such that $\sum_{h=1}^{\nu} \alpha_h q_{i_h} z_1^{\ell} z_2^{\ell}$ and, *a fortiori*, $\sum_{h=1}^{\nu} \alpha_h q_{i_h} z_1^{\mu+\ell} z_2^{\mu+\ell}$ belong to I_+.
Since the monomials

$$q_{i_1} z_1^{\mu+\ell} z_2^{\mu+\ell}, q_{i_2} z_1^{\mu+\ell} z_2^{\mu+\ell}, \ldots, q_{i_\nu} z_1^{\mu+\ell} z_2^{\mu+\ell}$$

are linearly independent modulo I_+, we have $\alpha_h = 0$, $h = 1, 2, \ldots, \nu$, and $q_{i_h}, h = 1, 2, \ldots, \nu$ are linearly independent modulo $I_\pm$.

It remains to show that they generate $A_\pm$ modulo $I_\pm$ To that purpose, consider any polynomial $p \in A_\pm$. Then there exists a positive integer ℓ such that $z_1^{\ell} z_2^{\ell} p \in A_+$. Therefore

$$\begin{aligned}(z_1^{\ell}z_2^{\ell}p)z_1^{\mu}z_2^{\mu} &= \sum_{h=1}^{\nu} \alpha_h q_{i_h} z_1^{\mu} z_2^{\mu} = \sum_{h=1}^{\nu} \beta_h q_{i_h} z_1^{\mu+\ell} z_2^{\mu+\ell} \quad \operatorname{mod} I_+\\ &= \sum_{h=1}^{\nu} \beta_h q_{i_h} z_1^{\mu+\ell} z_2^{\mu+\ell} \quad \operatorname{mod} I_\pm\end{aligned}$$

Upon multiplying on both sides by $z_1^{-\mu-\ell} z_2^{-\mu-\ell}$, we have

$$p = \sum_{h=1}^{\nu} \beta_h q_{i_h} \quad \mathrm{mod}\, \mathcal{I}_{\pm},$$

showing that the monomials q_{i_h} generate $A_{\pm}$ modulo $\mathcal{I}_{\pm}$.

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As a consequence of Proposition 3.2, the matrices N_1 and N_2 associated with the restrictions to $\mathcal{L}$ of ϕ_1 and ϕ_2, with respect to the basis (3.4) represent Z_1 and Z_2 with respect to the basis

$$([q_{i_1}], [q_{i_2}], \dots, [q_{i_\nu}]) \tag{3.9}$$

in $A_{\pm}/\mathcal{I}_{\pm}$. This result is almost obvious. Upon introducing the following isomorphism

$$\psi : \mathcal{L} \to A_{\pm}/\mathcal{I}_{\pm} : \sum_{h=1}^{\nu} \alpha_h (z_1^{\mu} z_2^{\mu} q_{i_h}) + \mathcal{I}_{+} \mapsto \sum_{h=1}^{\nu} \alpha_h [q_{i_h}]$$

one checks that the following diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\psi} & A_{\pm}/\mathcal{I}_{\pm} \\ \downarrow \phi_1 & & \downarrow Z_1 \\ \mathcal{L} & \xrightarrow{\psi} & A_{\pm}/\mathcal{I}_{\pm} \end{array}$$

commutes. With respect to the bases (3.4) and (3.9), ψ is represented by the identity matrix and therefore the same matrix N_1 represents both ϕ_1 and Z_1. Similarly ϕ_2 and Z_2 are both represented by N_2.

Remark In [11] it has been shown that the annihilating polynomials of M_1 and M_2 are exactly the polynomials of the ideal $\mathcal{I}_{+}$, i.e.

$$p(M_1, M_2) = 0 \Leftrightarrow p \in \mathcal{I}_{+}$$

It is quite natural to ask whether the Laurent polynomials in $\mathcal{I}_{\pm}$ do exhibit the characteristic property of annihilating the commutative invertible matrices N_1 and N_2. Actually this is true and we have

$$p(N_1, N_2) = 0 \Leftrightarrow p \in \mathcal{I}_{\pm} \tag{3.10}$$

To prove (3.10), we note first that, by Lemma 3.1, $p \in I_\pm$ if and only if there exists a pair of nonnegative integers i and j, such that $q(z_1, z_2) := z_1^i z_2^j p(z_1, z_2) \in I_+$. This in turn is equivalent to assume that $0 = q(M_1, M_2) = q(\phi_1, \phi_2)$ and therefore (3.10) can restated as follows

$$p(N_1, N_2) = 0 \;\Leftrightarrow\; q(M_1, M_2) = 0 \tag{3.11}$$

for some $q = z_1^j z_2^j p \in A_\pm$. To prove (3.11), assume first $q(M_1, M_2) = 0$. Then $q(\phi_1, \phi_2) = 0$ implies $q(\phi_1|_\mathcal{L}, \phi_2|_\mathcal{L}) = 0$ and, consequently, $0 = q(N_1, N_2) = N_1^i N_2^j p(N_1, N_2) = p(N_1, N_2)$, because of the invertibility of N_1 and N_2.

Viceversa, consider any polynomial $p \in A_\pm$ that annihilates the commutative pair N_1, N_2, i.e.

$$p(N_1, N_2) = 0 \qquad (3.12).$$

Select a pair of nonnegative integers h, k such that $p' = z_i^h z_2^k p$ is a polynomial in A_+. Rewrite p' as follows

$$p' = \sum_{j=1}^{m} \beta_j q_j + r, \quad r \in I_+$$

and let

$$q := z_1^\mu z_2^\mu p' = \sum_{j=1}^{m} \beta_j z_1^\mu z_2^\mu q_j + z_1^\mu z_2^\mu r$$

Note that (3.12) implies $p'(N_1, N_2) = q(N_1, N_2) = 0$ and $r \in I_+$ implies $r(M_1, M_2) = 0$. Restricting ϕ_1 and ϕ_2 to $\mathcal{L}$ gives $r(N_1, N_2) = 0$ and hence $\sum_{j=1}^m \beta_j q_j(N_1, N_2) = 0$. To prove that $q(M_1, M_2)$ is the zero matrix, we will show that $q(\phi_1, \phi_2)$ annihilates $q_i + I_\pm$, $i = 1, 2, \ldots m$. Actually we have

$$\begin{aligned} q(\phi_1, \phi_2)(q_i + I_+ &= \sum_{j=1}^{m} \beta_j \phi_1^\mu \phi_2^\mu q_j(\phi_1, phi_2)(q_i + I_+) + \phi_1^\mu \phi_2^\mu r(\phi_1, \phi_2)(q_i + I_+) \\ &= \sum_{j=1}^{m} \beta_j q_j(\phi_1, \phi_2)(z_1^\mu z_2^\mu q_i + I_+) \end{aligned}$$

Since $z_1^\mu z_2^\mu q_i + I_+$, $i = 1, 2 \ldots, m$ belong to $\mathcal{L}$ and $\sum_j \beta_j q_j(\phi_1, \phi_2)$ acts on $\mathcal{L}$ as the zero transformation, we are done.

|||||

3.3 Vector case

Suppose now that R is a $t \times q$ full column rank right prime matrix, describing a q variables behaviour $\mathcal{B}$. All concepts previously introduced for the scalar case have an immediate extension to the vector case. Let

$$A_+^q := \mathbf{R}^{1\times q}[z_1, z_2]$$

$$A_\pm^q := \mathbf{R}^{1\times q}[z_1, z_2, z_1^{-1}, z_2^{-1}]$$

and define the map

$$|\cdot| : A_\pm^q \to A_+^q : r \mapsto |r| := z_1^i z_2^j r,$$

where i and j are the minimum degrees of r w.r. to the indeterminates z_1 and z_2 respectively. In case $p = 0$, we define $|p| = 0$.

Let $\mathcal{M}_\pm := (r_1, \ldots, r_t)_\pm$ be the module in $A_\pm^q$ generated by the rows of R and $\mathcal{M}_+ := (|r_1|, \ldots, |r_t|)_+$ the module in A_+^q generated by the rows of the matrix

$$\bar{R} := \begin{bmatrix} |r_1| \\ \vdots \\ |r_t| \end{bmatrix} = \Lambda R$$

where $\Lambda := \text{diag}\ \{z_1^{\nu_1} z_2^{\mu_1}, \ldots, z_1^{\nu_t} z_2^{\mu_t}\}$ and ν_i and μ_i satisfy $z_1^{\mu_i} z_2^{\nu_i} r_i = |r_i|$, $i = 1, \ldots, t$.

Lemma 3.2 (i) A row r belongs to $\mathcal{M}_\pm$ if and only if there exists a pair of integers (i, j) such that $z_1^i z_2^j r$ is in $\mathcal{M}_+$.

(ii) $A_\pm^q / \mathcal{M}_\pm$ is finite dimensional if and only if $A_+^q / \mathcal{M}_+$ is finite dimensional.

PROOF: (i) Obvious.

(ii) Suppose that $A_+^q / \mathcal{M}_+$ is finite dimensional. This implies that there exist polynomials $f_i(z_1)$ and $g_i(z_2)$, $i = 1, \ldots, q$, such that

$$f_i(z_1) e_i^T \in \mathcal{M}_+$$

$$g_i(z_2) e_i^T \in \mathcal{M}_+$$

where e_i is the element of the canonical basis of $\mathbf{R}^q$ with 1 in position i. It is easily seen that

$$\bigcup_{i=1}^{q} \{z_1^h z_2^k e_i^T + \mathcal{M}_+, 0 \le h < \deg f_i, 0 \le k < \deg g_i\}$$

constitutes a set of generators for $A^q_\pm/\mathcal{M}_\pm$.

Conversely suppose that $A^q_\pm/\mathcal{M}_\pm$ is finite dimensional and let D be any right factor of $\bar{R}$

$$\bar{R} = \hat{R}D.$$

It follows that $R = \Lambda^{-1}\bar{R} = \Lambda^{-1}\hat{R}D$, where Λ^{-1} is still a polynomial matrix with elements in $A_\pm$. Since the module $\mathcal{M}_\pm(D)$ generated by the rows of D, satisfies $\mathcal{M}_\pm \subseteq \mathcal{M}_\pm(D)$, we have

$$\dim A^q_\pm/\mathcal{M}_\pm \;\geq\; \dim A^q_\pm/\mathcal{M}_\pm(D)$$

and $A^q_\pm/\mathcal{M}_\pm(D)$ is finite dimensional. Therefore there exist polynomials $f_i(z_1)$ and $g_i(z_2)$,$i = 1, \ldots, q$, such that

$$f_i(z_1)e_i^T \in \mathcal{M}_\pm(D)$$

$$g_i(z_2)e_i^T \in \mathcal{M}_\pm(D)$$

and, consequently, there exist polynomial matrices H and K such that

$$HD = \text{diag}\{f_1(z_1), \ldots, f_q(z_1)\} \tag{3.13}$$

$$KD = \text{diag}\{g_1(z_2), \ldots, g_q(z_2)\} \tag{3.14}$$

(3.13) implies that det D is a polynomial in z_1 and (3.14) implies that det D is a polynomial in z_2. Therefore D is unimodular, $\bar{R}$ is right factor prime and $A^q_+/\mathcal{M}_+$ is finite dimensional.

|||||

Introduce a nondegenerate bilinear function

$$\langle\cdot,\cdot\rangle : A^q_\pm \times (\mathbf{R}^q)^{\mathbf{Z}\times\mathbf{Z}} \to \mathbf{R},$$

such that $\langle r, w\rangle = \sum r_{ij}w(i,j)$, where $r = \sum r_{ij}z_1^i z_2^j$ is a polynomial row in $A^q_\pm$ and $w \in (\mathbf{R}^q)^{\mathbf{Z}\times\mathbf{Z}}$

Then $(\mathbf{R}^q)^{\mathbf{Z}\times\mathbf{Z}}$ is isomorphic to the algebraic dual of $A^q_\pm$ and we still have

$$\mathcal{B} = \mathcal{M}^\perp_\pm$$

$$\mathcal{B}^\perp = \mathcal{M}^{\perp\perp}_\pm = \mathcal{M}_\pm.$$

Moreover $\mathcal{B}$ and $A^q_\pm/\mathcal{M}_\pm$ are finite dimensional isomorphic vector spaces.

As in the scalar case, let N_1 and N_2 be the matrices of the linear trasformations

$$Z_1 : A^q_{\pm}/\mathcal{M}_{\pm} \to A^q_{\pm}/\mathcal{M}_{\pm} : [r] \mapsto [z_1 r]$$

$$Z_2 : A^q_{\pm}/\mathcal{M}_{\pm} \to A^q_{\pm}/\mathcal{M}_{\pm} : [r] \mapsto [z_2 r]$$

w.r. to the basis $([r_1], \dots, [r_n])$ of $A^q_{\pm}/\mathcal{M}_{\pm}$. If the state vector relative to a signal $w \in \mathcal{B}$ is defined as

$$\mathbf{x}(h,k) := \begin{bmatrix} \langle Z_1^h Z_2^k [r_1] \ , \ w \rangle \\ \vdots \\ \langle Z_1^h Z_2^k [r_n] \ , \ w \rangle \end{bmatrix},$$

and C is a $q \times n$ constant matrix such that

$$\begin{bmatrix} [e_1^T] \\ \vdots \\ [e_q^T] \end{bmatrix} = C \begin{bmatrix} [r_1] \\ \vdots \\ [r_n] \end{bmatrix},$$

then we have

$$\mathbf{x}(h,k) := (N_1^T)^h (N_2^T)^k \mathbf{x}(0,0)$$

and

$$w(h,k) = C\mathbf{x}(h,k).$$

By applying the theory of Gröbner basis over the polynomial modules [12], a basis in $A^q_{+}/\mathcal{M}_{+}$ with elements of the type $z_1^h z_2^k e_i^T + \mathcal{M}_+$ is easily obtained. After computing the matrices M_1 and M_2 that represent the transformations

$$\phi_i : A^q_{+}/\mathcal{M}_{+} \to A^q_{+}/\mathcal{M}_{+} : r + \mathcal{M}_+ \mapsto z_i r + \mathcal{M}_+, \qquad i = 1,2$$

w.r. to that basis, the procedure for extracting N_1 and N_2 from M_1 and M_2 is the same introduced in the scalar case.

4 References

1. Rocha, P. and J.C. Willems (1991) : "Controllability of 2-D Systems" *IEEE Trans. on Automatic Control*, vol AC-36, pp.413-23.

2. Willems, J.C. (1989) : "Models for Dynamics", *Dynamics Reported*, Vol. 2, pp. 171-269.

3. Rocha, P. (1990) : *"Structure and Representation of 2-D Systems"*, PHD thesis, University of Groningen, The Netherlands.

4. Attasi, S. (1973) : "Systèmes Linéaires Homogènes à Deux Indices", Rapport Laboria, no. 31.

5. Roesser, R.P. (1975) : "A Discrete State Space Model for Linear Image Processing", *IEEE Trans. Aut. Cont.*, Vol. AC-20, no. 1, pp. 1-10.

6. Fornasini, E. and G. Marchesini (1976) : "State Space Realization Theory of Two-Dimensinal Filters", *IEEE Trans. Aut. Cont.*, Vol. AC-21, no. 4, pp. 484-492.

7. Lévy, B.C. (1981) : *"2-D Polynomial and Rational matices, and their Applications for the Modelling of 2-D Dynamical Systems"*, PHD thesis, Stanford University, U.S.A.

8. Kaczorek, T. (1988) : "Singular General Model of 2-D Systems and its Solutions", *IEEE Trans. Aut. Cont.*, Vol. AC-31, no. 11, pp. 1060-1061.

9. Lewis, F.L. and B.G. Mertios (1989) : "On the Analysis of Two-dimensional Discrete Singular Systems", preprint.

10. Buchberger, B. (1985) :"Gröbner Basis: An Algorithmic Method in Polynomial Ideal Theory", in N.K. Bose (ed.), *Multidimensional Systems Theory*, pp. 184-232, D. Reidel.

11. Fornasini, E. (1988) : "A Note on Output Feedback Stabilizability of Multivariable 2-D Systems", *Systems and Control Letters*, no. 10, pp. 45-50.

12. Möller, H.M. and F. Mora (1986) : "New Constructive Methods in Classical Ideal Theory", *Journal of Algebra*, no. 100, pp. 138-178.

13. Greub, W (1975) : "Linear Algebra", Springer Verlag, N.Y.