A FORMAL POWER SERIES APPROACH TO CANONICAL REALIZATION OF BILINEAR INPUT-OUTPUT MAPS

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INTRODUCTION

In a previous paper [1] we presented a contribution to the realization theory of bilinear discrete-time stationary input-output maps. In particular we shown that for bilinear i/o maps characterized by realizable series the set of Nerode states (or canonical states) can be embedded in a finite dimensional vector space. The dynamics of the state is then described by recursive equations of "bilinear structure".

This paper is a continuation of [1]. We extend the notions of reachability, controllability and observability in bounded time to canonical realizations of bilinear i/o maps and we prove that each one of these conditions is equivalent to assume that the i/o map is represented by a realizable series.

1. DEFINITIONS

Let K be a field and let U_1, U_2 and Y denote the following spaces:

(a) $U_1 = U_2 = \{ u \in K^{Z^-} \text{ with compact support} \}$

(b) $Y = \{ y \in K^{\mathbb{N} - \{0\}} \}$

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$U_1 \times U_2$ is termed the <u>input space</u> and Y the <u>output space</u>.

DEFINITION. A map $f: U_1 \times U_2 \to Y$ is a <u>bilinear discrete time, stationary i/o map</u> if it satisfies the following conditions:

(i) <u>bilinearity</u>

$$f(ku_1,u_2) = kf(u_1,u_2); \quad f(u_1,ku_2) = kf(u_1,u_2)$$

$$f(u_1 + v_1,u_2) = f(u_1,u_2) + f(v_1,u_2)$$

$$f(u_1,u_2 + v_2) = f(u_1,u_2) + f(u_1,v_2)$$

for any $k \in K$, $u_1, v_1 \in U_1$, $u_2, v_2 \in U_2$

(ii) <u>stationarity</u>

the map f is invariant under translation with respect to time in the following sense: the diagram

$$\begin{array}{ccc} U_1 \times U_2 & \xrightarrow{f} & Y \\ \sigma \downarrow & & \downarrow \sigma_* \\ U_1 \times U_2 & \xrightarrow{f} & Y \end{array} \tag{1.1}$$

commutes with respect to the shift operators σ and σ_* defined as

$$\sigma: ((\ldots,u_1(-1),\ u_1(0)),\ (\ldots,u_2(-1),\ u_2(0))) \to$$

$$((\ldots,u_1(-1),\ u_1(0),0),\ (\ldots,u_2(-1),\ u_2(0),0))$$

$$\sigma_*: (y(1),\ y(2),\ldots) \to (y(2),\ y(3),\ldots)$$

REMARK. The space $K[z_1^{-1}] \times K[z_2^{-1}]$ is endowed with a $K[\sigma]$ - module structure via the operation

$$\sigma: K[z_1^{-1}] \times K[z_2^{-1}] \to K[z_1^{-1}] \times K[z_2^{-1}]: (p_1,p_2) \mapsto (z_1^{-1}p_1, z_2^{-1}p_2);$$

in this way $U_1 \times U_2$ can be identified to the $K[\sigma]$-module $K[z_1^{-1}] \times K[z_2^{-1}]$.

It is also obvious in what sense the ring of formal power series in one indeterminate z $K[[z]]$ ("causal" power series), denoted by $K_c[[z]]$, is a $K[z^{-1}]$-module and is identified to the $K[\sigma_*]$-module Y.

Polynomials and power series are then alternative ways of viewing the elements of U_1, U_2 and Y; thus we shall not distinguish between the two representations.

REMARK. The polynomial notation is particularly useful in representing i/o relations. In fact, let $(z_1z_2)K[[z_1,z_2]]$ be the ring (without identity) of "causal" power series, denoted by $K_c[[z_1,z_2]]$. Consider $s \in K_c[[z_1,z_2]]$, $s = \Sigma s_{ij}\ z_1^i z_2^j$ and assume that the coefficient s_{ij} represents the output at time 1 for inputs $u_1 = z_1^{-i}$, $u_2 = z_2^{-j}$. Then for any input pair $u_1 \in K[z_1^{-1}]$, $u_2 \in K[z_2^{-1}]$ the output can be represented by the formal power series in (z_1z_2) given by:

$$y = f(u_1,u_2) = \sum_{1}^{\infty}{}_r\ y(r)(z_1z_2)^r = s\ u_1(z_1^{-1})u_2(z_2^{-1}) \odot \sum_{1}{}_r (z_1z_2)^r$$

where $\odot$ denotes the Hadamard product:

$$\Sigma_{ij}a_{ij}z_1^i z_2^j \odot \Sigma_{ij}b_{ij}z_1^i z_2^j = \Sigma_{ij}a_{ij}b_{ij}z_1^i z_2^j$$

We have thus introduced a biunique correspondence between i/o maps and "causal" power series in two indeterminates.

The Nerode equivalence relation $\underset{N}{\sim}$ [2] is naturally defined in $U_1 \times U_2$: two input pairs (u_1,u_2) and (v_1,v_2) are Nerode equivalent iff the output sequences $f(u_1,u_2)$ and $f(v_1,v_2)$ are the same and remain the same whenever both (u_1,u_2) and (v_1,v_2) are followed by an arbitrary input pair $(w_1,w_2) \in U_1 \times U_2$. More precisely:

$$(u_1,u_2) \underset{N}{\sim} (v_1,v_2) \quad \text{iff} \quad f(\sigma^k(u_1,u_2) + (w_1,w_2)) = \\ = f(\sigma^k(v_1,v_2) + (w_1,w_2))$$

$\forall k \in \mathbb{N}$, $\forall (w_1,w_2) \in U_1 \times U_2$ with length $(w_1,w_2) \triangleq (\max(\deg w_1, \deg w_2))+1 \leqslant k$.

We denote the Nerode equivalence classes by $[u_1,u_2]$:

$$[u_1,u_2] = \{(v_1,v_2) \in U_1 \times U_2 : (v_1,v_2) \underset{N}{\sim} (u_1,u_2)\}.$$

The map f is then factorized as in the following commutative diagram:

$$\begin{tikzcd} U_1 \times U_2 \arrow[r, "f"] \arrow[d, "\nu"'] & Y \\ X_N \triangleq U_1 \times U_2/\underset{N}{\sim} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

The set X_N defined by

$$X_N = \{[u_1,u_2] : (u_1,u_2) \in U_1 \times U_2\} = (U_1 \times U_2)/\underset{N}{\sim}$$

is called the <u>Nerode</u> <u>(or canonical)</u> <u>state space</u>.

2. EMBEDDING OF X_N IN A FINITE DIMENSIONAL VECTOR SPACE

The following three equivalence relations defined on U_1, U_2 and $U_1 \times U_2$ respectively play an essential role in our study (see [3,4]):

(1) $\quad u_1 \underset{1}{\sim} v_1 \quad \text{iff} \quad f(\sigma^k u_1, w_2) = f(\sigma^k v_1, w_2), \ \forall k, \ \forall w_2 \in U_2, \ \deg w_2 < k$

(2) $\quad u_2 \underset{2}{\sim} v_2 \quad \text{iff} \quad f(w_1, \sigma^k u_2) = f(w_1, \sigma^k v_2), \ \forall k, \ \forall w_1 \in U_1, \ \deg w_1 < k$

(3) $\quad (u_1,u_2) \underset{3}{\sim} (v_1,v_2) \quad \text{iff} \quad f(u_1,u_2) = f(v_1,v_2).$

In [4] it has been proved that

$$(u_1,u_2) \underset{N}{\sim} (v_1,v_2) \quad \text{iff} \quad u_1 \underset{1}{\sim} v_1,\ u_2 \underset{2}{\sim} v_2, (u_1,u_2) \underset{3}{\sim} (v_1,v_2)$$

The quotient spaces $X_1 = U_1/\underset{1}{\sim}$ and $X_2 = U_2/\underset{2}{\sim}$ are naturally endowed with the structure of a linear space. In general the set $(U_1 \times U_2)/\underset{3}{\sim}$ does not admit such a structure but a standard algebraic construction allows the embedding of $U_1 \times U_2$ (i.e. $K[z_1^{-1}] \times K[z_2^{-1}]$) in the tensor space $U_1 \otimes U_2$ (i.e. $K[z_1^{-1}, z_2^{-1}]$). It follows that there exists a linear map $f_\otimes$ making the following diagram commutative

(2.1)
$$\begin{tikzcd}
U_1 \times U_2 \arrow[d, "\otimes"'] \arrow[rd, "f"] & \\
U_1 \otimes U_2 \arrow[r, "f_\otimes"] \arrow[d, "\nu_3"'] & Y \\
(U_1 \otimes U_2)/\ker f_\otimes \arrow[ru, "\bar{f}_\otimes"'] &
\end{tikzcd}$$

where ν_3 is onto and $\bar{f}_\otimes$ is one-to-one.

The map $f_\otimes$ induces an equivalence relation in $U_1 \otimes U_2$ and it can be verified immediately $(u_1,u_2) \underset{3}{\sim} (v_1,v_2)$ iff $u_1 \otimes u_2 = v_1 \otimes v_2 \pmod{f_\otimes}$. Thus the equivalence classes under $\underset{3}{\sim}$ are naturally embedded in the linear space $X_3 = (U_1 \otimes U_2)/\ker f_\otimes$ and the linear space $X_1 \oplus X_2 \oplus X_3$ furnishes a natural embedding for the canonical space X_N.

In [1] we related the finite dimensionality of $X_1 \oplus X_2 \oplus X_3$ to some properties of the series characterizing the i/o map f. Since some results presented there are necessary to a better comprehension of what follows, we devote the remaining of this section to recall them.

Denote by $K[(z)]$ the ring of rational power series in the indeterminate z and by $K[(z_1,z_2)]$ the ring of rational power series in z_1 and z_2.

The subring of $K[(z_1,z_2)]$ generated by $K[(z_1)]$, $K[(z_2)]$ and $K[(z_1z_2)]$ is denoted by $K^{real}[(z_1,z_2)]$ (the ring of "realizable" power series). The elements in $K^{real}[(z_1,z_2)]$ are power series expansions of the rational functions whose denominators can be factored in the form $p_1(z_1^{-1})p_2(z_1^{-1})p(z_1^{-1}z_2^{-1})$. A further characterization of realizable series is given by the following theorem:

THEOREM 2.1. Let $s \in K[[z_1,z_2]]$, $s = \Sigma_{ij}\, s_{ij} z_1^i z_2^j$ and define the following three families of formal power series r_i, c_j and d_{ij} in one indeterminate

$$r_i = \sum_{o}^{\infty}{}_k\, s_{i,i+k}\, z^k, \qquad i = 1,2,\ldots \qquad \text{"row series"}$$

$$c_j = \sum_{o}^{\infty}{}_k\, s_{j+k,j}\, z^k, \qquad j = 1,2,\ldots \qquad \text{"column series"}$$

$$d_{ij} = \sum_{o}^{\infty} s_{i+k,j+k} z^k, \qquad i,j = 0,1,2,\ldots \qquad \text{"diagonal series"}$$

Then $s \in K^{real}[(z_1,z_2)]$ if and only if r_i, c_j, d_{ij} are power series expansions of rational functions in one indeterminate having common denominator.

The connections among dimensions of X_1, X_2 and X_3 and the structure of the formal power series s are clarified by Lemmas 2.1, 2.2 and Theorem 2.2 below.

LEMMA 2.1. Let $s \in K_c[[z_1,z_2]]$ represent a bilinear i/o map $f: U_1 \times U_2 \to Y$. Then X_3 is finite dimensional iff the diagonal series d_{ij}, $i,j = 1,2,\ldots$ are power series expansions of rational functions having common denominator.

LEMMA 2.2. Let $s \in K_c[[z_1,z_2]]$ represent a bilinear i/o map f:

$U_1 \times U_2 \to Y$. Then the space $X_1 (X_2)$ is finite dimensional iff the column series c_j, $j = 1,2,\ldots$ (row series r_i, $i = 1,2,\ldots$) are power series expansions of rational functions having common denominator.

THEOREM 2.2. Let $s \in K_c[[z_1,z_2]]$ represent a bilinear i/o map f: $U_1 \times U_2 \to Y$. Then $X_1 \oplus X_2 \oplus X_3$ is finite dimensional iff s is a realizable power series.

3. REACHABILITY AND CONTROLLABILITY OF X_N IN BOUNDED TIME

The canonical state space X_N is intrinsically reachable by definition. Thus it is worth while to investigate if X_N is also controllable and, further, if there is an upper bound for the lengths of inputs needed to reach (control) reachable (controllable) states. To be more precise we introduce the following definitions.

DEFINITION 3.1. X_N is reachable in time m if each Nerode equivalence class $[u_1,u_2] \in X_N$ contains at least one input of length less than $m + 1$. X_N is reachable in bounded time if it is reachable in time m for some m.

DEFINITION 3.2. X_N is controllable (to zero state) in time k if for each Nerode equivalence class $[u_1,u_2]$ there exists at least one input (w_1,w_2) of length less that k+1 such that $(\sigma^k(u_1,u_2)+(w_1,w_2)) \in [0,0]$. X_N is controllable in bounded time if it is controllable in time k for some k.

Reachability and controllability in bounded time are characteristic properties of the canonical state space of bilinear i/o maps represented by realizable series. This fact is stated in the following

THEOREM 3.1. (reachability and controllability in b.t.). Let $s \in K_c[[z_1,z_2]]$ represent a bilinear i/o map f. Then the following conditions are equivalent:

(i) $s \in K_c^{real}[[z_1, z_2]]$

(ii) X_N is reachable in bounded time

(iii) X_N is controllable in bounded time

The equivalence (i) $\Leftrightarrow$ (ii) is proved in [1]. Before proceeding to prove (i) $\Leftrightarrow$ (iii), we shall derive two technical lemmas:

LEMMA 3.1. Let $f\colon U_1 \times U_2 \to Y$ be a bilinear i/o map. Then there exist polynomials $\omega_j \in K[z_j^{-1}]$, $j = 1,2$, such that

(i) X_j is a $K[z_j^{-1}]$-module isomorphic to $U_j/(\omega_j)$, $j = 1,2$

(ii) X_j is finite dimensional over K iff $\omega_j \neq 0$, $j = 1,2$

PROOF. Define the map $f_1\colon U_1 \to K[[z]]^{1 \times \infty}$ by the assignment

$$f_1(u_1) = (f(z_1^{-1}u_1, 1),\ f(z_1^{-2}u_1, 1) \ldots).$$

The linear space $K[[z]]^{1 \times \infty}$ admits the structure of a $K[z^{-1}]$ - module with scalar multiplication $z_1^{-1}(s_1, s_2, \ldots) = (s_2, s_3, \ldots)$. Hence the map f_1 is a $K[z_1^{-1}]$-morphism. For by definition of module scalar multiplication:

$$\begin{aligned} f_1(z_1^{-1}u_1) &= (f(z_1^{-2}u_1, 1), f(z_1^{-3}u_1, 1), \ldots) = \\ &= z_1^{-1}(f(z_1^{-1}u_1, 1), f(z_1^{-2}u_1, 1), \ldots). \end{aligned}$$

We therefore have the following commutative diagram

$$\begin{tikzcd} K[z_1^{-1}] \simeq U_1 \arrow[r, "f_1"] \arrow[d, "\nu_1"'] & K[[z]]^{1\times\infty} \\ U_1/\ker f_1 \arrow[ur, "\bar{f}_1"'] & \end{tikzcd}$$

where $U_1/\ker f_1$ is naturally endowed with $K[z^{-1}]$-module structure and $\ker f_1$ is an ideal in $K[z_1^{-1}]$.

A similar argument is used to prove the case $j = 2$. It is immediately verified that $u_1 \underset{1}{\sim} v_1$ if and only if $u_1 - v_1 \in \ker f_1$. Hence $X_1 \simeq U_1/\ker f_1$ is finite dimensional if and only if $\ker f_1 = (\omega_1) \neq (0)$. Moreover $\dim X_1 = \deg \omega_1$.

LEMMA 3.2. Let $f\colon U_1 \times U_2 \to Y$ be a bilinear i/o map, let X_i, $i = 1,2$ be finite dimensional and let ω_i, $i = 1,2$, be as in Lemma 3.1. Then the image under $f_\otimes$ of the ideal $\omega_1\omega_2\, K[z_1^{-1}, z_2^{-1}]$ is infinite dimensional iff $\dim X_3 = \infty$.

PROOF. Take any monomial $z_1^{-k}z_2^{-h}$ and make the following decompositions:

$$\begin{aligned} z_1^{-k} &= \omega_1 q_1 + r_1\,, && \deg r_1 < \deg \omega_1 \\ z_2^{-h} &= \omega_2 q_2 + r_2\,, && \deg r_2 < \deg \omega_2 \\ q_i &= z_i^{-a} p_i + g_i, && \deg g_i < a \qquad\qquad i = 1,2 \end{aligned}$$

with $a = \max(\deg \omega_1, \deg \omega_2)$.

Hence

$$\begin{aligned} f_\otimes(z_1^{-k}z_2^{-h}) &= f_\otimes((\omega_1 z_1^{-a}p_1 + \omega_1 g_1 + r_1)(\omega_2 z_2^{-a}p_2 + \omega_2 g_2 + r_2)) = \\ &= f_\otimes(\omega_1\omega_2 z_1^{-a}z_2^{-a}p_1p_2) + f_\otimes(\omega_1 g_1 r_2) + f_\otimes(\omega_2 g_2 r_1) + f_\otimes(r_1 r_2) \quad . \end{aligned}$$

Since the degrees of r_i and g_i are less than a,

$f_{\otimes}(z_1^{-k}z_2^{-h} - \omega_1\omega_2 z_1^{-a}z_2^{-a}p_1p_2)$, $h,k = 0,1,\ldots$ span a finite dimensional vector space.

PROOF of (i) $\Leftrightarrow$ (iii) (controllability in b.t)

(i) $\Rightarrow$ (iii). To prove this implication, we shall show that there exists an integer k such that for each input (u_1,u_2) we can find a pair (w_1,w_2) of length less than $k+1$ satisfying

$$(3.1) \quad \sigma^k(u_1u_2) + (w_1,w_2) \in [0,0]$$

Since by Lemma 3.1 ω_1 and ω_2 are non zero polynomials, (3.1) is equivalent to $z_1^{-k}u_1 + w_1 \in (\omega_1)$, $z_2^{-k}u_2 + w_2 \in (\omega_2)$, $f(z_1^{-k}u_1 + w_1, z_2^{-k}u_2 + w_2) = 0$.

Set max $(\deg \omega_1, \deg \omega_2) = a$ and denote by $\rho_i \in K[z_i^{-1}]$, $i = 1,2$, a pair of polynomials satisfying $u_1z_1^{-a} + \rho_1 \in (\omega_1)$, $\deg \rho_1 < a$, $u_2z_2^{-a} + \rho_2 \in (\omega_2)$, $\deg \rho_2 < a$. Observe now that reachability in time m implies the existence of polynomials $v_i \in (\omega_i)$, $\deg v_i < m$, $i=1,2$, such that:

$$(z_1^{-m}(u_1z_1^{-a} + \rho_1),\ z_2^{-m}(u_2z_2^{-a} + \rho_2)) \underset{N}{\sim} (v_1,v_2).$$

Hence setting $k = m+a$ and $(w_1,w_2) = (\rho_1z_1^{-m} + v_1, \rho_2z_2^{-m} - v_2)$, we see that $\sigma^k(u_1,u_2) + (w_1,w_2) \in [0,0]$. Consequently X_N is controllable (to zero state) in time k.

(iii) $\Rightarrow$ (i). Assume that X_N is controllable in time k. Obviously this implies controllability in time k of X_1 and X_2. Hence by Lemma 3.1, ω_i are non zero and $\dim X_i = \deg \omega_i = n_i < \infty$, $i = 1,2$.

By Lemma 3.2, X_3 is finite dimensional if $\dim f_{\otimes}((\omega_1\omega_2)) < \infty$, i.e. if $f_{\otimes}(\omega_1\omega_2z_1^{-i}z_2^{-j})$, $i,j = 0,1,\ldots$ span a finite dimensional vector space. Since X_N is controllable to zero in time k, for each monomial $z_1^{-i}z_2^{-j}$, there exist polynomials $w_i \in (\omega_i)$, $\deg w_i < k$, $i = 1,2$, such that

$$0 = f_{\otimes}((z_1 z_2)^{-k}(z_1^{-i} z_2^{-j} \omega_1 \omega_2) + w_1 w_2) =$$

$$= \sigma_*^k f_{\otimes}(z_1^{-i} z_2^{-j}) + f_{\otimes}(w_1 w_2).$$

Hence $\sigma_*^k\ f_{\otimes}(z_1^{-i} z_2^{-j} \omega_1 \omega_2)$, $i,j = 0,1,\ldots$ span a finite dimensional vector space an so do $f_{\otimes}(z_1^{-i} z_2^{-j} \omega_1 \omega_2)$, $i,j = 0,1,\ldots$

Thus $X_1 \oplus X_1 \oplus X_3$ is finite dimensional and by Theorem 2.2 s is a realizable series.

COROLLARY. X_N is connected in bounded time.

4. OBSERVABILITY OF X_N IN BOUNDED TIME

Let $\pi: K_c[[z]] \to K$, $\pi: \sum_{i=1}^{\infty} a_i z^i \to a_1$ and introduce the map $f_\pi = \pi \circ f$.

DEFINITION 4.1. Two states $[u_1,u_2]$ and $[v_1,v_2]$ in X_N are distinguishable in time m if there exist an integer k and an input (w_1,w_2) such that

$$\text{lenght } (w_1,w_2) \leqslant k \leqslant m$$

$$f_\pi(\sigma^k(u_1,u_2) + (w_1,w_2)) \neq f_\pi(\sigma^k(v_1,v_2) + (w_1,w_2))$$

The space X_N is called observable in time m when any two states are distinguishable in time m. X_N is observable in bounded time if it is observable in time m for some m.

A natural continuation of the programme of describing X_N in terms of its system theoretic properties is the description of the relationship between observability in bounded time and the structure of the i/o map. To be more precise we shall prove the following Theorem.

THEOREM 4.1. (observability in b.t.). Let $s \in K_c[[z_1,z_2]]$ represent a bilinear i/o map f. A necessary and sufficient condition that s be realizable is that X_N is observable in bounded time.

Necessity. Let s be realizable. Then by Theorem 2.2, $X_1 \oplus X_2 \oplus X_3$ is finite dimensional with $\dim X_i = n_i$, $i = 1,2,3$. Assume $[u_1,u_2] \neq$ $\neq [v_1,v_2]$. We therefore have three cases to consider.

If $(u_1,u_2) \underset{3}{\not\sim} (v_1,v_2)$, then $f(u_1,u_2) \neq f(v_1,v_2)$. Since Im $f_\otimes$ is a σ_*-invariant subspace of $K[[z]]$ having dimension n_3, it is easy to verify that

$$(4.1) \quad \sum_{1}^{n3}{}_i \ (z_1z_2)^i \odot (f(u_1,u_2) - f(v_1,v_2)) \neq 0$$

This implies that $[u_1,u_2]$ and $[v_1,v_2]$ are distinguishable in time n_3.

If $(u_1,u_2) \underset{3}{\sim} (v_1,v_2)$, assume $u_1 \underset{1}{\not\sim} v_1$. Hence $f_1(u_1) \neq f_1(v_1)$ implying $f(u_1z_1^{-k},1) \neq f(v_1z_1^{-k},1)$ for some positive integer $k \leqslant n_1$. Thus

$$f(\sigma^k \ (u_1,u_2) + (0,1)) \neq f(\sigma^k \ (v_1,v_2) + (0,1))$$

Recalling (4.1) we have

$$f_\pi(\sigma^{k+h} \ (u_1,u_2) + \sigma^h \ (0,1)) \neq f_\pi(\sigma^{k+h} \ (v_1,v_2) + \sigma^h \ (0,1))$$

for some non negative integer $h \leqslant n_1$. Hence $[u_1,u_2]$ and $[v_1,v_2]$ are distinguishable in time $n_3 + n_1$.

If $(u_1,u_2) \underset{3}{\sim} (v_1,v_2)$ and $u_2 \underset{2}{\not\sim} v_2$, we can use analogous arguments to show that $[u_1,u_2]$ and $[v_1,v_2]$ are distinguishable in time $n_3 + n_2$. It follows that $[u_1,u_2]$ and $[v_1,v_2]$ are distinguishable in time

$m = \max(n_3 + n_1, n_3 + n_2)$.

As a noticeable consequence of the proof above, there exists a finite set of experiments sufficient to distinguish two Nerode states in bounded time.

Sufficiency. Assume that X_N is observable in time m. We shall prove that $X_1 \oplus X_2 \oplus X_3$ is finite dimensional.

Suppose X_1 is infinite dimensional. Lets define the map $g: U_1 \to K^{m \times m}$, $g(\omega_1) = (k_{ij})_{i,j=1,\dots m}$, $k_{ij} = f_\pi(u_1 z_1^{-i-j+1}, z_2^{-j+1})$. Since $f_1(u_1) = 0$ implies $g(u_1) = 0$, then $\ker f_1 \subseteq \ker g$ and the quotient $(U_1/\ker f_1)/(\ker g/\ker f_1)$ is canonically isomorphic to $U_1/\ker g$.

This, together with the assumption $\dim X_1 = \dim(U_1/\ker f_1) = \infty$ gives that $\ker g/\ker f_1$ is infinite dimensional. Consequently we can find $u_1 \in U_1$ such that $g(u_1) \neq 0$ and $f_1(u_1) \neq 0$.

Then $[u_1, 0]$ and $[0,0]$ are indistinguishable in time m. For assuming $k \leqslant m$ we have

$$f_\pi(\sigma^k(u_1,0) + (\sum_0^{k-1}{}_i a_i z_1^{-i}, \sum_0^{k-1}{}_i b_i z_2^{-i})) -$$

$$-f_\pi(\sigma^k(0,0) + (\sum_0^{k-1}{}_i a_i z_1^{-i}, \sum_0^{k-1}{}_i b_i z_2^{-i})) =$$

$$= \sum_0^{k-1}{}_i b_i f_\pi(u_1 z_1^{-k}, z_2^{-i}) = 0$$

This contradicts the assumption. A similar result can be proved for X_2.

Assume now $\dim X_i = n_i \neq \infty$, $i = 1,2$ and $\dim X_3 = \infty$. Hence recalling Lemma 3.2, $f_\otimes((\omega_1\omega_2))$ is infinite dimensional. Let introduce the following linear maps

$$\bar{f}: K[z_1^{-1}, z_2^{-1}] \to K_c[[z]], \qquad \bar{f}(p) = f_\otimes(\omega_1\omega_2 p)$$

$$\mu: K_c[[z]] \to K^m, \qquad \mu(\sum_{i=1}^{\infty} a_i z^i) = (a_1, a_2, \dots a_m)$$

Now consider an (infinite) set $I \subseteq \mathbb{N}\times\mathbb{N}$ such that $\{\bar{f}(z_1^{-i}z_2^{-j}):(i,j)\in I\}$ is a Hamel basis for $\bar{f}(K[z_1^{-1},z_2^{-1}])$. Since the series s can be written in the form [5]:

$$s = (N + s^*)/\omega_1\omega_2$$

where $s^* = (z_1z_2)^{-1} \sum_{|h-k|\leqslant a} s'_{h,k}\, z_1^{-h}z_2^{-k}$, $a = \max(\deg\omega_1, \deg\omega_2)$, $N \in K[z_1^{-1},z_2^{-1}]$, it follows that $f_{\otimes}(\omega_1\omega_2z_1^{-i}z_2^{-j}) = 0$ when $|i-j| > a$. Hence each pair (i,j) in I satisfies the condition $|i-j| \leqslant a$.

Choose in I $m+1$ pairs $(i_1,j_2),(i_2,j_2),\dots,(i_{m+1},j_{m+1})$ so that

$$(4.2)\qquad \begin{aligned} &i_2,\ j_2 > \max(i_1,j_1) + a \\ &i_3,\ j_3 > \max(i_2,j_2) + a \\ &\dots\dots\dots\dots\dots\dots \\ &i_{m+1},j_{m+1} > \max(i_m,j_m) + a \end{aligned}$$

Since the range of μ is m-dimensional we have that $\pi \circ \bar{f}(\sum_{k=1}^{m+1} \alpha_k z_1^{-i_k}\, z_2^{-j_k}) \neq 0$ for some list of scalars $\alpha_1, \alpha_2, \dots \alpha_{m+1}$ not all zero. On the other hand $\bar{f}(\sum_{k=1}^{m+1} \alpha_k z_1^{-i_k}\, z_2^{-j_k}) \neq 0$.

We shall now use the sets of indices and of scalars introduced above to construct an input pair $(\omega_1v_1,\omega_2v_2)$ which does not belong to $[0,0]$ and is indistinguishable from $(0,0)$ in time m.

In fact consider the polynomials

$$v_1 = \sum_{k=1}^{m+1} z_1^{-i_k}, \qquad v_2 = \sum_{h=1}^{m+1} \alpha_h z_2^{-j_h}$$

By (4.2), $v_1v_2 - \sum_{k=1}^{m+1} \alpha_k z_1^{-i_k}\, z_2^{-j_k}$ is an element of ker $\bar{f}$ and hence

$$0 \neq \bar{f}\left(\sum_{k=1}^{m+1} \alpha_k z_1^{-i_k} z_2^{-i_k}\right) = \bar{f}(v_1 v_2) = f(\omega_1 v_1, \omega_2 v_2)$$

$$0 = \mu \circ \bar{f}\left(\sum_{k=1}^{m+1} \alpha_k z_1^{-i_k} z_2^{-i_k}\right) = \mu \circ f(\omega_1 v_1, \omega_2 v_2)$$

We therefore see that $(\omega_1 v_1, \omega_2 v_2)$ and $(0,0)$ are not equivalent under Nerode equivalence. However they are indistinguishable in time m; for

$$f_\pi(\sigma^k(\omega_1 v_1, \omega_2 v_2) + (w_1, w_2)) = f_\pi(w_1, w_2)$$

if $0 \leqslant k \leqslant m$ and length $(w_1, w_2) < k$.

This contradicts the assumption.

REMARK 1. A finite number of experiments is sufficient to observe X_N in bounded time. In fact, as proved in the necessity part of Theorem 4.1, these experiments correspond to apply a family of inputs $(0,0)$, $(0,z_2),\dots,(0,z_2^{n_1}),(z_1,0),\dots,(z_1^{n_2},0)$ and then to find the outputs in an interval of length n_3.

REMARK 2. A similar finite procedure can be adopted to characterize the i/o maps represented by realizable series. For, by Theorem 2.1 a realizable series in the indeterminates z_1 and z_2 can be computed from the first $2n_3$ coefficients of the series in one indeterminate $f(z_1^{-i},1)$, $i = 0,1,\dots, 2n_1$, and $f(1,z_2^{-i})$, $i = 1,\dots, 2n_2$;

The integers n_1, n_2, n_3 are the degrees of the lowest recurrence polynomials for "column", "row" and "diagonal" series respectively.

5. CONCLUSIONS

We have proved in this paper that the following conditions are equivalent:

(i) the i/o map $f: U_1 \times U_2 \to Y$ is represented by a realizable series

(ii) the canonical realization X_N is reachable in bounded time

(iii) the canonical realization X_N is controllable in bounded time

(iv) the canonical realization X_N is observable in bounded time.

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