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ON THE INTERNAL STRUCTURE OF BILINEAR INPUT-OUTPUT MAPS

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INTRODUCTION

This paper deals with some observations arising in the realization of bilinear input-output maps. It concerns essentially with some internal structural properties which are consequences of the definition of the state by the most natural way i.e. Nerode equivalence classes.

Most questions are yet to be studied in depth but the results are sufficient to give a picture of the problems.

Let K be a field and $\mathbb{Z}$ the ring of integers. A bilinear zero-state discrete-time input-output map is a map $f : U \times U \longrightarrow Y$ defined as follows :

i) $U = \{u : u \in K^{\mathbb{Z}}, \text{ card } (\text{supp } u) \lneq \aleph_0\}$; is naturally endowed with the structure of K-module.

ii) $Y = \{y : y \in K^{\mathbb{Z}}\}$

iii) $f : U \times U \longrightarrow Y$ (input-output map) has the following properties :

i) $\min \text{ supp } f(u_1,u_2) > \max \text{ supp}(u_1,u_2) \triangleq \text{supp } u_1 \cup \text{supp } u_2$

ii) f is bilinear :

$$f(ku_1,u_2) = kf(u_1,u_2) \; , \; k \in K \; , \; u_1,u_2 \in U$$

$$f(u_1,hu_2) = hf(u_1,u_2) \; , \; h \in K \; , \; u_1,u_2 \in U$$

$$f(u_1+v_1,u_2) = \left[f(u_1,u_2)+ f(v_1,u_2)\right]_{\text{supp } f(u_1+v_1,u_2)}$$

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$$f(u_1,u_2+v_2) = \left[f(u_1,u_2) + f(u_1,v_2)\right]_{\text{supp } f(u_1,u_2+v_2)}$$

$u_1,u_2,v_1,v_2 \in U$

The zero-state response of a bilinear map to an arbitrary pair of input sequences (u_1,u_2) with left compact support, is given by:

(1) $$y(r) = f(T_r u_1, T_r u_2)(r) \quad , \quad r \in \mathbb{Z}$$

where:

(2) $$T_r u_i(t) = \begin{cases} u_i(t) & , \ t < r \\ 0 & , \ t \geqslant r \end{cases} \quad , \quad i = 1,2$$

Denoting by σ the shift operator on $K^{\mathbb{Z}}$, the bilinear input-output map is stationary if it satisfies the condition :

(3) $$\sigma f(u_1,u_2) = f(\sigma u_1, \sigma u_2) \quad , \quad u_1,u_2 \in U$$

In this case $t = 0$ can be assumed as the max supp (u_1,u_2).

CHARACTERIZATION OF THE BILINEAR MAP

The input-output map defined in the previous section can be represented via a sequence of infinite matrices.

In particular the representation of the K-bilinear stationary input-output maps onto the infinite K-valued matrices is then biunique.

Actually it is immediate to observe that

$$f(u_1,u_2)(r) \quad , \quad r > \max \text{supp } (u_1,u_2)$$

can be considered as a bilinear functional

$$f(u_1,u_2)(r) \ : \ T_r[U] \times T_r[U] \longrightarrow K \ .$$

Proposition 1 : Let $\{\beta_i\}_{\ i = r-1,r-2,\dots}$ the usual basis in $T_r[U]$, where β_i is the sequence:

(4) $$\beta_i = (\delta_{ik})_{k = r-1,r-2,\dots}$$

and

(5) $$f(\beta_i, \beta_j)(r) \triangleq w_{ij}(r) \quad ,$$

then

$$(6)\quad f(u_1,u_2)(r)=\sum_{ij} w_{ij}(r)\, u_1(i)\, u_2(j) \quad , \quad u_1,u_2 \in T_r[U] \ .$$

The sequence w_{ij} comes out naturally depending on r since stationarity has not been assumed.

Obviously if we assume the bilinear i–o map to be stationary the w_{ij} dependence on r fails and the following Proposition holds.

Proposition 2 Let f be stationary. Then

$$(7)\qquad w_{i,j}(r) = w_{i-r,j-r}(0) \qquad , \qquad \forall\, i,j,r \in \mathbb{Z}$$

Proof. Direct application of invariancy definition to (6).

The stationarity assumption allows us to consider input sequences (u_1,u_2) having $t = 0$ as max $\mathrm{supp}(u_1,u_2)$. This has as consequence the possibility of evaluating the output of the bilinear map from the infinite matrix $W = (w_{ij})_{i,j<0}$. In this way the K-bilinear stationary i–o maps can be biuniquely represented onto the infinite K-valued matrices.

The input space being constituted by pairs of sequences with compact support in $\mathbb{Z}_-$, we shall adopt the usual polynomial representation :

$$(8)\quad u_1=(a^1_{-p},\, a^1_{-p+1},\ldots,\, a_0) \longmapsto p_1(z_1^{-1})=\sum_0^p {}_i\, a_i z_1^{-i},\ a_h= a^1_{-h}$$

$$(9)\quad u_2=(b^1_{-q},\, b^1_{-q+1},\ldots,b_0) \longmapsto p_2(z_2^{-1})=\sum_0^q {}_j\, b_j z_2^{-j} \ , \quad b_k= b^1_{-k}$$

Analogously the output space, constituted by sequences with support $\mathbb{Z}_{++}$ can be represented in $K[[z]]$.

It is now possible to give a global representation for the input-output map in terms of series and polynomials.

Proposition 3 Let (8), (9) and

$$(10)\qquad S(z_1,\, z_2) = \sum_1^{\infty} {}_{h,k}\, w_{-h,-k}(0)\, z_1^h\, z_2^k \ =$$

$$= \sum_{1}^{\infty} {}_{h,k}\, s_{h,k}\, z_1^h z_2^k \quad , \quad w_{-h,-k}(0) \triangleq s_{h,k}$$

Then :

$$(11) \quad f(p_1(z_1^{-1}),\ p_2(z_2^{-1})) = \underset{>0}{\text{diag}}\ S(z_1,z_2) p_1(z_1^{-1})\ p_2(z_2^{-1})$$

Proof.

$$(12) \qquad y(m) = \sum_{0}^{-\infty} {}_{i,j}\, w_{ij}(m)\ a_i^1\ b_j^1 =$$

$$= \sum_{0}^{-\infty} {}_{i,j}\, w_{i-m,\ j-m}(0)\ a_i^1\ b_j^1 = \sum_{0}^{+\infty} {}_{h,k}\, s_{h+m,k+m}\, a_h b_k$$

But

$$(13) \quad \underset{>0}{\text{diag}}\quad S(z_1,z_2)\ p_1(z_1^{-1})\ p_2(z_2^{-1}) =$$

$$= \sum_{1}^{\infty} {}_m \Big(\sum_{0}^{\infty} {}_{ij}\, s_{i+m,\,j+m}\ a_i\ b_j\Big)(z_1\ z_2)^m$$

INTERNAL PROPERTIES

The most natural way to attack the realization problem is by introducing the Nerode equivalence [2,3]:

$$(14) \qquad (p_1,p_2) \underset{N}{\sim} (\hat{p}_1,\hat{p}_2) \iff f(p_1 \circ \sigma_1, p_2 \circ \sigma_2) =$$

$$= f(\hat{p}_1 \circ \sigma_1, \hat{p}_2 \circ \sigma_2) \ , \quad \forall \sigma_1 \in K[z_1^{-1}],\ \forall \sigma_2 \in K[z_2^{-1}]$$

where

$$(15) \qquad p_1(z_1^{-1}) \circ \sigma_1(z_1^{-1}) = p_1(z_1^{-1})\ z_1^{-(c+1)} + \sigma_1(z_1^{-1})$$

$$p_2(z_2^{-1}) \circ \sigma_2(z_2^{-1}) = p_2(z_2^{-1})\ z_2^{-(c+1)} + \sigma_2(z_2^{-1})$$

$$c = \max(\deg \sigma_1, \sigma_2)$$

and similar expressions for $\hat{p}_1 \circ \sigma_1$ and $\hat{p}_2 \circ \sigma_2$

It is known that [4] :

(16) $$(p_1,p_2) \underset{N}{\sim} (\hat{p}_1,\hat{p}_2) \Longleftrightarrow (1),2),3))$$

1) $f(p_1\ z_1^{-c}, p_2\ z_2^{-c}) = f(\hat{p}_1\ z_1^{-c},\ \hat{p}_2\ z_2^{-c}) \qquad \forall\ c \geqslant 0$

2) $f(p_1\ z_1^{-c-1},\ \sigma_2) = f(\hat{p}_1\ z_1^{-c-1},\ \sigma_2)\ ,\ \forall \sigma_2 \in K[z_2^{-1}]\ c=\deg\ \sigma_2$

3) $f(\sigma_1, p_2\ z_2^{-c-1}) = f(\sigma_1, \hat{p}_2\ z_2^{-c-1})\ ,\ \forall \sigma_1 \in K[z_1^{-1}]\ \ c=\deg\ \sigma_1$

The N-state space (Nerode-state space) is the set of Nerode equivalence classes :

(17) $$X = (U \times U)/\underset{N}{\sim} = \left\{ [u_1,u_2] : (u_1,u_2) \in U \times U \right\}$$

In the linear case the N-state space X can be endowed with the linear vector space structure. If we represent the i-o linear map by a formal power series $\sum_1 {}_i\ a_i\ z^i$ in $K[z]$ the following facts are equivalent [5,6] :

i) $\dim\ X < \infty$

ii) $\text{card}\ ([0]) > 1$

iii) $\sum_1 {}_i\ a_i z^i = \dfrac{P(z^{-1})}{Q(z^{-1})}$

iv) $(a_i)_{i>0}$ is a H-sequence (°)

v) X is a $K[z^{-1}]$ torsion module

In the bilinear case the N-state space X cannot be endowed with a K-module structure and consequently there is no way to extend some of the previous statements to the bilinear case.

Nevertheless it is still interesting to investigate how the structure of the bilinear i-o map and the structure of the N-state space are correlated.

(°) by "H-sequence" we intend a sequence whose Hankel matrix has finite rank [7] .

We recall now some properties of H-sequences and introduce some notations to simplify the formalism.

1 – (s_i) is a H-sequence iff there exist $(r+1)$ numbers $a_0, \dots, a_r$ not all zero such that :

$$(18) \qquad \sum_{0}^{r}{}_{i}\, a_i\, s_{q-i} = 0 \;, \qquad q = r+1, \dots\dots$$

we call the ordered set $(a_0, \dots a_r)$ an annihilating set of the H-sequence.

2 – The annihilating sets constitute a principal ideal $((a_0, \dots, a_r))$ in the ring of finite sequences with Cauchy product. This ideal is generated by a "minimal annihilating set" (i.e. of minimal length).

3 – Consider a j-indexed family of H-sequences (s_{ij}), $j \in \mathbb{N}$ with the corresponding annihilating ideals I_j; the principal ideal $\bigcap_j I_j$ is the "annihilating ideal of the family".

4 – Given a double sequence (s_{ij}), a finite matrix $A = (a_{rs})$ is a (p,q)-annihilating matrix for (s_{ij}) if

$$\sum_{0}^{n,m}{}_{rs}\, a_{rs}\, s_{p+r,q+s} = 0$$

Lemma 1. Let S as in (10). The annihilating ideal of the j-indexed family (s_{ij}), $j \geqslant 1$, $i > j$ is different from (0) iff there exists $p_1(z_1^{-1}) \in K[z_1^{-1}]$, $p_1 \neq 0$, such that $(p_1(z_1^{-1}), 0) \in [0,0]$.

Proof. Let $p_1(z_1^{-1}) = \sum_{0}^{n}{}_{r}\, a_r\, z_1^{-r}$ and $(p_1, 0) \in [0,0]$. For every $\sigma_2(z_2^{-1})$, (16) and (11) give

$$(19) \quad \operatorname*{diag}_{>0} \sum_{1}^{\infty}{}_{ij}\, s_{ij} z_1^{i} z_2^{j} \sum_{0}^{n}{}_{r}\, a_r\, z_1^{-r} z_1^{-c-1} \sum_{0}^{c}{}_{s} b_s z_2^{-s} = 0, \forall b_s, \forall c \geqslant 0$$

$$\sum_{0}^{c}{}_{s} \; b_s \; \underset{>0}{\mathrm{diag}} \sum_{ijr} s_{ij} \; a_r \; z_1^{i-r-c-1} \; z_2^{j-s} = 0$$

$$\sum_{0}^{n}{}_{r} \; s_{t+r+c+1,\,t+s} \; a_r = 0 \quad , \; \forall t > 0, \;\; \forall c \geqslant 0, \forall s, \; 0 \leqslant s \leqslant c$$

It is then immediate that

$$(20) \quad \sum_{0}^{n}{}_{r} \; s_{t+1+c+r,\,t} \; a_r = 0 \; , \quad \forall c \geqslant 0 \; , \quad \forall t > 0$$

Conversely, let (20) hold and assume $p_1(z_1^{-1}) = \sum_{0}^{n}{}_{r} a_r \; z_1^{-r}$.

Then recalling (16) and straightforward checking that

$$(21) \; f(p_1(z_1^{-1}) \; z_1^{-c-1} \; , \; \sigma_2(z_2^{-1})) = 0, \forall \sigma_2 \in K[z_2^{-1}], \quad c = \deg \sigma_2$$

it follows that the input pair $(p_1(z_1^{-1}),0) \in [0,0]$.

<u>Lemma 2.</u> Let S as in (10). The annihilating ideal of the i-indexed family (s_{ij}), $i \geqslant 1$, $j > i$ is different from (0) iff there exists $p_2(z_2^{-1}) \in K[z_2^{-1}]$, $p_2 \neq 0$, such that $(0,p_2(z_2^{-1})) \in [0,0]$.

<u>Proof.</u> As in Lemma 1.

<u>Theorem 1.</u> Let S as in (10) and $p_1(z_1^{-1}) = \sum_{0}^{n}{}_{r} a_r z_1^{-r}$, $p_2(z_2^{-1}) = \sum_{0}^{m}{}_{s} b_s z_2^{-s}$. The input pair $(p_1(z_1^{-1}),\; p_2(z_2^{-1})) \in [0,0]$ iff

i) $(a_0,\ldots,a_n)$ is an annihilating set for the j-indexed family (s_{ij}), $j \geqslant 1$, $i > j$,

ii) $(b_0,\ldots,b_m)$ is an annihilating set for the i-indexed family (s_{ij}), $i \geqslant 1$, $j > i$

iii) the matrix $A = (a_i b_j)$ is pp-annihilating for (s_{ij}), $\forall p \geqslant 1$

<u>Proof.</u> By definition $(p_1,p_2) \in [0,0]$ iff

$$(22)\qquad f(p_1\circ\sigma_1, p_2\circ\sigma_2) = f(\sigma_1,\ \sigma_2), \forall\,(\sigma_1, \sigma_2)\in K[z_1^{-1}]\times K[z_2^{-1}]$$

Recalling (16), (22) implies :

$$(23)\qquad f(p_1(z_1^{-1})\ z_1^{-c-1},\ \sigma_2(z_2^{-1})) = 0\ ,\quad c = \deg\sigma_2$$

$$(24)\qquad f(\sigma_1(z_1^{-1}), p_2(z_2^{-1})z_2^{-c-1}) = 0\quad ,\quad c = \deg\sigma_1$$

$$(25)\qquad f(p_1(z_1^{-1})\ z_1^{-c}\ ,\ p_2(z_2^{-1})\ z_2^{-c}) = 0\ ,\quad \forall c \geqslant 0\ .$$

(23) and (24) imply i) and ii) by Lemma 1 and Lemma 2.

(25) implies

$$(26)\ \underset{>0}{\text{diag}} \sum_{1}^{\infty}{}_{ij}\, s_{ij} z_1^i z_2^j \sum_{0}^{n}{}_r\, a_r z_1^{-r} \sum_{0}^{m}{}_s\, b_s z_2^{-s} z_1^{-c} z_2^{-c} = 0\ ,\ \forall c \geqslant 0$$

$$\sum_{1}^{\infty}{}_t\Big(\sum_{0}^{n}{}_r \sum_{0}^{m}{}_s\, a_r b_s s_{t+c+r,\,t+c+s}\Big)(z_1 z_2)^t = 0 \quad \forall c \geqslant 0$$

and

$$(27)\ \sum_{0}^{n}{}_r \sum_{0}^{m}{}_s\, a_r b_s s_{t+r,\,t+s} = 0 \qquad \forall\, t > 0$$

The converse is proved assuming $p_1(z_1^{-1})$ and $p_2(z_2^{-1})$ having as coefficient sets the annihilating sets.

Remark. The assumption that the zero state $[0,0]$ contains at least one element of the form (p_1,p_2) with $p_1,p_2 \neq 0$ is a necessary condition for controllability to zero state. This condition has a direct implication on the operator S as proved in Theorem 1.

Lemma 3. Let S satisfy i) and ii) in Theorem 1.
Then

$$(28)\ S(z_1,z_2) = \frac{N(z_1^{-1},z_2^{-1})}{p_1(z_1^{-1})p_2(z_2^{-1})} + \frac{(z_1 z_2)^{-1}}{p_1(z_1^{-1})p_2(z_2^{-1})} \sum_{\substack{1\\ -m\leqslant h-k\leqslant n}}^{\infty}{}_{h,k}\, s^1_{h,k}\, z_1^h z_2^k$$

Conversely if S has the structure (28) then i) and ii) hold.

Proof. $S(z_1,z_2)p_1(z_1^{-1})p_2(z_2^{-1}) = N(z_1^{-1},z_2^{-1}) + S^1(z_1,z_2)$.

It is immediate to check that $S^1(z_1,z_2)$ is a double series with $s^1_{h,k}=0$, $-m > h-k > n$ and that $N(z_1^{-1}, z_2^{-1}) \in K[z_1^{-1},z_2^{-1}]$, with $\deg N(z_2^{-1})(z_1^{-1}) < \deg p_1$, $\deg N(z_1^{-1})(z_2^{-1}) < \deg p_2$

The converse is a consequence of the fact that for every polynomial $p(z^{-1})$, the sequence corresponding to the series $1/_{p(z^{-1})} = \sum_1 {}_i s_i z^i$ has the coefficients of $p(z^{-1})$ as minimal annihilating set.

Theorem 2. $(p_1,p_2) \in [0,0]$, $p_1,p_2 \neq 0$ iff S can be represented as in (28) and $s^1_{h,h} = 0$, $\forall h > 1$.

Proof. S represented by (28) implies i) and ii) of Theorem 1; using (25) and (26); $s^1_{h,h} = 0$, $\forall h > 1$ gets iii).

Conversely if $(p_1,p_2) \in [0,0]$, $p_1,p_2 \neq 0$ by Theorem 1, i) and ii) are satisfied and by Lemma 3 the structure (28) holds. $s_{h,h} = 0\ \forall h > 1$ follows from $\underset{>0}{\mathrm{diag}}\, S\, p_1 p_2 = 0$.

Corollary. Consider now the two minimal annihilating sets (possibly zero)

$$(29) \qquad (a^1_0,\dots,a^1_N), \quad \text{for } (s_{ij}), \quad j \geq 1\ , \quad i > j$$

$$(30) \qquad (b^1_0,\dots,b^1_M), \quad \text{for } (s_{ij}), \quad i \geq 1, \quad j > i$$

with the associated polynomial $p_{1m}(z_1^{-1})$ and $p_{2m}(z_2^{-1})$.

The state $[0,0]$ belongs to the ideal $(p_{1m}) \oplus (p_{2m})$ and contains the ideals $(p_{1m}) \oplus (0)$ and $(0) \oplus (p_{2m})$. If $p_{1m}, p_{2m} \neq 0$, S can be represented as in (28), with -of course- $p_1 = p_{1m}$ and $p_2 = p_{2m}$.

As recalled before, in the linear case the rational structure of the i-o map is equivalent to other fundamental properties of the system that can be evidentiated from the i-o map. In the bilinear case the rationality condition does not come out so directly from i-o characteristics. It is still possible to give an interpretation in this direction referring to a finite space repetition of the system [1] .

<u>Theorem 3.</u> Let S as in (28) and $p_1 = p_{1m}$, $p_2 = p_{2m}$. Assume S be the series expansion of a rational function $L(z_1^{-1}, z_2^{-1})/D(z_1^{-1}, z_2^{-1})$, $\deg L(z_1^{-1})(z_2^{-1}) < \deg D(z_1^{-1})(z_2^{-1})$, $\deg L(z_2^{-1})(z_1^{-1}) < \deg D(z_2^{-1})(z_1^{-1})$. Then

$$(31)\quad S(z_1,z_2) = \frac{N(z_1^{-1},z_2^{-1})}{p_{1m}(z_1^{-1})p_{2m}(z_2^{-1})} + \frac{(z_1z_2)^{-1}}{p_{1m}(z_1^{-1})p_{2m}(z_2^{-1})}\Big(\frac{P_0((z_1z_2)^{-1})}{Q_0((z_1z_2)^{-1})} + $$
$$+ \frac{1}{z_1^{-1}}\,\frac{P_1((z_1z_2)^{-1})}{Q_1((z_1z_2)^{-1})} + \dots\dots + \frac{1}{z_1^{-N}}\,\frac{P_N((z_1z_2)^{-1})}{Q_N((z_1z_2)^{-1})} +$$
$$+ \frac{1}{z_2^{-1}}\,\frac{\hat{P}_1((z_1z_2)^{-1})}{\hat{Q}_1((z_1z_2)^{-1})} + \dots\dots + \frac{1}{z_2^{-N}}\,\frac{\hat{P}_N((z_1z_2)^{-1})}{\hat{Q}_N((z_1z_2)^{-1})}\Big),$$

$\deg P_i < \deg Q_i$, $\deg \hat{P}_i < \deg \hat{Q}_i$. (by semplicity of notations M=N)

<u>Proof.</u> (Hint) An infinite matrix (s_{ij}) is associated to a series expansion of a proper rational function

$$\frac{R(z_1^{-1},z_2^{-1})}{Q(z_1^{-1},z_2^{-1})} = \sum_{1}^{\infty}{}_{ij}\ s_{ij}\ z_1^i\ z_2^j$$

iff there exists a pq-annihilating matrix for any (p,q), p or q ≥ 1.

Consider the series $\sum_{\substack{1 \\ |h-k|\leq n}}^{\infty}{}_{h,k}\ s^1_{hk}\ z_1^h\ z_2^k$ which is now a rational function.

By assigning appropriate values to p and q one proves that the side located diagonals in the matrix $(s^1_{h,k})$ are H-sequences and then can be represented by rational functions in $(z_1z_2)^{-1}$.

The procedure is repeated for the infinite matrix obtained deleting the above mentioned diagonals.

An i-o map having the structure (31) can be of course realized by connecting a finite number of linear maps and multipliers.

<u>Theorem 4.</u> Let S as in (31). Then there exist $(p_1,p_2) \in [0,0]$; $p_1,p_2 \neq 0$.

<u>Proof.</u> See [1] .

Theorem 4 shows that the rationality condition with i) and ii) of Theorem 1 are sufficient to guarantee that there exist $p_1,p_2 \neq 0$ such that $(p_1,p_2) \in [0, 0]$.

Before concluding we give a rough sketch of some consequences implied by the introduction of a state basing on the structure (31).

Assume S as in (31) and introduce the following equivalence relation on $K[z_1^{-1}] \times K[z_2^{-1}]$: two input polynomial pairs are L-equivalent, $(p_1,p_2) \underset{L}{\sim} (\hat{p}_1,\hat{p}_2)$, iff every linear subsystem, appearing in the structure (31), reaches in t = 0 the same state, starting from zero state.

<u>Theorem 5.</u> $(p_1,p_2) \underset{L}{\sim} (\hat{p}_1,\hat{p}_2) \Rightarrow (p_1,p_2) \underset{N}{\sim} (\hat{p}_1,\hat{p}_2)$

<u>Proof.</u> See [1] .

The set of L-classes can be obviously embedded in the direct sum of the state spaces of the linear subsystems. The L-classes constitute a finer partition of the input space than Nerode equivalence classes.

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