Observability and extendability of finite support nD behaviors

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Abstract In this contribution the local structure of finite support nD signals is analysed according to the behavioral approach. Observability and local detectability are introduced, and characterized in terms of polynomial matrix descriptions. The problem of extending into a legal trajectory a set of data that satisfies the parity checks on a suitable subset of $\mathbb{Z}^n$ is then considered and necessary and sufficient conditions for its solution provided. The main properties of locally undetectable behaviors and their connections with the notion of constrained variables are investigated. A general representation result for finite support behaviors is finally discussed.

1. Introduction and preliminary definitions

Multidimensional behavior theory is concerned with the representation, analysis and recognition of the trajectories which characterize the dynamics of a discrete multidimensional (nD) system.

In recent times, research efforts mostly concentrated on 2D behaviors with infinite support trajectories, which constitute a natural, yet non trivial, extension of Willems classical approach to 1D systems modelling. A general introduction to the subject was first provided by P.Rocha [6], while a detailed analysis of specific topics like controllability and autonomy was carried on in [1, 7].
A different point of view has been adopted in [2], where 2D behaviors have been interpreted as convolutional codes, and generation, detection and realization problems revisited in a coding theoretic setting.

The large number of applications involving signals with dimension n greater than 2, has called for a generalization of 2D behavior theory to higher dimensions. Somehow unexpectedly, this extension has to face with severe difficulties, connected with the algebra of nD polynomial matrices and, in particular, with some factorizations and primeness definitions that make their first appearance in this context [10].

A different topic of research is constituted by nD behaviors with finite supports sequences. Unlike 1D signals, which are usually parametrized on an infinite time set, nD signals are often functions of spatial coordinates, and consequently their supports are generally compact, at least along some directions. 2D finite support behaviors have been introduced in [9], by resorting to the algebraic duality between polynomials and formal power series. This kind of approach, however, tends to disguise the meaning of certain properties in terms of trajectories structure. A first attempt to perform a direct analysis of finite multidimensional behaviors has been presented in [3], where concepts like (external) controllability, observability and extendability have been defined in terms of elementary operations on the trajectories, and related to the algebraic properties of the (Laurent) polynomial matrices involved in their representation.

The purpose of this article is to focus on some features of finite support nD behaviors which seem particularly relevant for trajectories recognition. Two different situations will be considered, namely the ideal case when all the sequence samples are given, and the more realistic one when only a partial set of data is available. As we shall see, observability corresponds to the possibility of checking whether a sequence is a behavior trajectory, by resorting to a finite number of linear parity checks. On the other hand, if one deals with an observable behavior and the parity checks are fulfilled by a partial set of data, stronger requirements are necessary in order to conclude that the data can be completed into a legal trajectory.

Finally, we will introduce the class of unconstrained behaviors, that are endowed with properties somehow opposite with respect to observability, and show that every behavior can be represented as the intersection of an observable and an unconstrained behavior.

Before proceeding we briefly summarize some facts about nD signals representation. More details on this topic can be found in [3, 9]. An nD finite support *behavior with p components* $\mathcal{B}$ over the field $\mathbb{F}$ is a set of finite support sequences (trajectories), with values in $\mathbb{F}^p$, which is closed w.r.t. linear combinations and shifts along the coordinate axes of $\mathbb{Z}^n$.

Let $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$ denote the ring of Laurent polynomials (*L-polynomials*) in the indeterminates $z_1, ..., z_n$, ($\mathbf{z}$, for short). By representing nD signals via formal power series in $\mathbf{z}$, we set a bijective correspondence between finite support sequences of length p and elements of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$, so that every nD behavior $\mathcal{B}$ can be viewed as a $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]$-submodule of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$. As

$\mathcal{B}$ is finitely generated, there is a finite set of column vectors $\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m$ in $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ s.t.

$$\begin{aligned}\mathcal{B} &\equiv \left\{\sum_{i=1}^{m} \mathbf{g}_i u_i : u_i \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]\right\} \qquad (1)\\ &= \{\mathbf{w} = G\mathbf{u} : \mathbf{u} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^m\} =: \mathrm{Im}G.\end{aligned}$$

We call $G := \mathrm{row}\{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_m\} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{p\times m}$ a *generator matrix* of $\mathcal{B}$.

Two L-polynomial matrices G_1 and G_2, generate the same behavior if and only if $G_1 P_1 = G_2$ and $G_2 P_2 = G_1$ for suitable L-polynomial matrices P_1 and P_2. Consequently, all generator matrices of $\mathcal{B}$ have the same rank r, (called the *rank of* $\mathcal{B}$), over the field of rational functions $\mathbb{F}(\mathbf{z})$. A behavior $\mathcal{B}$ of rank r is *free* if it admits a full column rank generator matrix, i.e. a generator matrix G with r columns.

2. Controllability and observability

In the 1D case (external) controllability expresses the possibility to steer any past evolution in $(-\infty, t]$ into any other trajectory on $[t+\delta, +\infty)$, provided that $\delta > 0$ is sufficiently large. In a nD context the notions of "past" and "future" are elusive and unsuitable for classifying and processing the available data. What seems more reasonable, instead, is to investigate the independence of the values a trajectory $\mathbf{w}$ assumes on a pair of disjoint subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbb{Z}^n$, provided that their distance

$d(\mathcal{S}_1, \mathcal{S}_2) := \min\{\sum_{i=1}^n |h_i^{(1)} - h_i^{(2)}|, \mathbf{h}^{(\ell)} \in \mathcal{S}_\ell, \ell = 1, 2\}$,

is large enough.

(C) [Controllability] *A finite behavior* $\mathcal{B}$ *is* controllable *if there exists an integer* $\delta > 0$ *such that, for any pair of nonempty subsets* $\mathcal{S}_1, \mathcal{S}_2$ *of* $\mathbb{Z}^n$, *with* $d(\mathcal{S}_1, \mathcal{S}_2) \geq \delta$, *and any pair of trajectories* $\mathbf{w}_1$ *and* $\mathbf{w}_2 \in \mathcal{B}$, *there exists* $\mathbf{v} \in \mathcal{B}$ *such that the restrictions of* $\mathbf{v}$ *to the subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$, $\mathbf{v}|\mathcal{S}_1$ *and* $\mathbf{v}|\mathcal{S}_2$, *satisfy*

$$\mathbf{v}|\mathcal{S}_1 = \mathbf{w}_1|\mathcal{S}_1 \quad \text{and} \quad \mathbf{v}|\mathcal{S}_2 = \mathbf{w}_2|\mathcal{S}_2. \qquad (2)$$

While for infinite support behaviors controllability constitutes quite a strong requirement, which entails nontrivial consequences on the structure of a dynamical system, finite support behaviors are always controllable, as it immediately follows from the image representation (1) they always possess. So, in the finite support case controllability analysis is not an issue. However, basing on the same kind of elementary operations on the trajectories adopted to define controllability, different properties, endowing a behavior with distinguished features, can be highlighted.

According to some recent works of Forney et al. [5], observability is naturally introduced without reference to the concept of state and hence as an "external" notion, and it formalizes the possibility of pasting into a legal sequence any pair of trajectories that take the same values on a sufficiently large subset of $\mathbb{Z}^n$.

(O) [Observability] *A finite behavior* $\mathcal{B}$ *is* observable *if there is an integer* $\delta > 0$ *s.t., for any pair of nonempty subsets* $\mathcal{S}_1, \mathcal{S}_2$ *of* $\mathbb{Z}^n$, *with* $d(\mathcal{S}_1, \mathcal{S}_2) \geq \delta$, *and any pair of trajectories* $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathcal{B}$, *satisfying* $\mathbf{w}_1|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{w}_2|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2)$, *the trajectory*

$$\mathbf{v}(\mathbf{h}) = \begin{cases} \mathbf{w}_1(\mathbf{h}) & \mathbf{h} \in \mathcal{S}_1 \\ \mathbf{w}_1(\mathbf{h}) = \mathbf{w}_2(\mathbf{h}) & \mathbf{h} \in \mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) \\ \mathbf{w}_2(\mathbf{h}) & \mathbf{h} \in \mathcal{S}_2 \end{cases} \qquad (3)$$

is an element of $\mathcal{B}$.

When a pair of trajectories $\mathbf{w}_1$ and $\mathbf{w}_2$ of some behavior $\mathcal{B}$ satisfies condition $\mathbf{w}_1|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathbf{w}_2|\mathcal{C}(\mathcal{S}_1 \cup \mathcal{S}_2)$, and we sample the composite signal $\mathbf{v}$, given in (3) by means of a moving window, whose diameter does not exceed δ, the window content always appears as the restriction of a behavior trajectory. In general, however, if no particular assumption is introduced on $\mathcal{B}$, we cannot conclude that $\mathbf{v}$ is a behavior trajectory.

Interestingly enough, the possibility of giving a bound on the size of the windows one has to look at when deciding whether a signal belongs to $\mathcal{B}$ is equivalent to observability. So, observability expresses a sort of "localization" of the system laws, which is extremely useful when one devises a procedure for deciding whether a generic signal is a legal trajectory.

Denoting by $\mathcal{B}|\mathcal{S} := \{\mathbf{w}|\mathcal{S} : \mathbf{w} \in \mathcal{B}\}$ the set of all restrictions to $\mathcal{S}$ of behavior trajectories, the above localization property can be formalized as follows:

(LD) [Local-detectability] *A finite behavior* $\mathcal{B}$ *is* locally-detectable *if there is an integer* $\nu > 0$ *such that every signal* $\mathbf{w}$ *satisfying* $\mathbf{w}|\mathcal{S} \in \mathcal{B}|\mathcal{S}$ *for every* $\mathcal{S} \subset \mathbb{Z}^n$ *with* $\mathrm{diam}(\mathcal{S}) \leq \nu$, *is in* $\mathcal{B}$.

Observability and local detectability properties, whose equivalence is stated in the following proposition, correspond to the possibility of expressing a behavior $\mathcal{B}$ as the kernel of a polynomial matrix, and hence to describe the trajectories of $\mathcal{B}$ as the solutions of a finite set of recursive equations.

Proposition 1 [3] *Let* $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$ *be a finite support behavior. The following facts are equivalent:*
(i) $\mathcal{B}$ *is observable;*
(ii) $\mathcal{B}$ *is locally detectable;*
(iii) there exist $h \in \mathbb{N}$ *and an L-polynomial matrix* $H^T \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^{h\times p}$ *s.t.*

$$\mathcal{B} = \ker H^T := \{\mathbf{w} \in \mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p : H^T\mathbf{w} = 0\}. \quad \blacksquare$$

Further characterizations of observability can be derived when adopting a different approach, which consists in regarding behaviors with p components as elements in the lattice of submodules of $\mathbb{F}[\mathbf{z},\mathbf{z}^{-1}]^p$. As we shall see, observable behaviors enjoy very special ordering properties among the elements of the lattice.

Given $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, the *orthogonal behavior* of $\mathcal{B}$ is defined as

$$\mathcal{B}^\perp := \{\mathbf{s} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : \mathbf{s}^T \mathbf{w} = \mathbf{0}, \forall \mathbf{w} \in \mathcal{B}\}. \quad (4)$$

As a submodule of $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, it can be represented as the image of some matrix $H \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times q}$, namely $\mathcal{B}^\perp = \mathrm{Im} H$. On the other hand, condition $\mathbf{s}^T \mathbf{w} = \mathbf{0}$, $\forall\, \mathbf{s} \in \mathcal{B}^\perp$, needs not imply $\mathbf{w} \in \mathcal{B}$. So, in general

$$\mathcal{B}^{\perp\perp} := \{\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : \mathbf{s}^T \mathbf{w} = \mathbf{0}, \forall\, \mathbf{s} \in \mathcal{B}^\perp\} = \ker H^T$$

represents a proper extension of $\mathcal{B}$, of the same rank r, and it is easy to realize that $\mathcal{B}^{\perp\perp}$ is the smallest observable behavior including $\mathcal{B}$.

For an arbitrary behavior $\mathcal{B}$, $\mathcal{B}^{\perp\perp}$ represents a proper extension of the same rank. Keeping in with the same spirit, one may investigate how a behavior is affected by other "extension operations" that merge lattice elements into larger ones of the same rank. There are essentially two natural ways to perform these extensions: one consists in embedding $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, and therefore each of its submodules, in the rational vector space $\mathbb{F}(\mathbf{z})^p$, the other in considering $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ as a submodule of $\mathcal{F}_\infty^p$, the set of nD trajectories with p components, whose supports possibly extend to the whole space $\mathbb{Z}^n$. Once a behavior $\mathcal{B}$ with p components is given, in the first case we have to consider the smallest vector subspace of $\mathbb{F}(\mathbf{z})^p$ including $\mathcal{B}$

$$\mathcal{B}_{\mathrm{rat}} := \Big\{ \sum_{i=1}^{r} \mathbf{w}_i a_i : \mathbf{w}_i \in \mathcal{B},\ a_i \in \mathbb{F}(\mathbf{z}),\ r \in \mathbb{N} \Big\}, \quad (5)$$

and confine our attention to the submodule $\mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ of finite support sequences. This in general properly includes $\mathcal{B}$, and hence is a larger element of the lattice. In the second case, we merge $\mathcal{B}$ in

$$\mathcal{B}_\infty := \Big\{ \sum_{i=1}^{r} \mathbf{w}_i a_i : \mathbf{w}_i \in \mathcal{B},\ a_i \in \mathcal{F}_\infty,\ r \in \mathbb{N} \Big\}, \quad (6)$$

the smallest $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$-submodule of $\mathcal{F}_\infty^p$ which includes $\mathcal{B}$. Again, one has to consider only the set of its finite elements $\mathcal{B}_\infty \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, which clearly includes all trajectories of $\mathcal{B}$. The following proposition provides a fairly complete picture of the lattice conditions observability relies on.

Proposition 2 *Let $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ be a behavior of rank r. The following statements are equivalent:*

(i) $\mathcal{B}$ is observable;

(ii) $\mathcal{B} \equiv \mathcal{B}_\infty \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$;

(iii) $\mathcal{B} \equiv \mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$;

(iv) $\mathcal{B}$ is maximal in the set of all submodules of $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ of rank r;

(v) $s\mathbf{w} \in \mathcal{B} \;\Rightarrow\; \mathbf{w} \in \mathcal{B}$, for every $\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ and every nonzero $s \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$;

(vi) $\mathcal{B} = \mathcal{B}^{\perp\perp}$.

PROOF (i) $\Rightarrow$ (ii) As $\mathcal{B}$ is observable, there exists $H \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times q}$ such that $\mathcal{B} = \ker H^T$. If $\mathbf{w} \in \mathcal{B}_\infty \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, then $\mathbf{w} = \sum_i \mathbf{w}_i a_i$, $a_i \in \mathcal{F}_\infty, \mathbf{w}_i \in \mathcal{B}$, and therefore $H^T \mathbf{w} = H^T \left(\sum_i \mathbf{w}_i a_i \right) = \sum_i (H^T \mathbf{w}_i) a_i = \mathbf{0}$. Thus $\mathbf{w} \in \ker H^T = \mathcal{B}$, which implies $\mathcal{B} \supseteq \mathcal{B}_\infty \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$. The reverse inclusion is obvious.

(ii) $\Rightarrow$ (iii) Follows immediately from $\mathcal{B} \subseteq \mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p \subseteq \mathcal{B}_\infty \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$.

(iii) $\Rightarrow$ (iv) If $\mathcal{B}' \supseteq \mathcal{B}$ and $\mathrm{rank}\mathcal{B}' = \mathrm{rank}\mathcal{B}$, it is clear that $\mathcal{B}$ and $\mathcal{B}'$ generate the same $\mathbb{F}(\mathbf{z})$-subspace of $\mathbb{F}(\mathbf{z})^p$ and, consequently, $\mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p = \mathcal{B}'_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$. So, the inclusions chain $\mathcal{B}_{\mathrm{rat}} \cap \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p \supseteq \mathcal{B}' \supseteq \mathcal{B}$ and assumption (3) together imply $\mathcal{B}' = \mathcal{B}$, which means that $\mathcal{B}$ is maximal.

(iv) $\Rightarrow$ (v) Suppose $s\mathbf{w} \in \mathcal{B}$, $s \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$. The behavior $\mathcal{B}'$ generated by $\mathcal{B}$ and $\mathbf{w}$ has the same rank of $\mathcal{B}$, and hence, by the maximality assumption, coincides with $\mathcal{B}$.

(v) $\Rightarrow$ (vi) As $\mathcal{B}$ and $\mathcal{B}^{\perp\perp}$ have the same rank r and $\mathcal{B}^{\perp\perp} \supseteq \mathcal{B}$, both behaviors generate the same $\mathbb{F}(\mathbf{z})$-subspace of $\mathbb{F}(\mathbf{z})^p$. In particular, $\mathbf{w} \in \mathcal{B}^{\perp\perp}$ implies $\mathbf{w} \in (\mathcal{B}^{\perp\perp})_{\mathrm{rat}} = \mathcal{B}_{\mathrm{rat}}$. So, there exist $p_i, s_i \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$ and $\mathbf{w}_i \in \mathcal{B}$, s.t. $\mathbf{w} = \sum_{i=1}^{r} \mathbf{w}_i\, p_i/s_i$, which implies $s\mathbf{w} \in \mathcal{B}$, $s = \ell.\mathrm{c.m.}\{s_i\}$. By (5), $\mathbf{w} \in \mathcal{B}$, too.

(vi) $\Rightarrow$ (i) Since $\mathcal{B}^\perp$ is a submodule of $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, there exists a suitable L-polynomial matrix H such that $\mathcal{B}^\perp = \mathrm{Im} H$. So $\mathcal{B}^{\perp\perp} = (\mathcal{B}^\perp)^\perp = \{\mathbf{w} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p : \mathbf{v}^T \mathbf{w} = \mathbf{0}, \forall \mathbf{v} \in \mathrm{Im} H\} = \ker H^T$. By (vi), $\mathcal{B}$ coincides with $\ker H^T$, and hence is observable. ■

3. Trajectories recognition and extendability

The problem of recognizing whether a given sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ is an element of $\mathcal{B}$ arises both in fault detection and coding contexts. It can be managed by resorting to a linear filter (residual generator or syndrome former) that produces an identically zero output provided that the input is an admissible trajectory of $\mathcal{B}$. From an abstract point of view, the filter design reduces to find a "complete" set $\mathcal{P}$ of finite support sequences (*parity checks*) endowed with the property that their convolution with the elements of $\mathcal{B}$ is zero. More precisely, we require that $\mathbf{v}$ belongs to $\mathcal{B}$ if and only if $\mathbf{p}^T \mathbf{v} = 0$ for every $\mathbf{p} \in \mathcal{P}$, or equivalently the coefficient of $\mathbf{z}^{\mathbf{i}}$ in $\mathbf{p}^T$, $(\mathbf{p}^T \mathbf{v}, \mathbf{z}^{\mathbf{i}})$, is zero for every $\mathbf{p} \in \mathcal{P}$ and $\mathbf{i} \in \mathbb{Z}^n$.

As clarified by Proposition 1, the case when all the trajectories of a behavior $\mathcal{B}$ can be recognized by resorting to a suitable family of parity checks occurs if and only if $\mathcal{B}$ is observable. When no a priori information on the support of a trajectory is given, however, a positive outcome of the parity checks, performed on some window $\mathcal{S}$, does not guarantee that a behavior sequence can be found, interpolating the data available on $\mathcal{S}$. So, in general, the checking procedure should be extended to the whole space $\mathbb{Z}^n$. A noteworthy exception is represented by the case when $\mathcal{S}$ is surrounded by a sufficiently large boundary region where the signal is zero. If so, extending the data out of $\mathcal{S}$ via the identically zero sequence leads to a signal which satisfies the parity checks all over $\mathbb{Z}^n$. Clearly, it would be highly desirable if the extension into a legal trajectory could be accomplished without any particular assumption on the data values in the boundary region. A thorough discussion of this problem is based on the definition of what we precisely mean by " satisfying the parity checks" on a set $\mathcal{S} \subset \mathbb{Z}^n$.

Definition 1 *Let $\mathcal{B} = \ker H^T$ be an observable behavior. A sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ satisfies the parity checks of $\mathcal{B}$ in $\mathbf{h} \in \mathbb{Z}^n$ if*

$$\left(H^T\mathbf{v}, \mathbf{z}^{\mathbf{i}}\right) = 0, \quad \forall\, \mathbf{i} \in \mathbf{h} + \operatorname{supp}(H^T), \qquad (7)$$

where $\mathbf{h} + \operatorname{supp}(H^T) := \{\mathbf{h} + \mathbf{j} : \mathbf{j} \in \operatorname{supp}(H^T)\}$. In general, if $\mathcal{S}$ is any subset of $\mathbb{Z}^n$, $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}$ on $\mathcal{S}$ if satisfies them in every point of $\mathcal{S}$.

Letting $H^T := \sum_{\mathbf{j}} H_{\mathbf{j}}^T \mathbf{z}^{\mathbf{j}}$, the above condition reduces to the following system of linear equations

$$\sum_{\mathbf{j} \in \operatorname{supp}(H^T)} H_{\mathbf{j}}^T \mathbf{v}(\mathbf{i} - \mathbf{j}) = 0, \ \forall\, \mathbf{i} \in \mathcal{S} + \operatorname{supp}(H^T), \quad (8)$$

and hence to the system of all difference equations which involve the sample $\mathbf{v}(\mathbf{h})$.

Once the parity checks have been successfully performed on a sequence $\mathbf{v}$ in a subset $\mathcal{S}$, the natural question arises whether the data on $\mathcal{S}$ can be extended into a legal trajectory, namely, if there exists a behavior signal that fits on $\mathcal{S}$ the available data. In general, observability is not enough to guarantee this possibility, which depends on stronger assumptions on the behavior structure. Even then, however, it may be necessary to discard some samples in the border of $\mathcal{S}$.

(E) [Extendability] *An observable behavior $\mathcal{B} = \ker H^T$ is* extendable *if there is an integer $\varepsilon > 0$ such that, for every subset $\mathcal{S} \subset \mathbb{Z}^n$ and every $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, which satisfies on*

$$\mathcal{S}^\varepsilon := \{(h_1, h_2, ..., h_n) \in \mathbb{Z}^n : d((h_1, h_2, ..., h_n), \mathcal{S}) \leq \varepsilon\},$$

the parity checks of $\mathcal{B}$, a trajectory $\mathbf{w} \in \mathcal{B}$ can be found s.t. $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$.

As clarified in Proposition 3, extendable behaviors are described by left zero-prime (ℓZP) parity check matrices, i.e. matrices with an L-polynomial left inverse.

Proposition 3 [3] *A finite behavior $\mathcal{B}$ is extendable if and only if $\mathcal{B} = \ker H^T$, for some ℓZP matrix H^T.*

In the definition of extendability no constraints are assumed on the shape and cardinality of the set $\mathcal{S}$ where the parity checks are performed. As a counterpart of adopting this general setting, only generator matrices with strong structural requirements possess this feature. If we agree to extend into behavior sequences only data sets which fulfill the parity checks on particular subsets of $\mathbb{Z}^n$, we can partly relax the requirements on the generator matrices. The subsets of $\mathbb{Z}^n$ we will refer to are (infinite) cylinders with either one-dimensional or $n-1$-dimensional bases, enveloping a given finite set $\mathcal{S}$. More precisely, 1-cylinders enveloping $\mathcal{S}$ are defined as

$$C_i(\mathcal{S}) := \{\mathbf{h} \in \mathbb{Z}^n : h_i = k_i,\ \exists \mathbf{k} \in \mathcal{S}\},\ i = 1, 2, \ldots, n,$$

while $n-1$-cilinders are

$$C_{i^c}(\mathcal{S}) := \{\mathbf{h} \in \mathbb{Z}^n : \mathbf{h}_{i^c} = \mathbf{k}_{i^c},\ \exists \mathbf{k} \in \mathcal{S}\}, i = 1, 2, \ldots, n,$$

where $\mathbf{h}_i^c$ denotes the subset of $\{h_1, h_2, \ldots, h_n\}$ complementary to $\{h_i\}$.

($\mathbf{E}_1$-$\mathbf{E}_{n-1}$) [1- and n-1-Extendability] *An observable behavior $\mathcal{B} = \ker H^T$ is* 1-extendable *($n-1$-*extendable*) if there is an integer $\varepsilon > 0$ s.t., for every finite subset $\mathcal{S} \subset \mathbb{Z}^n$ and every $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, if $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}$ on the 1-cylinder $C_i(\mathcal{S}^\varepsilon)$, (on the $n-1$-cylinder $C_{i^c}(\mathcal{S}^\varepsilon)$), for some $i \in \{1, 2, \ldots, n\}$, some $\mathbf{w} \in \mathcal{B}$ can be found s.t. $\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S}$.*

The following notions of minor and variety prime matrices, based on the solvability of certain Bézout equations in polynomial rings that properly include $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]$, are equivalent to more familiar definitions [8], based on the coprimeness of the maximal order minors of G and on the cardinality of the corresponding algebraic varieties, respectively.

Definition 2 *Let G be a full column rank matrix, and consider the Bézout equation $XG = I_m$.*

i) G is right minor prime (rMP), if the equation is solvable in the rings $\mathbb{F}(\mathbf{z}_i^c)[z_i, z_i^{-1}]$, $i = 1, 2, ..., n$;

ii) G is right variety prime (rVP) if the equation is solvable in $\mathbb{F}(z_i)[\mathbf{z}_i^c, (\mathbf{z}_i^c)^{-1}]$, $i = 1, 2, ..., n$.

Proposition 4 *Let $\mathcal{B} = \operatorname{Im} G$ be a free behavior, $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times r}$ a full column rank matrix.*

i) $\mathcal{B}$ is 1-extendable $\Leftrightarrow$ $\mathcal{B}$ is observable $\Leftrightarrow$ G is rMP;

ii) $\mathcal{B}$ is $n-1$-extendable $\Leftrightarrow$ G is rVP.

PROOF *i)* 1-extendability implies observability, by definition. If $\mathcal{B}$ is observable, it is maximal among the submodules of $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ of rank r. Since maximal modules of a given rank that are free admit rMP generator matrices [4], $\mathcal{B} = \operatorname{Im}\bar{G}$, $\bar{G}$ rMP. So, $G = \bar{G}\bar{P}$

and $\bar{G} = GP$, $P, \bar{P} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{r \times r}$, implies that G is rMP, too.

If G is rMP, $\mathcal{B}$ is a maximal module of rank r and hence is observable. As G is rZP in $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]$, $i = 1, 2, \ldots, n$, by Proposition 3, the behaviors $\mathcal{B}_i := \mathrm{Im}_{\mathbb{F}(z_i^c)[z_i, z_i^{-1}]} G$, $i = 1, 2, \ldots, n$, satisfy definition (E) for suitable $\varepsilon_i > 0$ and hence are extendable (and observable) in a 1D context. Let ε be the maximum of the ε_i. If we represent $\mathcal{B}$ as the kernel (in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$) of some L-polynomial matrix H^T, it can be shown [4] that $\mathcal{B}_i \equiv \ker_{\mathbb{F}(z_i^c)[z_i, z_i^{-1}]} H^T = \{\mathbf{w} \in \mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p : H^T \mathbf{w} = 0\}$. Consider, now, a finite set $\mathcal{S} \subset \mathbb{Z}^n$ and some $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ which satisfies the parity checks of $\mathcal{B}$ in $C_i(\mathcal{S}^\varepsilon)$, for some $i \in \{1, 2, \ldots, n\}$. As an element of $\mathbb{F}(z_i^c)[z_i, z_i^{-1}]^p$, $\mathbf{v}$ satisfies the parity checks of $\mathcal{B}_i$ on the one-dimensional projection I_i^ε of $\mathcal{S}^\varepsilon$ into the i-th coordinate axis, and hence there is $\tilde{\mathbf{w}} = G\tilde{\mathbf{u}}$, $\tilde{\mathbf{u}} \in \mathbb{F}(z_i^c)[z_i, z_i^{-1}]^m$, in $\mathcal{B}_i$ s.t. $\tilde{\mathbf{w}}|I_i = \mathbf{v}|I_i$. So, as n-dimensional sequences, $\mathbf{v}$ and $\tilde{\mathbf{w}}$ coincide on $C_i(\mathcal{S})$. If r is the radius of a ball $B(0, r)$ centered in the origin which includes $\mathrm{supp}(G)$, clearly the values of $\tilde{\mathbf{w}}$ in $\mathcal{S}$ only depends on the values of $\tilde{\mathbf{u}}$ in $\mathcal{S}^r$. Thus, the finite sequence $\mathbf{u}$, which coincides with $\tilde{\mathbf{u}}$ on $\mathcal{S}^r$ and is zero elsewhere, produces a behavior sequence $\mathbf{w} = G\mathbf{u}$ which coincides with $\mathbf{v}$ on $\mathcal{S}$.

ii) Following an analogous reasoning one shows that G rVP implies $\mathcal{B}$ $n-1$-extendable. For sake of brevity, the proof of the converse is omitted. Interested readers are referred to [4]. ■

4. Behavior decomposition

The scope of this section is to take a first step towards a structural analysis of finite support behaviors. Behavior structure theory aims to describe general behaviors in terms of some simpler ones, simpler in some perceptible way, perhaps in terms of concreteness, perhaps in terms of tractability. Of essential importance, after one has decided upon these simpler objects, is to find a method of passing down to them and to discover how they weave together to yield the general behavior with which we began. Observable behaviors constitute good candidates for these simpler objects, as each behavior can be embedded into an observable one. In order to represent a general behavior $\mathcal{B}$, then, we have to slice out of its enveloping observable behavior $\mathcal{B}^{\perp\perp}$ a certain part. This can be done by intersecting $\mathcal{B}^{\perp\perp}$ with a suitable, nonnecessarily unique, element of a behavior class that exhibits properties which are as far as possible from local detectability.

Definition 3 *Let $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ be a finite support behavior and $\{i_1, i_2, \ldots, i_r\}$, $r < p$, a subset of $\{1, 2, \ldots, p\}$. We call $w_{i_1}, w_{i_2}, \ldots, w_{i_r}$* constrained variables *of $\mathcal{B}$ if for every pair of trajectories $\mathbf{v}, \mathbf{v}' \in \mathcal{B}$, $v_j = v'_j$ for every $j \notin \{i_1, i_2, \ldots, i_r\}$ implies $\mathbf{v} = \mathbf{v}'$.*

As shown in the following lemma, the maximum number of constrained variables of a behavior $\mathcal{B}$ in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ can be expressed in terms of the rank and the number of components of $\mathcal{B}$.

Lemma 5 *Let $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ be a behavior of rank r. The maximum number of constrained variables of $\mathcal{B}$ coincides with $p - r$.*

PROOF Let $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ be a generator matrix of $\mathcal{B}$ and suppose, for sake of simplicity, that the first r rows of G are linearly independent, so that in

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{matrix} \}r \\ \}p-r \end{matrix},$$

G_1 has full row rank. The components w_i, $i = r+1, r+2, \ldots, n$, are constrained variables. If not, there would be a trajectory $\mathbf{w} = \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2 \end{bmatrix}$ in $\mathcal{B}$, with $\mathbf{w}_2 \neq 0$, and hence an L-polynomial vector $\mathbf{u} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m$ s.t. $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \mathbf{u} = \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_2 \end{bmatrix}$. This is a contradiction, however, since $\mathrm{rank}G_1 = \mathrm{rank}G$ implies $(\mathrm{Im}G_1^T)^\perp = (\mathrm{Im}G^T)^\perp$.

It remains to prove that the number of constrained variables cannot exceed $p - r$. Suppose, instead, that $k > p - r$ variables of $\mathcal{B}$, say the last k, are constrained, and partition the generator matrix G into

$$G = \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix} \begin{matrix} \}p-k \\ \}k \end{matrix}.$$

As $r = \mathrm{rank}\ G > \mathrm{rank}\hat{G}_1$, $\ker\hat{G}_1$ properly includes $\ker G$. Consequently, there exists $\mathbf{u}$ s.t. $\hat{G}_2\mathbf{u} \neq \mathbf{0}$ and both $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{0} \\ \hat{G}_2\mathbf{u} \end{bmatrix}$, $\hat{G}_2\mathbf{u} \neq \mathbf{0}$, are in $\mathcal{B}$, which contradicts the assumption that the last k components are constrained. ■

A behavior $\mathcal{B}$ devoid of constrained variables exhibits the very peculiar feature that for every finite set $\mathcal{S} \subset \mathbb{Z}^n$, $\mathcal{B}|\mathcal{S}$ coincides with $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p \mid \mathcal{S}$. This property, which appears somehow opposite to local detectability, makes it is impossible to recognize the trajectories of $\mathcal{B}$ by resorting to a local checking procedure.

(LU) [Local-undetectability] *A behavior $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ is* locally undetectable *if there exists $\delta > 0$ s.t. for every sequence $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ and every set $\mathcal{S} \subset \mathbb{Z}^n$, a trajectory $\mathbf{w} \in \mathcal{B}$ can be found, satisfying*

$$\mathbf{w}|\mathcal{S} = \mathbf{v}|\mathcal{S} \quad \text{and} \quad \mathrm{supp}(\mathbf{w}) \subseteq \mathcal{S}^\delta. \qquad (9)$$

Proposition 6 *Let $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ be a finite support behavior. The following facts are equivalent:*

i) $\mathcal{B}$ is devoid of constrained variables;

ii) $\mathcal{B}$ is the image of some L-polynomial matrix $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ with rank p;

iii) $\mathcal{B}$ is locally undetectable.

PROOF i) $\Leftrightarrow$ ii) Immediate from Lemma 5.

ii) $\Leftrightarrow$ iii) Assume $\mathcal{B} = \text{Im}G$, for some $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ whose rank is less than p, and let $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$, $\text{supp}(\mathbf{v}) \subset B(0, \rho)$, be an L-polynomial vector satisfying $\mathbf{v}^T G = 0$. Consider, then, a set $\mathcal{T}$ and a sequence $\mathbf{u} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ s.t. $\text{supp}(\mathbf{u}) \subseteq \mathcal{T}$ and $\text{supp}(\mathbf{v}^T\mathbf{u}) \cap \mathcal{T} \neq \emptyset$, and let $\mathcal{S} := \mathcal{T}^\rho$. If property (LU) holds for some $\delta > 0$, there exists $\mathbf{w} = \mathbf{u} + \mathbf{r}$ in $\mathcal{B}$ with $\text{supp}(\mathbf{r}) \subseteq \mathcal{S}^\delta \setminus \mathcal{S}$. As $\mathbf{w} = G\mathbf{a}$, for some $\mathbf{a} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^m$, it follows that $0 = \mathbf{v}^T\mathbf{w} = \mathbf{v}^T\mathbf{u} + \mathbf{v}^T\mathbf{r}$. This is not possible, however, since $\mathcal{T}$ intersects the support of $\mathbf{v}^T\mathbf{u}$ without intersecting $\text{supp}(\mathbf{v}^T\mathbf{r})$.

On the other hand, if $\mathcal{B} = \text{Im}G$, for some $G \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^{p \times m}$ of rank p, every $\mathbf{v} \in \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ can be obtained as the image of some vector $\mathbf{u} \in \mathbb{F}(\mathbf{z})^p$, i.e. $\mathbf{v} = G\mathbf{u}$. Consider an arbitrary finite set $\mathcal{S}$ and a power series expansion of $\mathbf{u}$ with support in a suitable cone of $\mathbb{Z}^n$, and let $\bar{\mathbf{u}} := \mathbf{u}|\mathcal{S}^\varepsilon$, where $\varepsilon := \text{diam}(\text{supp}G)$. Then $\bar{\mathbf{v}} := G\bar{\mathbf{u}}$ is a behavior sequence which coincides with $\mathbf{v}$ on $\mathcal{S}$ and whose support is included in $\mathcal{S}^{2\varepsilon}$. So, (9) holds with $\delta = 2\varepsilon$. ∎

Proposition 7 *For every behavior $\mathcal{B} \subseteq \mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ there exist an observable behavior $\mathcal{B}_0$ and a locally undetectable behavior $\mathcal{B}_{\text{lu}}$ in $\mathbb{F}[\mathbf{z}, \mathbf{z}^{-1}]^p$ s.t.*

$$\mathcal{B} = \mathcal{B}_0 \cap \mathcal{B}_{\text{lu}}. \tag{10}$$

Moreover, $\mathcal{B}_0$ is uniquely determined as $\mathcal{B}^{\perp\perp}$, the smallest observable behavior including $\mathcal{B}$.

PROOF Let $\mathcal{B} = \text{Im}G$ and $\mathcal{B}_0 := \mathcal{B}^{\perp\perp} = \ker H^T$. Clearly, $\mathcal{B}_0$ is an observable behavior including $\mathcal{B}$. If G has rank r, we can assume, for sake of simplicity, that its first r rows are linearly independent. So, G can be partitioned as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{matrix} \}r \\ \}p-r \end{matrix},$$

where G_1 is a full rank matrix. Let $G_{\text{lu}} := \text{block} - \text{diag}\{G_1, I_{p-r}\}$, and $\mathcal{B}_{\text{lu}} := \text{Im}G_{\text{lu}}$. Clearly, $\mathcal{B}_{\text{lu}}$ is a locally undetectable behavior, and it includes $\mathcal{B}$ as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & I_{p-r} \end{bmatrix} \begin{bmatrix} I \\ G_2 \end{bmatrix}.$$

So, one obviously gets $\mathcal{B} \subset \mathcal{B}_0 \cap \mathcal{B}_{\text{lu}}$.
To prove the reverse inclusion, consider $\mathbf{w} \in \mathcal{B}_0 \cap \mathcal{B}_{\text{lu}}$. Clearly, $\mathbf{w}$ satisfies $H^T\mathbf{w} = 0$ and can be expressed as $\mathbf{w} = \begin{bmatrix} G_1\mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$. Factorizing G into the product of a (full column rank) right factor prime matrix $\bar{G}$ and a full row rank rational matrix Q [8], one gets

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = G = \bar{G}Q = \begin{bmatrix} \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} Q.$$

As the columns of $\bar{G}$ generates the $\mathbb{F}(\mathbf{z})$-vector space orthogonal to the rows of H^T, there exists $\mathbf{v} \in \mathbb{F}(\mathbf{z})^r$ s.t. $\mathbf{w} = \bar{G}\mathbf{v}$. But then $\bar{G}_1\mathbf{v} = G_1\mathbf{u}_1 = \bar{G}_1 Q\mathbf{u}_1$ implies $\mathbf{v} = Q\mathbf{u}_1$, and thus $\mathbf{u}_2 = \bar{G}_2\mathbf{v} = \bar{G}_2 Q\mathbf{u}_1$ and

$$\mathbf{w} = \begin{bmatrix} G_1\mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} Q\mathbf{u}_1 G\mathbf{u}_1.$$

This implies that $\mathbf{w}$ is in $\mathcal{B}$.

It remains to prove the uniqueness of $\mathcal{B}_0$ in the above representation. Suppose, by contradiction, that $\mathcal{B} = \hat{\mathcal{B}}_0 \cap \hat{\mathcal{B}}_{\text{lu}}$, for some observable behavior $\hat{\mathcal{B}}_0 \neq \mathcal{B}^{\perp\perp}$ and some locally undetectable behavior $\hat{\mathcal{B}}_{\text{lu}}$. As $\mathcal{B}^{\perp\perp}$ is the smallest observable behavior including $\mathcal{B}$ and is maximal in the class of modules of rank r, $\hat{\mathcal{B}}_0$ must have rank greater than r. Consequently, $(\hat{\mathcal{B}}_0)_{\text{rat}} \supset (\mathcal{B}^{\perp\perp})_{\text{rat}}$. On the other hand $(\mathcal{B}_{\text{lu}})_{\text{rat}} = (\hat{\mathcal{B}}_{\text{lu}})_{\text{rat}} = \mathbb{F}(\mathbf{z})^p$, and therefore $(\mathcal{B}^{\perp\perp})_{\text{rat}} \cap (\mathcal{B}_{\text{lu}})_{\text{rat}} = (\mathcal{B}^{\perp\perp})_{\text{rat}} \subset (\hat{\mathcal{B}}_0)_{\text{rat}} = (\hat{\mathcal{B}}_0)_{\text{rat}} \cap (\hat{\mathcal{B}}_{\text{lu}})_{\text{rat}}$. But this is not possible, as $\mathcal{B}^{\perp\perp} \cap \mathcal{B}_{\text{lu}} = \mathcal{B} = \hat{\mathcal{B}}_0 \cap \hat{\mathcal{B}}_{\text{lu}}$ should imply [4] $(\mathcal{B}^{\perp\perp})_{\text{rat}} \cap (\mathcal{B}_{\text{lu}})_{\text{rat}} = (\hat{\mathcal{B}}_0)_{\text{rat}} \cap (\hat{\mathcal{B}}_{\text{lu}})_{\text{rat}}$. ∎

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