DISSIPATIVITY IN THE SYSTEM THEORY CONTEXT: AN OUTLINE

Ettore Fornasini (^)

THIS WORK WAS SUPPORTED BY GNAS-CNR

(^) Istituto di Elettrotecnica e di Elettronica
Via Gradenigo 6/A
Università di Padova

SOMMARIO

Alcuni argomenti classici della teoria della passività vengono introdotti e discussi facendo ricorso a metodi della Teoria dei Sistemi.

In particolare, viene investigata la funzione che i concetti di controllabilità e di osservabilità svolgono nel correlare le proprietà nel dominio del tempo con quelle nel dominio s.

ABSTRACT

Some classical topics of passivity theory are introduced and discussed in a system theoretic context. In particular, time domain and Laplace transform domain passivity properties are related, resorting to the controllability and observability concepts.

INTRODUCTION

The investigation of "passive" physical structures is a traditional field of Electrical Engineering. Its origins arose from electrical network analysis, and a lot of results were obtained in connection with the well known problems of "physical realizability". Recently the introduction of system theoretic methods made possible some more steps in the formal explanation of dissipativity and energy concepts as well as the introduction of very general synthesis techniques. In particular, the papers by Willems [W 1,2] , Anderson [A 1] and others provided a clean and rigorous set up for the study of input/output dissipativity and state model properties. In this context the determination of energy functions of linear systems has been connected with the solution of suitable optimum least square problems.

These mostly theoretical results have been followed by several applications: it will be sufficient to mention here the introduction of general synthesis techniques for passive m-ports, both reciprocal and non reciprocal, on the basis of the so called positive real lemma.

The aim of this report is to present in an organized way several interesting results in the theory of linear invariant dissipative systems .

The first chapter gives a brief introduction to the most relevant concepts in dissipativity theory. Its kernel is the analysis of the constraints the dissipativity requirement imposes on the system matrices. Great emphasis is given to clarifying the relations among controllability observability and dissipativity, and to extending all results to the generalized linear systems which constitute the natural framework for network theory.

The second chapter relates the dissipativity property of a linear system to the analytical structure of its transfer matrix.

Finally, the third chapter is devoted to the connections among energy functions and matrices A,B,C,D_0 expressed by the positive real lemma. An extension of the energy concept is also considered in connection with spectral factorization analysis.

I. DISSIPATIVE DYNAMICAL SYSTEMS

I.1. General properties of dissipative dynamical systems

The aim of this section is to recall in a very compact way how the concepts of energy and of passivity can be introduced in a system theoretic context. The definitions and the properties we will consider do not refer in any way to the linearity hypothesis. Furthermore the assumption that we deal with continuous time systems (instead of discrete systems) could be dropped whithout seriously changing in the picture.

In the next sections we will refer instead to linear systems: the results we will derive there are crucial in the synthesis of linear passive electrical networks.

Except for some details, we will follow closely the rigorous and clean formulation of Willems [W 2] .

Let Σ be a regular continuous system, and define on the cartesian product of input and output alphabets UxY a real valued scalar function

$$\begin{aligned} w&: U \times Y \longrightarrow \mathbb{R} \\ w&: (\alpha,\beta) \mapsto w(\alpha,\beta) \end{aligned} \tag{1.1}$$

We will call w a supply,(or supply function) if for any finite time interval $[t_0,t_1]$, any input u and any initial state $x_0 = x(t_0)$

$$\int_{t_0}^{t_1} \left| w(u(t),y(t)) \right|_{x_0 \rightarrow} dt < \infty \tag{1.2}$$

(y denotes the output which corresponds to x_0 and u). In other words the supply function is locally absolutely integrable for any u and x_0.

Clearly for any given Σ there is an infinite set of possible supply functions. In the sequel we will assume that a well defined w has been chosen as supply function, and we will denote by (Σ,w) the system Σ with the supply w. It will be clear from the definition of dissipativity, that the supply function we choose is critical in determining the properties of the system (Σ,w).

In almost every case, w is chosen in such a way that it has a concrete physical meaning. For instance, when dealing with electrical networks, U and Y denote the vectors of instantaneous currents and voltages at the ports of the networks, and w(i,v) is just

$$w(i,v) = i^T v$$

which denotes the power which flows into the network.

The following definition formalizes the intuitive concept of dissipative system, we think of as a physical structure which accepts some work at the device terminals and accumulates this work as "internal energy". The internal energy can be subsequently (partially or totally) given back.

Def.1. A dynamical system (Σ,w) is dissipative (with respect to the supply function w) if there exists a non negative function

$$S: X \to \mathbb{R}_+ \tag{1.4}$$

such that, for any time interval $[t_o,t_1]$, for any state $x_o \varepsilon X$ and for any input u

$$\int_{t_o}^{t_1} w(u(t),y(t))dt \geq S(x_1)-S(x_o) \tag{1.5}$$

where $y(t)$ denotes the output at time t which corresponds to the input u and to the initial state $x(t_o) = x_o$. Every function S satisfying condition (1.5) is called an "energy function".

When the initial state x_o and the terminal state x_1, coincide, we write the left hand side in (1.5) as

$$\oint w \, dt$$

so that (1.5) becomes

$$\oint w(t)dt \geq 0 \tag{1.6}$$

Remark. A dissipative system (Σ,w) admits several energy functions. In fact if S is an energy function of (Σ,w), $S+k$ is an energy function too, for any positive k. In general S is not determined up a positive additive constant, and the problem of determining the set of all energy functions of (Σ,w) is often very hard.

In checking the dissipativity property of (Σ,w) a very useful state function is the "available storage" S_d. This function can be defined for every system - indipendently on its dissipativity properties: intuitively $S_d(x)$ represents a measure of the maximum work the system Σ can deliver to some external device when starting from the state x.

Def.2. Let (Σ,w) be a dynamical system with a supply function w. The available storage function S_d is the map (°)

$$S_d: X \to R_+^e$$
$$x_o \mapsto S_d(x_o) \triangleq \sup_{\substack{x_o \to \\ t_1 \geq t_o}} - \int_{t_o}^{t_1} w(t)dt \tag{1.7}$$

The supremum is evaluated along all trajectories starting from the state x_o at time t_o.

The following Theorem shows how available storage is related to dissipativity. The proof is given by [W.2] and can be also found in [F.1].

Theor.1 A system (Σ,w) is dissipative if and only if its available storage is everywhere finite

$$0 \leq S_d < \infty \tag{1.8}$$

In this case the function S_d is an energy function and

$$S_d \leq S \tag{1.9}$$

holds for any other energy functions S.

Theorem 1 gives a test for checking dissipativity and tells us also that the set of energy functions (when not empty) has a minimum element. As a matter of fact, in most problems the energy function one deals with is not the available storage, since in general the system is not assumed to be able to give back all stored energy.

In order to simplify our considerations, from now on we will keep in force the following

(°) R_+^e denotes the set on non negative real numbers extended with the symbol $+\infty$.

Assumption 1. <u>There exixsts at least one state x^* in X such that $S_d(x^*)=0$</u>

If (Σ,w) is dissipative with respect to the energy function S and S coincides with S_d, then S has an absolute minimum point in x^*: $S(x^*)=0$. In the sequel we will take into account only those energy functions S wich are zero wherever S_d is zero:

$$S_d(x) = 0 \Rightarrow S(x) = 0 \tag{1.10}$$

Then we have the definition of passive system

Def.3. <u>A dynamical system (Σ,w) is passive if</u>

i) <u>there exists at least one state x^* such that</u>

$$S_d(x^*) = 0,$$

ii) <u>(Σ,w) is dissipative and the energy function S we associate to Σ satisfies (1.10).</u>

If x^* is a state with zero energy, then

$$\int_{t_0}^{t} w(t)dt \geq 0 \quad x^* \rightarrow$$

Let (Σ,w) be completely reachable from some state $\bar{x}$ and let the set of its energy functions be not empty. Then Σ has a minimum energy function (S_d) and also a maximum energy function, as a consequence of the next Theorem

Theor.2 [W.2] <u>Let X be completely reachable by $\bar{x}$ in X. Then (Σ,w) is dissipative if and only if there is a real constant K such that</u>

$$\inf_{\substack{\bar{x}\rightarrow x \\ t_1 \geq t_0}} \int_{t_0}^{t_1} w(t)dt \geq K \tag{1.11}$$

<u>for any state x and for any trajectory from $\bar{x}$ to x.</u>

Def. 4 <u>Let x^* in X satisfy $S_d(x^*)=0$. The required supply function (from the state x^*) is a map</u>

$$S_{r,x^*} : X \rightarrow \mathbb{R}_+^e \tag{1.12}$$

defined as follows

$$S_{r,x^*}(x) = \begin{cases} +\infty \text{ if } x \text{ is not reachable from } x^* \\ \inf\limits_{\substack{x^*\to x \\ t_1 \to t_o}} \int_{t_o}^{t_1} w(t)dt \text{ if } x \text{ is reachable from } x^* \end{cases}$$

The number $\inf\limits_{x^*\to x} \int_{t_o}^{t_1} w(t)dt$ is evaluated along all trajectory from x^* to x: clearly this number exists if and only if x can be reached from x^*, and is non negative by the assumption $S_d(x^*)=0$.

Theor.3 (DISSIPATION INEQUALITY) Let (Σ,w) be a dissipative dynamical system, and let x^* satisfy $S_d(x^*)=0$. Then for any energy function S one has (°)

$$0 \leq S_d(x) \leq S(x) \leq S_{r,x^*}(x) \tag{1.13}$$

Moreover, if Σ is completely reachable from x^*, S_{r,x^*} is an energy function.

proof. (1.13) is proven by

$$S(x) - S(x^*) = S(x) \leq \inf\limits_{\substack{x^*\to x \\ t_1 \geq t_o}} \int_{t_o}^{t_1} w(t)dt = S_{r,x^*}(x)$$

In fact for any trajectory from x^* to x one has

$$\int_{t_o \; x^*\to x}^{t} x(t)dt \geq S(x) - S(x^*)$$

The proof of Theorem 2 also shows that

$$S_d(x^*) + \inf\limits_{\substack{x^*\to x \\ t_1 \geq t_o}} \int_{t_o}^{t_1} w(t)dt \triangleq S_{r,x^*}(x)$$

is an energy function. One can refer to [F.1] for the details. □

(°) Recall that the energy functions we consider are zero wherever S_d is zero.

The dissipation inequality can be rephrased as follows: "In a dissipative dynamical system (Σ,w) the stored energy in any state x is not less than the available storage in x and is not greater than the work one requires for reaching x from a zero energy state".

Note that every energy function S satisfies (1.13), but the converse is not true, that is a function of: $X \to \mathbb{R}_+$ which satisfies

$$S_d(x) \le f(x) \le S_{r,x^*}(x)$$

for any x in X is not necessarily an energy function.
It is easily shown that the set of all possible energy functions of a dissipative dynamical system is a convex set. Hence, if X is reachable from x and if $0 \le \beta \le 1$

$$\beta S_d + (1+\beta) S_{r,x^*}$$

is still an energy function.

I.2. Dissipative lossless systems

Def.1 Let (Σ,w) be a dissipative system, and let S be one of its energy functions.
(Σ,w) is "lossless" (with respect to S) if for any state x_o in X and any input driving Σ from x_o at time t_o to x_1 at time t_1 one has

$$\int_{t_o}^{t_1} w(t)dt = S(x_1) - S(x_o) \qquad (2.1)$$

In other words, along any system trajectory the increase of the stored energy is equal to the supply the system gets by describing the trajectory.

If the state set X is connected, the lossless property does not depend on the specific energy function one considers. In fact the energy function is unique, as is proved by the following

Theor.1 [W.2] Let (Σ,w) be a passive dynamical system and assume the state set X to be connected.
If (Σ,w) is lossless (with respect to the energy function S), then

i) $S_d(x) = S(x) = S_{r,x^*}(x)$ for any $x \in X$

ii) $S(x) = \int_{0\; x^* \to x}^{t1} w(t)dt = - \int_{0\; x \to x^*}^{t2} w(t)dt$ for any $x \in X$, and for any trajectory from x^* to x (from x to x^*).

proof. Definitions of available storage and required supply and relation (2.1) imply

$$S_d(x) = -\inf_{x \to} \int w(t)dt = -\inf_{y} \left[S(y)-S(x)\right] = S(x) - S(x^*) = S(x)$$

$$S_{r,x^*}(x) = \inf_{x^* \to x} \int w(t)dt = \inf \left[S(x) - S(x^*)\right] = S(x) - S(x^*) = S(x)$$

so that (i) is a trivial consequence. Moreover the energy function is unique, becouse of the dissipation inequality. Relations (ii) follow from (2.1), recalling that $S(x^*) = S_d(x^*) = 0$. □

Remark. If a dynamical system satisfies $S_d(x) = S_{r,x^*}(x)$, then the energy function is unique. However in general this is not sufficient to guarantee the lossless property. In fact the existence of optimal trajectory characterized by identical increments of the available storage and of the required supply does not imply that every trajectory exhibits this kind of property.

I.3. Dissipativity conditions for a linear system

Let us consider a linear system $\Sigma = (A,B,C,D_o)$

$$\dot{x} = Ax + Bu$$
$$y = Cx + D_o u$$

with m inputs and m outputs, and introduce the following standard supply function:

$$w(t) = u^T(t)y(t) \qquad (3.1)$$

When we deal with linear systems we are in a position to explicitly determine the structure of an energy function. In fact the following theorem holds

Theor.1 Let $\Sigma = (A,B,C,D_o)$ be dissipative. Then the available storage is a non-negative quadratic form

$$S_d(x) = x^T \frac{\Pi_d}{2} x, \qquad \Pi_d \geq 0 \qquad (3.2)$$

proof (hint). A map $f: \mathbb{R}^n \to \mathbb{R}$ is a quadratic form if and only if i) $f(\lambda x) = \lambda^2 f(x)$, ii) $f(x_1) + f(x_2) = \frac{1}{2}\left[f(x_1+x_2) + f(x_1-x_2)\right]$.
When S_d is finite, one checks that it satisfies conditions i) and ii). Non negativeness is trivially implied by $S_d(\cdot) \geq 0$. □

The structure of matrices A, B, C, D_o is strictly related to the dissipativity of the system Σ. Theorem 1 provides a first fundamental relation between system matrices and energy functions: a more complete set of relations will be considered in chap. III.

Theor.2 A linear system $\Sigma = (A, B, C, D_o)$ is dissipative if and only if the inequality in the unknown matrix K

$$\begin{bmatrix} D_o + D_o^T & C - B^T K \\ C^T - KB & -KA - A^T K \end{bmatrix} \geq 0 \tag{3.3}$$

admits some solutions $\Pi \geq 0$. The set of quadratic forms $x^T \frac{\Pi}{2} x$ associated with the non negative solutions is the set of quadratic energy functions of Σ.

proof. If Σ is dissipative, there is at least one quadratic energy function, namely S_d. Let $S(x) = x^T \frac{\Pi}{2} x$ be any quadratic energy function. Then for any $x(t_o) \in X$ and for any input u we have

$$\int_{t_o}^{t_o+\varepsilon} u^T y dt \geq x(t_o+\varepsilon)^T \frac{\Pi}{2} x(t_o+\varepsilon) - x(t_o)^T \frac{\Pi}{2} x(t_o) \tag{3.4}$$

Taking the limit as $\varepsilon \to 0_+$

$$2u^T(t_o)Cx(t_o) + 2u^T(t_o)D_o u(t_o) \geq \dot{x}^T(t_o)\Pi x(t_o) + x^T(t_o)\Pi \dot{x}(t_o) =$$
$$= x^T(t_o)(A^T\Pi + \Pi A)x(t_o) + x^T(t_o)\Pi B u(t_o) + u^T(t_o)B^T \Pi x(t_o)$$

so that we obtain

$$\left[u^T(t_o), x^T(t_o)\right] \begin{bmatrix} D_o + D_o^T & -B^T\Pi + C \\ C^T - \Pi B & -\Pi A - A^T \Pi \end{bmatrix} \begin{bmatrix} u(t_o) \\ x(t_o) \end{bmatrix} \geq 0 \tag{3.5}$$

Since $u(t_o)$ and $x(t_o)$ are arbitrary, the matrix Π satisfies the inequality (3.3).
Conversely, if $\Pi \geq 0$ is a solution of (3.3)

$$u^T Cx + u^T D_o u \geq x^T \Pi A x + x^T \Pi B u$$

and

$$\dot{x} = Ax + Bu , \quad y = Cx + D_o u$$

imply

$$u^T y \geq \frac{d}{dt} x^T \frac{\Pi}{2} x \qquad (3.6)$$

The integration of (3.6) shows that the quadratic form $x^T \frac{\Pi}{2} x$ is an energy function:

$$\int_{t_o}^{t_1} u^T y \, dt \geq x^T \frac{\Pi}{2} x \Big|_{t_o}^{t_1} = x^T(t_1) \frac{\Pi}{2} x(t_1) - x^T(t_o) \frac{\Pi}{2} x(t_o) \qquad (3.7)$$

□

It is worth while to point out what constraints controllability and observability assumptions introduce on the set of non-negative solutions of (3.3).

The proof of the following Theorem is identical with that of Theor. 1.

Theor.3 Let $\Sigma = (A,B,C,D_o)$ be dissipative and controllable. Then the required supply (from zero state) is a non-negative quadratic function:

$$S_{r,o}(x) = x^T \frac{\Pi_r}{2} x, \qquad \Pi_r \geq 0 \qquad (3.8)$$

Since non negative solutions of (3.3) biuniquely correspond with quadratic energy functions, the dissipation inequality leads to the following

Coroll.1 Let $\Sigma = (A,B,C,D_o)$ be dissipative and controllable. Then Π_r is the l.u.b. of the set of non-negative solutions of (3.3).

An observability assumption implies that the origin is a strong minimum of any energy function

Theor 4 Let $\Sigma = (A,B,C,D_o)$ be dissipative and observable. Then for any energy function $S(\cdot)$

$$S(x) = 0 \Rightarrow x = o$$

proof. Suppose that S is zero for some $x_0 \neq 0$. For every input u and for every instant $t_1 \geq 0$ we obtain

$$\int_{0,\,x_o\to}^{t} u^T y \, dt = \int_{0,\,x_o\to}^{t} u^T Cx + u^T D_o u \, dt \geq 0 \qquad (3.9)$$

By introducing the family of inputs

$$u_\varepsilon = \varepsilon C\exp(At)x_o \qquad \varepsilon \quad R$$

(3.9) becomes

$$0 \leq \int_0^{t_1} u_\varepsilon^T C\left(\exp A^T t\; x_o + \int_0^t \exp A(t-\sigma)Bu_\varepsilon(\sigma)d\sigma\right) + u_\varepsilon^T D_o u_\varepsilon dt =$$

$$= \varepsilon \int_0^{t_1} x_o^T \exp(A^T t)C^T C\exp(At)x_o dt + \qquad (3.10)$$

$$+ \varepsilon^2 \int_0^{t_1} u_1^T C \int_0^t \exp A(t-\sigma)Bu_1(\sigma)d\sigma + u_1^T D_o u_1 dt$$

When t_1 is fixed, the two integrals in the right hand side of (3.10) are two constant terms which we will denote by m and n respectively. Hence

$$0 \leq \varepsilon m + \varepsilon^2 n, \qquad \forall \varepsilon \in \mathbb{R}$$

which implies $n \geq 0$ and

$$m = 0 = x_o^T \int_0^{t_1} \exp(A^T t)C^T C\exp(At)dt\; x_o$$

This shows that Σ is not observable, contrary to the assumptions. □

The converse of Theorem 4 holds too. In fact, if $x_o \neq 0$ is indistinguishable from the zero state, the available storage is zero in x_o. Hence an energy function exists which amihilates in two points (at least), and consequently we have

Coroll.2 Let $\Sigma = (A,B,C,D_o)$ be dissipative. Then it is observable if and only if non negative solutions of (3.3) are positive definite.

We shall now establish a dissipativity condition which refers to the supply integral when the system starts from zero initial state. Later (Chap. II) we shall use extensively this condition in the dissipativity analysis of transfer functions.

We emphasize that controllability is needed in proving the equivalence between dissipativity and inequality (3.11).

Theor.5 Let $\Sigma = (A,B,C,D_o)$ be dissipative and assume $x(0) = 0$. Then for any $u \in \mathcal{U}$ and for any $t_1 \geq 0$

$$\int_0^{t_1} u^T y\; dt \geq 0 \qquad (3.11)$$

Conversely, let (3.8) hold and Σ be controllable. Then Σ is dissipative.

proof. Suppose first that Σ is dissipative and assume $x(0) = 0$. When one replaces an input u by the input $ku (k \in \mathbb{R})$, the corresponding output y becomes ky. Consequently, if for some u we have

$$\int_{0 \rightarrow}^{t_1} u^T(t)y(t)dt = h < 0$$

for an input ku we have

$$\int_{0 \rightarrow}^{t_1} ku^T(t)ky(t)dt = k^2 h < 0$$

This would imply that the available storage in the zero state is infinite

$$S_d(0) = - \inf_{\substack{0 \rightarrow 0 \\ t_1 \geq 0}} \int_0^{t_1} w(t)dt = + \infty \qquad (3.13)$$

contrary to the assumption of dissipativity.

Conversely, if Σ is controllable and (3.11) holds, theorem 2 of sec . 1 can be applied. In fact the state space X is reachable from the zero state and one has

$$\inf_{\substack{0 \rightarrow \\ t_1 \geq 0}} \int_0^{t_1} u^T y \, dt \geq 0 \qquad \square$$

When Σ is not controllable, in general the second part of the theorem does not hold. For instance, the electrical network of fig. 1, which includes a driven generator, does not constitute a dissipative dynamical system. Ob-

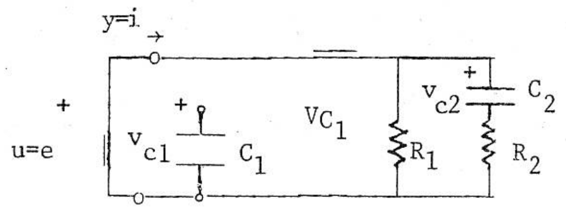


Fig. 1

serve that (3.11) still holds and that the state space is not completely controllable.

I.4 Lossless linear systems

Let Σ be a dissipative linear system. It is well known that Σ is lossles (with respect to an energy function S) if

$$\int_{t_o}^{t_1} w(t)dt = S(x_1) - S(x_o) \tag{4.1}$$

for every input u which drives the system from the state x_o in t_o to the state x_1 in $t_1 \geq t_o$.

Theor.1 If the equation in the unknown matrix K

$$\left[\begin{array}{c|c} D_o + D_o^T & C - B^T K \\ \hline C^T - KB & -KA - A^T K \end{array}\right] = 0 \tag{4.2}$$

admits some solution $\Pi \geq 0$, then $\Sigma = (A,B,C,D_o)$ is lossless dissipative with respect to the energy function $S(x) = x^T \frac{\Pi}{2} x$. Conversely if Σ is lossless dissipative with respect to $S(x) = x^T \frac{\Pi}{2} x$, then Π is a non negative solution of (4.2).

proof. Suppose that $\Pi \geq 0$ is a solution of (4.2). Following the pattern of the proof of Theorem 2, section 3, one gets

$$\int_{t_o}^{t_1} u^T y \, dt = x^T \frac{\Pi}{2} x \Big|_{t_o}^{t_1}$$

Hence Σ is lossless dissipative. On the other hand, suppose that Σ is lossless dissipative with respect to the non negative quadratic form $x^T \frac{\Pi}{2} x$. That Π is a solution of 4.2, follows from

$$\int_{t_o}^{t_1} w(t)dt = x^T \frac{\Pi}{2} x \Big|_{t_o}^{t_1}$$

The solution of 4.2 is unique when Σ is controllable. □

Theor.2 If Σ is a lossless dissipative linear system then each closed trajectory in X satisfies

$$\oint w(t)dt = 0$$

If Σ is controllable and dissipative, and if

$$\oint w(t)dt = 0 \tag{4.3}$$

holds for each closed trajectory in X, then Σ is lossless proof. The first part is obvious. To prove the second part, we resort to the connectedness of X. If Σ is not lossless with respect to some energy function S along some trajectory the strong inequality holds

$$\int_{\substack{t_o \\ x_o \to x_1}}^{t_1} w(t)dt > S(x_1) - S(x_o) \tag{4.4}$$

Taking now a trajectory from x_1 to x_o, dissipativity implies

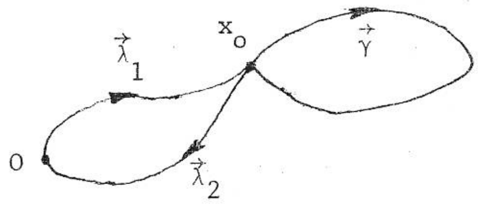
$$\int_{\substack{t_o \\ x_1 \to x_o}}^{t_2} w(t)dt \geq S(x_o) - S(x_1) \tag{4.5}$$

Concatenating the two trajectory one gets

$$\oint w(t)dt > 0$$

contrary to the assumption. □

Remark. Condition (4.3) refers to the set of all closed trajectories in X. It is easy to see that one could restrict (4.3) to the set of trajectories which have their origin and their end in the zero state. In fact, suppose



$\vec{\gamma}$ is a cycle which "starts" and "ends" in x_o. By the connectedness of X, there is a trajectory λ_1 from 0 to x_o as well a trajectory λ_2 from x_o to 0. Clearly

$$\oint_{\vec{\lambda}_1+\vec{\gamma}+\vec{\lambda}_2} w(t)dt = 0$$

$$\oint_{\vec{\lambda}_1+\vec{\lambda}_2} w(t)dt = 0 \tag{4.6}$$

hold, so that along $\vec{\gamma}$ trajectory

$$\oint_{\vec{\gamma}} w(t)dt = 0$$

is verified.

I.5 Dissipativeness in generalized linear systems

It is well known that the model of linear system we introduced in sec.3 in general is not suitable for modelling a linear invariant electric network [A.1].

This drawback can be overcome by extending the class of dynamical systems up to include the so called "generalized linear systems" (GLS)

$$\dot{x} = Ax + Bu$$
$$y = Cx + \sum_{i=0}^{d} D_i \frac{d^i u}{dt^i} \tag{5.1}$$

Dissipativity definitions and theorems require some adjustments when GLS are considered. The changes could be justified also from an heuristic point of view: if we pursue in adopting $u^T y$ as supply function we should avoid the possibility of impulses in the output function y, which imply a not infinitesimal energy flux through the system gates during an infinitesimal time interval.

Consequently we restrict the set at possible inputs of the system (5.1) to the d-times differentiable functions which fulfill the "initial" conditions:

$$u(0_-) = u(0_+);\ u'(0_-) = u'(0_+);\ldots;\ u^{(d-1)}(0_-) = u^{(d-1)}(0_+)$$

In this way no impulses are included in the output

$$y = Cx + \sum_{i=0}^{d} D_i \frac{d^i u}{dt^i}$$

Moreover, the stored energy at time 0 is assumed to depend on the state $x(0)$ as well as on the values $u(0_-)\ldots u^{(d-1)}(0_-)$, all of which determine the behaviour of Σ on the closed interval $[0,+\infty)$.

We therefore give the following

Def.1 A GLS $(A,B,C,D_o,\ldots D_d)$ with m inputs and m outputs is dissipative with respect to the supply

$$w = u^T y$$

if there exists a function

$$S:\ R^n \times \underbrace{R^m \times \ldots\ R^m}_{d\ \text{times}} \rightarrow R_+$$

$$S:\ (x(0_-), u(0_-), \ldots u^{(d-1)}(0_-)) \rightarrow S(x(0_-), \ldots u^{d-1}(0_-))$$

<u>such that for any $t_1 \geq 0$ and any d times differentiable function</u> $u \in \mathcal{U}_{[0,t_1]}$ <u>which satisfies the conditions</u>

$$u(0) = u(0\_), u'(0) = u'(0\_), \ldots u^{(d-1)}(0) = u^{(d-1)}(0\_) \quad (5.2)$$

<u>one has</u>

$$\begin{aligned} &\int_0^{t_1} \langle u^T y \rangle dt \geq S(x(t_1), u(t_1), \ldots u^{(d-1)}(t_1)) - \\ &-S(x(0), u(0\_), \ldots, u^{(d-1)}(0\_)) \end{aligned} \quad (5.3)$$

Starting from its d-th derivative $u^{(d)}$, a function u can be reconstructed when $u(0), u'(0) \ldots u^{(d-1)}(0)$ are known. Assuming that the input u of the GLS $\Sigma = (A, B, C, D_o, D_1, \ldots D_d)$ satisfies (5.2), the standard linear system $\bar{\Sigma} = (F, G, H, J)$

$$\begin{aligned} \frac{dz}{dt} &= Fz + Gv \\ y &= Hz + Jv \end{aligned} \quad (5.4)$$

with

$$F = \begin{bmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & \mathbb{1} & \ldots & 0 \\ & & & \ddots & \mathbb{1} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mathbb{1} \end{bmatrix}, \quad H = \begin{bmatrix} C & D_o & \ldots & D_{d-1} \end{bmatrix}, \quad J = D_d \quad (5.5)$$

and

$$z(0) = \begin{bmatrix} x(0) \\ u(0\_) \\ \vdots \\ u^{(d-1)}(0\_) \end{bmatrix} \quad (5.5')$$

and $v(t) = u^{(d)}(t)$ gives the same output $y(t)$ as the original system Σ does when its input is $u(t)$.

Therefore the supply of the GLS Σ, $w = u^T y$, can be viewed in the LS $\bar{\Sigma}$ as a function of the state z and of the input v

$$v = z^T \begin{bmatrix} 0 & 0 & \ldots & 0 \\ C & D_o & \ldots & D_{d-1} \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix} z + z^T \begin{bmatrix} 0 \\ D_d \\ 0 \\ \vdots \\ 0 \end{bmatrix} v \quad (5.6)$$

The definition of available storage extends to GLS in a natural way:

Def.2 The available storage of a GLS $= (A,B,C,D_o,\dots D_d)$ with supply $w = u^T y$ is the map

$$S_d: R^n \times \underbrace{R^m \times \dots R^m}_{d\ \text{times}} \to R^e_+ :$$

$$(x(0),u(0\_),\dots u^{(d-1)}(0\_)) \to \sup_{\substack{u\in\tilde{\mathcal{U}}\\ t_1\geq 0}} -\int_0^{t_1} u^T y\ dy \quad (5.7)$$

where $\tilde{\mathcal{U}}$ denotes the set of d-times differentiable function which satisfy conditions (5.2).

Following the pattern of the proofs given for standard dynamical systems, it is easy to show that S_d is everywhere finite if and only if Σ is dissipative.

Moreover when Σ is dissipative, S_d is an energy function expressed by a non negative definite quadratic form:

$$S_d(x(0),u(0\_),\dots u^{(d-1)}(0\_)) = \left[x^T(0),u^T(0\_)\dots\right] \frac{\Pi}{2} \begin{bmatrix} x(0) \\ u(0\_) \\ \vdots \\ u^{(d-1)}(0\_) \end{bmatrix} \quad (5.8)$$

The interest of the following Theorem is in that is restricts to the first derivative the dependence of y on the input derivatives, in any dissipative GLS.

Theor.1 A dissipative GLS $\Sigma = (A,B,C,D_o,D_1,\dots D_d)$ satisfies

$$D_2 = D_3 = \dots = D_d = 0 \quad (5.9)$$

proof. Since S_d is an energy function, (5.8) gives

$$\int_0^{\varepsilon} u^T y\ dt \geq \left[x^T(t),u^T(t),\dots\right] \frac{\Pi}{2} \left.\begin{bmatrix} x(t) \\ u(t) \\ \vdots \\ u^{(d-1)}(t) \end{bmatrix}\right|_0^{\varepsilon} \quad (5.10)$$

for every input u which satisfies conditions (5.2).

Let $d > 1$ and $D_d \neq 0$. Taking in (5.10) the limit as Σ goes to zero, from (5.6) we have the inequality

$$z^T(0)\begin{bmatrix} 0 & 0 \dots & 0 \\ C & D_o \cdots & D_{d-1} \\ \dots & \dots & \dots \\ 0 & 0 \dots & 0 \end{bmatrix} z(0) + z^T(0)\begin{bmatrix} 0 \\ D_d \\ \vdots \\ 0 \end{bmatrix} v(0) \geq z^T(0)\frac{\Pi}{2}\,\dot z(0) + \dot z^T(0)\,\frac{\Pi}{2} z(0) =$$

$$= z^T(0)\,\Pi\, F\, z(0) + z^T(0)\,\Pi\, G\, v(0)$$

By partitioning Π conformably with the block partition of F, G, etc., the inequality above can be rewritten as

$$z^T(0)\left\{\begin{bmatrix} 0 & 0 \dots & 0 \\ C & D_o \cdots & D_{d-1} \\ \dots & \dots & \dots \\ 0 & 0 \dots & 0 \end{bmatrix} - \Pi F\right\} z(0) \geq z^T(0)\begin{bmatrix} \Pi_{d+1,1} \\ \Pi_{d+1,2} - D_d \\ \vdots \\ \Pi_{d+1,d+1} \end{bmatrix} v(0) \qquad (5.11)$$

Since (5.12) has to be satisfied for every choice of v(0), the row vector

$$z^T(0)\begin{bmatrix} \Pi_{d+1,1} \\ \Pi_{d+1,2} - D_d \\ \vdots \\ \Pi_{d+1,d+1} \end{bmatrix} \qquad (5.12)$$

is the zero vector, and since there is no constraint on $z(0)$ this shows that

$$\Pi_{d+1,1} = 0, \quad \Pi_{d+1,2} = D_d, \quad \Pi_{d+1,3} = \Pi_{d+1,4} = \dots \Pi_{d+1,d+1} = 0 \qquad (5.13)$$

The structure of matrix Π is the following

$$\Pi = \left[\begin{array}{cccc|c} & & & & 0 \\ & & * & & D_d^T \\ & & & & 0 \\ & & & & \vdots \\ \hline 0 & D_d & 0 \dots & 0 & 0 \end{array}\right] \Bigg\} \text{(d+1) blocks} \qquad (5.14)$$

which agrees with condition $\Pi \geq 0$ if $D_d = 0$. □

The only alternate possibility is to assume $d = 1$. This implies that $D_d = D_1$ has diagonal position in (5.14), and the resulting structure of Π is compatible with the assumption $\Pi \geq 0$.

When $d = 1$, (5.11) simplifies as follows

$$z^T(0)\left(\begin{bmatrix} 0 & 0 \\ C & D_o \end{bmatrix} - \Pi F\right) z(0) \geq z^T(0) \begin{bmatrix} \Pi_{21} \\ \Pi_{22} - D_1 \end{bmatrix} v(0) \tag{5.15}$$

By (5.15) and $\Pi \geq 0$ we get the following conditions on the blocks of Π

$$\Pi_{21} = \Pi_{12} = 0$$
$$\Pi_{22} = D_1 \geq 0$$
$$\Pi_{11} \geq 0$$

After replacing F with its explicit formula (5.5) we derive

$$\begin{bmatrix} -\Pi_{11}A - A^T\Pi_{11} & C^T - \Pi_{11}B \\ C - B^T \Pi_{11} & D_o + D_o^T \end{bmatrix} \geq 0 \tag{5.16}$$

This shows that relation (3.3), which characterizes a dissipative LS, is a necessary dissipativity condition for a GLS. We have therefore proved the necessity part of the following

Theor.2 <u>A GLS $\Sigma = (A,B,C,D_o,D_1)$ is dissipative if and only if there exists a non negative definite matrix Π which satisfies the following inequality</u>

$$\left[\begin{array}{c|cc} D_1 & 0 & 0 \\ \hline 0 & D_o + D_o^T & C - B^T\Pi \\ 0 & C^T - \Pi B & -\Pi A - A^T\Pi \end{array}\right] \geq 0 \tag{5.17}$$

For the sufficiency part, observe that non negative definiteness of

$$\begin{bmatrix} D_o + D_o^T & C - B^T\Pi \\ C^T - \Pi B & -\Pi A - A^T\Pi \end{bmatrix}$$

implies

$$u^T D_o u + u^T Cx \geq x^T \Pi Ax + x^T \Pi Bu \tag{5.18}$$

for any $u \in \mathbb{R}^m$ and for any $x \in \mathbb{R}^n$.

If u, x, y are relative to the GLS

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + D_o u + D_1 \frac{du}{dt}$$

then the relations

$$u^T(y - D_1\dot{u}) \geq x^T \Pi \dot{x}$$

$$u^T y \geq \frac{d}{dt}\left(u^T \frac{D_1}{2} u + x^T \frac{\Pi}{2} x\right)$$

follow from (5.18) and imply

$$\int_{t_o}^{t_1} u^T y \, dt \geq \left(u^T \frac{D_1}{2} u + x^T \frac{\Pi}{2} x\right)\Big|_{t_o}^{t_1}$$

From this we conclude (cfr. characterization of energy functions) that

$$S(x,u) = \frac{1}{2} \begin{bmatrix} x^T u^T \end{bmatrix} \begin{bmatrix} \Pi & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix}$$

is an energy function of Σ.

□

Coroll.1 <u>A necessary condition for dissipativeness of the GLS</u> $\Sigma = (A,B,C,D_o, \ldots D_d)$ <u>is</u>

$$\int_0^{t_1} u^T y \, dt \geq 0 \qquad (5.19)$$
$$0 \to$$

<u>for any input u satisfying</u> $u(0) = u'(0) = \ldots u^{(d-1)}(0) = 0$. <u>Condition (5.19) is also sufficient if</u> Σ <u>is controllable</u>.

II. AN s DOMAIN ANALYSIS OF DISSIPATIVE LINEAR SYSTEMS

II.1 Bounded real and positive real matrices

In this section we briefly recall some properties of the classes of bounded real (BR) matrices and positive real (PR) matrices we will use in the analysis of dissipative linear systems. As we shall subsequently show, PR matrices characterize the input/output maps of these systems as well as BR matrices characterize their scattering relations.

Def.1 Let $S(s)$ be an $m \times m$ matrix of rational functions of a complex variable s, with real coefficients (real rational matrix). $S(s)$ is 'bounded real' if

i) all elements of $S(s)$ are analytic in $Re(s) > 0$

ii) for $Re(s) > 0$, $\mathbb{I} - \check{S}^T(s)S(s)$ is non negative definite Hermitian

Condition ii) could appear troublesome because of its lack of symmetry. Actually this formulation is not intrinsic to BR property, as is shown by the following

Theor.1 [N.1] The matrix $\mathbb{I}-\check{S}^T S$ is non negative definite if and only the same property holds for $\mathbb{I} - S\check{S}^T$.

When $S(s)$ is B.R., its analiticity region extends to the imaginary axis. On the other hand, when one knows that $S(s)$ is analytic for $Re(s) > 0$, a B.R. test can be proved which reduces to a test relative to the imaginary axis. These facts are formally stated in the following

Theor.2 [N.1] An $m \times m$ real rational matrix $S(s)$ is BR if and only if

i) all elements of $S(s)$ are analytic in $Re(s) \geq 0$

ii) $\mathbb{I}-\check{S}^T(j\omega)S(j\omega) \geq 0$ for all real ω.

Def.2 An $m \times m$ real rational matrix $W(s)$ is positive real if

i) all elements of $W(s)$ are analytic in $Re\ s > 0$

ii) for $Re(s) > 0$, $\check{W}^T(s) + W(s)$ is non negative definite Hermitian

For testing the P.R. character of a real rational matrix $W(s)$ we can resort to the following

Theor.3 [N.1] An $m \times m$ real rational matrix $W(s)$ is PR if and only if

i) $W(s)$ is analytic in $Re(s) > 0$

ii) the poles (if any) on the imaginary axis and at infinity are simple poles, and the residue matrices are non negative definite Hermitian

iii) $\check{W}^T(j\omega)+W(j\omega)\geq 0$ for all real ω, with $j\omega$ not a pole of $W(s)$

Theor.4 [N.1] Assume the $m\times m$ real rational matrix $W(s)$ admit an inverse Then $W(s)$ is P.R. if and only if so is $W^{-1}(s)$.

P.R. matrices and B.R. m trices are closely related, as the following theorem shows.

Theor.5 [N.1, F.1] Let $W(s)$ be a P.R. matrix. Then $\mathbb{1}+W(s)$ has an inverse, and the matrix

$$S(s) = (\mathbb{1}-W(s))(\mathbb{1}+W(s))^{-1} \tag{5.1}$$

is B.R.

On the other hand, let $S(s)$ be a B.R. matrix and assume that $\mathbb{1}+S(s)$ has an inverse. Then the matrix

$$W(s) = \left[\mathbb{1}+S(s)\right]^{-1}\left[\mathbb{1}-S(s)\right] \tag{5.2}$$

is P.R.

As a consequence, if matrices $S(s)$ and $W(s)$ are real rational and relation (5.1) or relation (5.2) holds, then $S(s)$ is B.R. if and only if $W(s)$ is P.R.

We end with a specialization of the B.R. and P.R. matrices, we will exploit when dealing with lossless linear systems.

Def.3 An $m\times m$ matrix $S(s)$ is lossless bounded real (LBR) if

i) $S(s)$ is B.R.

ii) $1-\check{S}^T(j\omega)S(j\omega) = 0$ for all real ω

Theor.6 An $m\times m$ real rational matrix $S(s)$ is LBR if and only if

i) $S(s)$ is analytic in $\mathrm{Re}(s)\geq 0$

ii) $1-S^T(-s)S(s) = 0$

Def.4 An $m\times m$ matrix $W(s)$ is lossless positive real (LPR) if

i) $W(s)$ is P.R.

ii) $\check{W}^T(j\omega)+W(j\omega)=0$ for all real ω such that $j\omega$ is not a pole of $W(s)$.

Theor.7 An $m \times m$ real rational matrix $W(s)$ is LPR if and only if

i) all poles of $W(s)$ are pure imaginary (^) and the residue matrices are non negative definite Hermitian

ii) $W^T(-s) + W(s) = 0$

Coroll.1 Under the hypothesis of Theorem 5, $S(s)$ is LBR if and only if $W(s)$ is LPR.

(^) for this purpose, the point at infinity is considered pure imaginary

II.2 The scattering matrix

The concepts we introduce here in a purely formal way have an interesting physical counterpart in electrical network theory.
However, since our aim is to give some definitions and to derive some abstract results we will need in the investigation of dissipative linear systems, we do not give a concrete interpretation to the notion of scattering matrix, and we refer to [C.3] for its physical meaning.

Let Σ be a dynamical system with m inputs and m outputs, and consider the following vectors

$$\begin{aligned} v_i &= \frac{1}{2}(u+y) \qquad \text{(incident wave)} \\ v_r &= \frac{1}{2}(u-y) \qquad \text{(reflected wave)} \end{aligned} \tag{2.1}$$

The supply function of Σ

$$w = u^T y = v_i^T v_i - v_r^T v_r$$

is represented as the difference between the square norm of the incident wave and the square norm of the reflected wave.
If Σ is a (generalized) linear system and the initial conditions are zero the relation between the L-transforms of u and y can be expressed by the transfer matrix W(s):

$$Y(s) = W(s)U(s) \tag{2.2}$$

Under the same assumptions, one could expect that a completely similar relation holds, when the Laplace transforms $V_i(s)$ and $V_r(s)$ of v_i and v_r respectively are considered. As a matter of fact if the matrix $\mathbb{1}+W(s)$ admits an inverse, the "scattering matrix"

$$S(s) = (1-W(s))(1+W(s))^{-1} \tag{2.3}$$

allows us to write

$$V_r(s) = S(s)V_i(s)$$

The following theorem introduces a class of transfer matrices which admit associated scattering matrices.

Theor.1 Let $W(s) = D_1 s + D_o + C(sI-A)^{-1}B$ be the transfer matrix of a given GLS $\Sigma = (A,B,C,D_o,D_1)$. Then

i) if $W(s)$ is P.R., Σ admits a BR scattering matrix

ii) if Σ admits a B.R. scattering matrix, $W(s)$ is P.R.

The proof is a trivial consequence of sec.1, Theorem 4. □

The possibility of resorting to scattering matrices in the analysis of dissipative linear systems is the cornerstone of the next sections. This is a consequence of the following

Theor.2 Let $\Sigma = (A,B,C,D_o \ldots D_d)$ be a GLS with m outputs and m outputs. If the inequality

$$\int_{0}^{t_1} u(t)^T y(t)dt \geq 0$$
$$0 \rightarrow$$

holds for any $t_1 \geq 0$ and any input $u \in \mathscr{U}$ such that $u(0) = u'(0) = \ldots$
$\ldots = u^{(d-1)}(0) = 0$, then Σ admits a scattering matrix $S(s)$.

proof. Since (2.2) gives the relation between scattering matrix and transfer matrix, $S(s)$ is well defined if and only if $1+W(s)$ is an invertible matrix. If $1+W(s)$ does not admit an inverse, there is a column vector $U(s) \neq 0$, whose elements are real rational functions, which satisfies

$$(W(s) + \mathbb{1})U(s) = 0 \tag{2.4}$$

Is it no restriction to suppose that

$$\lim_{s \to \infty} s^d U(s) = 0$$

Then $u(t) = \mathscr{L}^{-1}(U(s))$ is zero for $t = 0$ as well as its first d-1 derivatives. We therefore have

$$\begin{aligned} W(s)U(s) &= -U(s) \\ y(t) = \mathscr{L}^{-1}(W(s)) &= -\mathscr{L}^{-1}(U(s)) = -u(t) \end{aligned} \tag{2.5}$$

Since u is not the zero input, an instant $t_1 > 0$ can be choosen in such a way that

$$\int_{0}^{t_1} u^T y \, dt = \int_{0}^{t_1} u^T(-u)dt < 0$$
$$0 \rightarrow$$

contrary to the assumption.

□

II.3 Dissipativity conditions in the s-domain

In this section our primary concern will be with the conditions the dissipativity assumption imposes on the transfer matrices of a linear system $\Sigma = (A,B,C,D_o)$. The key result is well known, and says that scattering matrix of Σ has to be B.R. Some related results are also quoted; in particular sufficient conditions for dissipativity, based on the controllability assumption, are proved. It should be clear that the extension to GLSs is devoid of essential difficulties.

Let now $\Sigma = (A,B,C,D_o)$ accept complex valued input functions and give out complex valued output function. The underlying assumption is that $\mathrm{Re}\, y$ and $\mathrm{Im}\, y$ correspond respectively to the real inputs $\mathrm{Re}\, u$ and $\mathrm{Im}\, u$, for zero initial state.

Lemmas 1 and 2 constitute the bridge for connecting real input and complex input results. Their proofs are quite simple and will be omitted .

Lemma 1 Suppose that for any real input u we have

$$\int_0^{t_1} u^T(t)y(t)\,dt \underset{0 \to}{\geq} 0 \tag{3.1}$$

Then for any complex valued input u

$$\mathrm{Re} \int_0^{t_1} \check{u}^T(t)y(t)\,dt \underset{0 \to}{\geq} 0 \tag{3.2}$$

and conversely.

Lemma 2 Suppose that for any complex valued input u we have

$$\mathrm{Re} \int_0^{t_1} \check{u}^T(t)y(t)\,dt \underset{0 \to}{\geq} 0 \tag{3.3}$$

Then the incident and the reflected waves satisfy

$$\int_0^{t_1} (\check{v}_i^T v_i - \check{v}_r^T v_r)\,dt \underset{0 \to}{\geq} 0 \tag{3.4}$$

and conversely.

Theor.1 Let $\Sigma = (A,B,C,D_o)$ be dissipative. Then its scattering matrix S(s) is B.R.

Proof. Dissipativity implies

$$\int_0^{t_1} \langle u^T y \rangle\, dt \underset{0 \to}{\geq} 0$$

which guarantees the existence of S(s). The theorem splits in two propositions:

i) S(s) analytic for $\mathrm{Re}(s) > 0$

ii) $\mathbb{1}-\check{S}^T(s)S(s)$ is nonnegative definite hermitian for $\mathrm{Re}(s) > 0$

We first prove the assertion about analyticity. By lemma 2

$$\int_0^{t_1} \check{v}_i^T v_i - \check{v}_r^T v_r \, dt \geq 0 \qquad t_1 \geq 0$$
$$0 \to$$

holds for any incident wave v_i. If v_i is in $(L^2 \cap \mathscr{D}_0)^m$ (^) one gets

$$\infty > \int_0^{\infty} \check{v}_i^T \, v_i dt \geq \int_0^{\infty} \check{v}_r^T v_r dt \geq 0 \tag{3.5}$$
$$0 \to \qquad 0 \to$$

showing that v_r belongs to the same space. Now the L-transforms of the functions in $L^2 \cap \mathscr{D}_0$ are analytic for $\mathrm{Re}(s) > 0$ and there exist functions in $L^2 \cap \mathscr{D}_0$ whose L-transform has no zeros in $\mathrm{Re}(s) > 0$ (see for instance [N.1]). From

$$V_r(s) = S(s)V_i(s) \tag{3.6}$$

the (j,k) entry of S(s) is given by

$$\left[S(s)\right]_{jk} = \left[V_r(s)\right]_j \Big/ \left[V_i(s)\right]_k \Bigg|_{\left[V_i(s)\right]_h = 0, \quad h \neq k} \tag{3.7}$$

Then point i) is proved when considering $\left[V_i(s)\right]_k$ devoid of zeros for $\mathrm{Re}(s) > 0$.

We now come to the assertion concerning non negative definiteness of $\mathbb{1} - \check{S}^T(s)S(s)$ for $\mathrm{Re}(s) > 0$. The proof resorts to a family of complex valued inputs.

Let $s_0 = \sigma_0 + j\omega_0$, $\sigma_0 > 0$, and let $z \in \mathbb{C}^m$. The complex incident wave

$$v_i(t) = z \, e^{s_0(t-\tau)} \pi_{[0,\tau]} \tag{3.8}$$

(where $\pi_{[0,\tau]}$ denotes the unitary rectangular impulse with support $[0,\tau]$) has the L-transform

---

(^) L^2 denotes the space of square integrable functions and $\mathscr{D}_0$ the space of regular functions whose support is included in $[0,+\infty)$.

$$V_i(s) = \frac{e^{-s_o\tau} - e^{-s\tau}}{s-s_o} z \tag{3.9}$$

so that the L-transform of the reflected wave (Σ is in the zero state at time zero) is

$$V_r(s) = S(s)z\,\frac{e^{-s_o\tau} - e^{-s\tau}}{s-s_o} \tag{3.10}$$

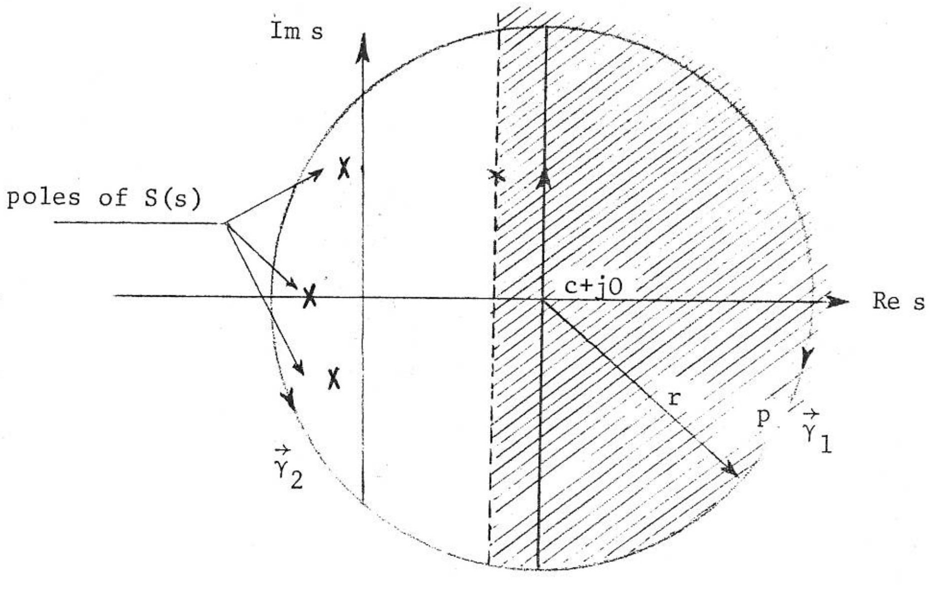
Let us apply the residue formula to obtain $\mathscr{L}^{-1}(V_r(s))$. First observe that $V_r(s)$ is analytic in $\mathrm{Re}(s-s_o) > 0$, so that we get $v_r(t)$ by integrating along a stright line in $\mathrm{Re}(s-s_o) > 0$:

$$v_r(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} V_r(s)e^{st}ds \tag{3.11}$$

For $t < \tau$ (3.11) splits as follows

$$v_r(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{S(s)z}{s-s_o}\, e^{st-s_o\tau}ds - \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{S(s)z}{s-s_o}\, e^{s(t-\tau)}ds \tag{3.12}$$

Now consider a circle centered in (c,0) and denote by $\vec{\gamma}_1$ and $\vec{\gamma}_2$ its semicircles in $\mathrm{Re}(s-c) > 0$ and $\mathrm{Re}(s-c) < 0$ respectively. As the radium r of the circle goes to infinity, one gets



$$\int_{\vec{\gamma}_1} \frac{S(s)z}{s-s_o}\, e^{s(t-\tau)}ds \to 0 \tag{3.13}$$

$$\int_{\vec{\gamma}_2} \frac{S(s)z}{s-s_o}\, e^{st-s_o}\, dt \to 0 \tag{3.14}$$

In fact if f is a function continuous for $|s|>R$, and we assume that

$$\lim_{|s|\to\infty} f(s) = 0$$

denoting by σ_r the semicircle of radius $r>R$ centered in (0,0) and contained in $\mathrm{Re}(s)\geq 0$ the integral $\int_{\sigma_r} f(s)e^{-s}ds$ goes to zero as $r\to\infty$ [C.1].

An easy application of residue method and formula (3.13) to the function $\{S(s)z/s-s_o\}e^{s(t-\tau)}$ give:

$$\begin{aligned} 0 &= \int_{c-j\infty}^{c+j\infty} \frac{S(s)z}{s-s_o} e^{s(t-\tau)}ds + \int_{\vec{\gamma}_1} \frac{S(s)z}{s-s_o} e^{s(t-\tau)}ds = \\ &= \int_{c-j\infty}^{c+j\infty} \frac{S(s)z}{s-s_o} e^{s(t-\tau)}ds = 0 \end{aligned} \tag{3.15}$$

This shows that the second integral in (3.12) is zero. For evaluating the first integral, we apply the residue method and formula (3.14) to the function: $\{S(s)z/s-s_o\}\ e^{st-s_o\tau}$.

$$\begin{aligned} v_r(t) &= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{S(s)z}{s-s_o} e^{st-s_o\tau}ds + \frac{1}{2\pi j}\int_{\vec{\gamma}_2} \frac{S(s)z}{s-s_o} e^{st-s_o\tau}ds = \\ &= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{S(s)z}{s-s_o} e^{st-s_o\tau}\ ds = \Sigma\,\mathrm{Res}\left(\frac{S(s)z}{s-s_o} e^{st-s_o\tau}\right) \end{aligned} \tag{3.16}$$

where the sum is taken over all poles of $\{S(s)z/s-s_o\}e^{st-s_o\tau}$ contained in $\mathrm{Re}\ s<c$. Let us split this sum as follows

$$v_r(t) = S(s_o)z\ e^{s_o(t-\tau)} + e^{-s_o\tau} \sum_{s_i\neq s_o} K_i z\ e^{s_i t} \tag{3.17}$$

where $K_i\,(i\neq 0)$ denote the residue of $S(s)/(s-s_o)$ in the pole $s_i\neq s_o$. Since $\mathrm{Re}\ s_i<\mathrm{Re}\ s_o$ for every $i\neq 0$, (3.17) implies

$$\begin{aligned} 0 &\leq \int_0^\tau \check{v}^T v_i - \check{v}_r^T v_r dt = \\ &= \int_0^\tau \check{z}^T z\ e^{2\mathrm{Re}s_o(t-\tau)}dt + \int_0^\tau \check{z}^T\check{S}^T(s_o)S(s_o)z\ e^{2\mathrm{Re}s_o(t-\tau)}dt + \\ &+ \int_0^\tau e^{-2\mathrm{Re}s_o}\Big(\sum_{ij\neq 0} \check{z}^T\check{K}_i^T K_j z\ e^{(\check{s}_i+s_j)t} + \sum_{i\neq 0} \check{z}^T\check{K}_i^T S(s_o)z\ e^{(s_o+\check{s}_i)t} + \\ &+ \sum_{i\neq 0} \check{z}^T\check{S}^T(s_o)K_i z\ e^{(\check{s}_o+s_i)t}\Big)dt \end{aligned} \tag{3.18}$$

As τ becomes large, third integral in (3.18) goes to zero, whence

$$0 \leq \lim_{\tau\to+\infty} \int_0^\tau \check{v}_i^T v_i - \check{v}_r^T v_r dt =$$

$$= \lim_{\tau\to+\infty} \check{z}^T(1 - \check{S}^T(s_o)S(s_o))z \cdot \int_0^\tau e^{2\mathrm{Re}s_o(t-\tau)} dt =$$

$$= \lim_{\tau\to+\infty} \check{z}^T(1 - \check{S}^T(s_o)S(s_o))z \frac{1-e^{-2\mathrm{Re}s_o\tau}}{2\mathrm{Re}s_o} =$$

$$= \check{z}^T(1 - \check{S}^T(s_o)S(s_o))\check{z} \frac{1}{2\mathrm{Re}s_o}$$

Since z is chosen arbitrary in $\mathbb{C}^m$, $1-\check{S}^T(s_o)S(s_o)$ is nonnegative definite hermitian.

$\square$

Theor.2 Let the scattering matrix $S(s)$ of $\Sigma = (A,B,C,D_o)$ be B.R. Then

$$\int_{0\to}^\tau \check{v}_i^T v_i - \check{v}_r^T v_r dt \geq 0, \quad \forall v_i, \ \forall \tau \geq 0.$$

proof. Recall that

(i) the Fourier transform of an L^2 function is an L^2 function [T.1]

(ii) the product of an L^2 function and a bounded locally integrable function is an L^2 function

(iii) $S(s)$ is analytic and bounded for any purely imaginary s.

Clearly, if v_i belongs to L^2, then $S(j\omega)V_i(j\omega) = V_r(j\omega)$ is in L^2, whence $v_r \in L^2$. Since L^2 functions satisfy Parseval equality

$$\int_{-\infty}^{+\infty} \check{f}^T f \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathscr{F}(\check{f})^T \mathscr{F}(f) d\omega \qquad f \in L^2 \tag{3.20}$$

assuming $v_i \in L^2$, $v_i(t) = 0 \ \forall t < 0$, the integral of the supply is expressed in the F-transform domain

$$\int_0^{+\infty} (\check{v}_i^T v_i - \check{v}_r^T v_r) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \check{V}_i^T(j\omega)(1-\check{S}^T(j\omega)S(j\omega))V_i(j\omega) d\omega \geqslant 0 \tag{3.21}$$

The last inequality is a consequence at the B.R. property of $S(s)$:

$$\check{v}^T(1 - \check{S}^T(j\omega)S(j\omega))v \geq 0 \qquad \forall v \in \mathbb{C}^m$$

Observe now that for any $\tau \geq 0$ the value of

$$\int_0^{\tau} \check{v}_i^T v_i - \check{v}_r^T v_r \, dt \Big|_{0\to}$$

does not depend on the behaviour of $v_i(\sigma)$ for $\sigma > \tau$. An incident wave v_i with support in $[0,\tau]$ is an element of L^2. We conclude that

$$0 \leq \int_0^{\tau} \check{v}_i^T v_i - \check{v}_r^T v_r \, dt \Big|_{0\to} + \int_{\tau}^{+\infty} - \check{v}_r^T v_r \, dt \Big|_{0\to} \leq \int_0^{\tau} \check{v}_i^T v_i - v_r^T v_r \, dt \Big|_{0\to} \tag{3.22}$$

□

As an immediate corollary from Theorem 2,

Coroll.1 *Let $\Sigma = (A,B,C,D_o)$ be completely controllable and assume Σ admits a B.R. scattering matrix. Then Σ is dissipative.* □

Summarizing the results above, we can say that dissipativity implies the existence of a BR scattering matrix, whereas a BR scattering matrix requires the controllability extra condition in ensuring dissipativity.

In section 1 we stated that a PR transfer function and the existence of a BR scattering matrix are equivalent facts. This allows us to easily reformulate. Thms 1, 2 and Coroll. 1 in terms of transfer matrices. Theorem 3 is a classic result in passivity theory.

Theor.3 *Let $\Sigma = (A,B,C,D_o)$ $(\Sigma = (A,B,C,D_o,\ldots D_d))$ be a completely controllable LS (GLS). Then the following propositions are equivalent:*

i) *Σ is dissipative*
ii) *W(s) is P.R.*
iii) *Σ admits a BR scattering matrix* □

If the controllability assumption does not hold the implications are as follows: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). In other words matrices W(s) and S(s), which characterize the input/output map, are unsuitable for testing dissipativity. When dealing with minimal realizations, the dissipativity conditions can be restated as follows:

Theor.4 *Let $\Sigma = (A,B,C,D_o)$ be a minimal realization of its transfer matrix W(s). Then Σ is dissipative if and only if the following conditions simultaneously hold:*

(i) Re $\lambda(A) \leq 0$

(ii) the purely imaginary poles of $W(s)$ are simple poles and the residue matrices are nonnegative definite Hermitian

(iii) $\check{W}^T(j\omega) + W(j\omega) \geq 0$ for all real ω, with $j\omega$ not a pole of $W(s)$.

□

II.4 s-domain lossless dissipativity conditions

The purpose of this section is the introduction of LBR and LPR matrices in the dissipativity analysis. As we shall show, LBR and LPR matrices play the same role in lossless linear systems as BR and PR matrices in general dissipative linear systems.

Theor.1 Assume an LS Σ to be lossless dissipative. Then its diffusion matrix $S(s)$ is LBR.

proof. Assume that an incident wave v_i induces a cyclic trajectory in the state space whose origin at $t=0$ and end at $t=t_1$ are both the zero state. If $v_i(t)=0$ for $t>t_1$, Parseval theorem and Theorem 5 in II.1 imply

$$0=\oint_{0\to 0} w(t)dt = \int_{-\infty \atop 0\to}^{+\infty} (\check{v}_i^T v_i - \check{v}_r^T v_r)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \check{V}_i^T(j\omega)(1-\check{S}^T(j\omega)S(j\omega))V_i(j\omega)d\omega \tag{4.1}$$

Now, for any z in $\mathbb{C}^m$ and any $\bar{\omega}$ in $\mathbb{R}$ there exists an incident wave v_i with support in $[0,t_1]$ which gives this kind of trajectory and satisfies the F-transform condition

$$\mathscr{F}(v_i)\Big|_{\omega=\bar{\omega}} = V_i(j\bar{\omega}) = z \tag{4.2}$$

Suppose the matrix $\mathbb{1} - \check{S}^T(j\bar{\omega})S(j\bar{\omega})$ is different from zero. By (4.2) there exists an incident wave v_i whose F-transform gives

$$\check{V}_i^T(j\bar{\omega})\left[1 - \check{S}^T(j\bar{\omega})S(j\bar{\omega})\right]V_i(j\bar{\omega}) \underset{\neq}{\geq} 0$$

Hence the integral (4.1) cannot be zero, a contradiction. The LBR property of $S(s)$ follows from BR property and

$$1 - \check{S}^T(j\omega)S(j\omega) = 0$$

□

By resorting to the results we proved in II.1 and II.3 we obtain directly theorem 2.

Theor.2 Let $\Sigma = (A,B,C,D_o)$ $(\Sigma = (A,B,C,D_o,D_1))$ be a completely controllable LS (GLS). Then the following propositions are equivalent:

i) Σ is lossless dissipative

ii) $W(s)$ is LPR

iii) the scattering matrix $S(s)$ exists and is LBR $\square$

III ALGEBRAIC STRUCTURE OF DISSIPATIVE LINEAR SYSTEMS

The results we quoted in chapter II restrict the class of transfer matrices which are realizable by dissipative (G)LS to the class of PR matrices. Moreover any controllable realization of these matrices is a dissipative system. This gives a rather satisfactory picture of the connections between the input/output behaviour and the "internal" dissipativity constraint.

The next chapter is devoted to examining closely several questions connected with the structure of system matrices A, B, C, D_o and their relationship with energy functions. We shall confine ourselves to standard linear systems: the extensions to GLS are quite obvious.

III.1 Positive real lemma and dissipation function

In section 3 of ch.1 we derived a linear matrix inequality which must be fulfilled by the A, B, C, D_o matrices of a dissipative linear system. This condition can be restated in a different form, involving a quadratic matrix inequality.

Theorem 1 gives a complete picture of the situation.

Theor.1 [W.2, A.1] For any $\Sigma = (A,B,C,D_o)$ the following are equivalent

(o) Σ is dissipative

(i) the set of matrices

$$\mathcal{S}_1 = \{\Pi : \Pi \geq 0,\ x^T \frac{\Pi}{2} x \text{ is an energy function of } \Sigma\}$$

is not empty

ii) the set $\mathscr{S}_2$ of nonnegative definite solutions of the inequality

$$\begin{bmatrix} D_o + D_o^T & C - B^T K \\ C^T - KB & -A^T K - KA \end{bmatrix} \geq 0 \tag{1.1}$$

is not empty

iii) the set $\mathscr{S}_3$ of the triples (Π, H, J), $\Pi \geq 0$ which satisfy the equalities

$$\begin{aligned} \Pi A + A^T \Pi &= -H^T H \\ \Pi B &= C^T - H^T J \\ J^T J &= D_o + D_o^T \end{aligned} \tag{1.2}$$

is not empty

iv) if $D_o + D_o^T$ is a unit, the set $\mathscr{S}_4$ of nonnegative definite solutions of

$$KA + A^T K + (KB - C^T)(D_o + D_o^T)^{-1}(B^T K - C) \leq 0 \tag{1.3}$$

is not empty; if $D_o + D_o^T$ is not a unit, the set $\mathscr{S}_4' = \lim_{\varepsilon \to 0^+} \mathscr{S}_{4\varepsilon}$ is not empty. $\mathscr{S}_{4\varepsilon}$ denotes the set of nonnegative definite solutions of

$$KA + A^T K + (KB - C^T)(D_o + D_o^T + \varepsilon \mathbb{1}_m)(B^T K - C) \leq 0 \tag{1.4}$$

The sets $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_3$ and $\mathscr{S}_4$ (or $\mathscr{S}_4'$) coincide.

proof. The equivalence of (o) with (i) and (ii) and the equality $\mathscr{S}_1 = \mathscr{S}_2$ have been proven in Ch. 1. Assume now $\Pi \geq 0$ is a solution of (1.1). The following factorization holds

$$\begin{bmatrix} D_o + D_o^T & C - B^T \Pi \\ C^T - \Pi B & -\Pi A - A^T \Pi \end{bmatrix} = \begin{bmatrix} J^T \\ H^T \end{bmatrix} \begin{bmatrix} J & H \end{bmatrix} \tag{1.5}$$

because the left hand side is nonnegative definite and this gives the implication ii)$\Rightarrow$iii). Implication iii)$\Rightarrow$ii) is trivial, and $\mathscr{S}_2 = \mathscr{S}_3$ follows immediately. Let now $D_o + D_o^T$ have full rank and (Π, H, J), $\Pi \geq 0$ satisfy (1.3). Since the spectrum of $J(J^T J)^{-1} J^T$ only constains the eigenvalues 0 and 1, $\mathbb{1}_m - J(J^TJ)^{-1}J^T$ is nonnegative definite and

$$H^TJ(J^TJ)^{-1}J^TH \leq H^TH$$

implies (iv):

$$\Pi A + A^T\Pi = -H^TH \leq -H^TJ(J^TJ)^{-1}J^TH = -(\Pi B - C^T)(D_o + D_o^T)^{-1}(B^T\Pi - C^T)$$

Conversely, let $\Pi \geq 0$ satisfy inequality (1.4). Then there exists a matrix N such that

$$\Pi A + A^T\Pi + (\Pi B - C^T)(D_o + D_o^T)^{-1}(B^T\Pi - C) = -N^TN$$

holds. (1.3) is proven by introducing the matrices

$$H^T = \left[-(\Pi B - C^T)(D_o + D_o^T)^{-1/2} \;\vdots\; N^T \right] \tag{1.6}$$

$$J = \left[(D_o + D_o^T)^{1/2} \;\vdots\; 0 \right] \tag{1.7}$$

As an easy consequence, when $D_o + D_o^T$ is a unit, the sets $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ coincide.

Finally assume iii) hold and let $D_o + D_o^T$ does not have full rank. For any $\varepsilon > 0$, $D_o + D_o^T + \varepsilon \mathbb{1}_m$ is full rank, so the system $(A, B, C, D_o + \frac{\varepsilon}{2}\mathbb{1}_m)$ (which is dissipative whenever Σ is) gives a non empty set $\mathcal{S}_{2\varepsilon}$ of nonnegative solutions of the inequality

$$0 \leq \begin{bmatrix} D_o + D_o^T + \mathbb{1}_m & C - B^TK \\ C^T - KB & -A^TK - KA \end{bmatrix} = \begin{bmatrix} \varepsilon \mathbb{1}_m & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D_o + D_o^T & C - B^TK \\ C^T - KB & -A^TK - KA \end{bmatrix},$$

Planly

$$\mathcal{S}_{2\varepsilon} = \mathcal{S}_{4\varepsilon} = \mathcal{S}_{3\varepsilon} \tag{1.8}$$

for any $\varepsilon > 0$, and $\mathcal{S}_{2\varepsilon}$ is an ε-indexed family of sets which monotonically decreases as $\varepsilon \to 0_+$.

Since the non empty set $\mathcal{S}_2$ is the intersection of the family

$$\mathcal{S}_2 = \lim_{\varepsilon \to 0^+} \mathcal{S}_{2\varepsilon} = \bigcap_{\varepsilon > 0} \mathcal{S}_{2\varepsilon} \tag{1.9}$$

we get from (1.8)

$$\mathcal{S}'_4 \triangleq \lim_{\varepsilon\to 0^+} \mathcal{S}_{4\varepsilon} = \lim_{\varepsilon\to 0^+} \mathcal{S}_{2\varepsilon} = \mathcal{S}_2 = \mathcal{S}_3$$

This gives also the converse (iv)$\Rightarrow$(iii).

The equivalence (o)$\Leftrightarrow$(iii) is usually called the Positive Real Lemma [A.1]. As we showed above, the matrix Π in any solution (Π,H,J) of the P.R. lemma is associated to an energy function

$$S(x) \triangleq x^T \frac{\Pi}{2} x$$

of the system. It is natural to ask whether we can attach some significa - tion to H and J matrices. To this purpose, define

$$d(x,u) \triangleq \frac{1}{2} \|Ju + Hx\|^2 = \frac{1}{2} (Ju + Hx)^T (Ju + Hx) \tag{1.10}$$

and observe that

$$w(u,y) = u^T y = \frac{d}{dt} (x^T \frac{\Pi}{2} x) + d(x,u) \tag{1.11}$$

along the solutions of the system equation $\dot{x} = Ax + Bu$. Hence $d(x,u)dt$ represents the irreversibly dissipated part of the supply $w(u,y)dt$ the system receives, i.e. the part the system is unable to store as internal energy. We will call $d(x,u)$ the <u>dissipation function</u>.

If $D_o + D_o^T$ has full rank, we have

$$d(x,u) \geq \frac{1}{2} \|u - R^{-1}(B^T\Pi - C)x\|_R^2 \tag{1.12}$$

by (1.3). Moreover, if Π satisfies (1.3) as an equality, then (1.12) too is satisfied as an equality and along the trajectory which correspond to

$$u - R^{-1}(B^T\Pi - C)\, x = 0$$

the input supply completely changes in internal energy. As it should be clear, this do not imply a lossless property. In fact the lossless condition

$$d(u,x)dt = 0 \tag{1.13}$$

should hold along all system trajectories.

Let us denote by $r \times m$ and $r \times n$ the dimensions of the matrices J and H respectively. Actually there is no upper bound on the integer r, whereas a lower bound is implicit in the rank of the (1.5) left hand side. We will consider again this problem: here we confine ourselves to a simple remark. If $D_o + D_o^T$ is full rank then $r \geq m$, and the bound m is attained if and only if Π satisfies (1.4) as an equality.

We conclude this section by outlinig some interesting relations which connect dissipativity, controllability, observability and the spectrum of the A matrix.

i) The A matrix of a dissipative linear observable system $\Sigma = (A,B,C,D_o)$ is simply stable (in the sense that all elements in exp At are bounded as $t \to +\infty$). More generally, simple internal stability is implied by the existence of a positive definite solution of the P.R. lemma .

ii) In some dissipative (even if controllable) linear systems there are eigenvalues of the A matrix with positive real part. It is sufficient to consider a controllable but not observable realization of a P.R. matrix W(s), in its Kalman canonical observability from

$$\Sigma_W = \left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} C, & 0 \end{bmatrix}, D_o \right)$$

There is no constraint on the spectrum of A_{22}.

iii) Assume the system $\Sigma = (A,B,C,D_o)$ is dissipative and the matrix $R = D_o + D_o^T$ is full rank.

- if Σ admits a positive definite energy function, $\hat{A} = A - BR^{-1}C$ is simply stable
- if moreover Σ is controllable and/or observable, $\hat{A}$ is asymptotically stable.

iv) If $\Sigma = (A, B, C, D_o)$ admits some definite positive energy function, even its dual system is dissipative. In particular the dual system of an observable dissipative linear system is dissipative.

III.2 Limit solutions of the P.R. lemma

Several equivalent dissipativity conditions have been proven in sec. 1, which relate the non negative definite solutions of matrix inequalities to the possible energy functions of a systems $\Sigma = (A,B,C,D_o)$. Our purpose now is to single out explicitely the minimal and the maximal solution of (1.1) (S_d and $S_{r,o}$ respectively) under the hypothesis that Σ is dissipative.

AVAILABLE STORAGE

As it is known, the available storage is a quadratic non negative definite function

$$S_d(x) = -\inf_{\substack{u \\ t_1 \geq 0}} \int_{0}^{t1} u^T y \, dt = x^T \frac{\Pi_d}{2} x \quad x \to \tag{2.1}$$

The matrix Π_d is the minimal non negative definite solution of (1.1) (or (1.2)). Assuming $R = D_o + D_o^T$ to be full rank, Π_d is even the minimal non neg. solution of (1.3) inequality

$$KA + A^T K + (KB-C^T)R-1(B^T K-C) \leq 0$$

Whe want to show that Π_d is the minimal non negative solution of the associated equation

$$KA + A^T K + (KB-C^T)R^{-1}(B^T K-C) = 0 \tag{2.2}$$

Observe that (2.2) can be arranged as

$$K(A-BR^{-1}C) + (A-BR^{-1}C)^T K + KBR^{-1}B^T K + C^T R^{-1} C = 0 \tag{2.3}$$

This clearly shows that (2.2) is a Riccati algebraic equation.

Theor.1 Assume $\Sigma = (A,B,C,D_o)$ be dissipative and $R = D_o + D_o^T$ be full rank. For any K_1, $0 \leq K_1 \leq \Pi_d$ the solution $\Pi(t,-K_1,0)$ of the differential Riccati equation

$$\begin{aligned} \frac{dK}{dt} &= -\hat{A}^T K - K\hat{A} + KBR^{-1}B^T K - \hat{Q} \\ \hat{A} &= A - BR^{-1}C, \quad \hat{Q} = -C^T R^{-1} C. \end{aligned} \tag{2.4}$$

converges to $-\Pi_d$ when $t \to -\infty$:

$$\lim_{t \to -\infty} \Pi(t, -K_1, 0) = -\Pi_d$$

proof. First introduce the following function $h: X \times \mathbb{R}_+ \to \mathbb{R}$:

$$h(x_o, t_1) = \inf_u \left(\int_o^{t_1} 2u^T y \, dt - x^T(t_1) K_1 x(t_1) \right)_{x_o \to} \tag{2.5}$$

The inequality chain

$$\begin{aligned} &\int_0^{t_1} 2u^T y \, dt - x^T(t_1) K_1 x(t_1) \geq \\ &\int_0^{t_1} 2u^T y \, dt - x^T(t_1) \Pi_d x(t_1) \geq -x_o^T \Pi_d x_o \end{aligned} \quad (x_o \to) \tag{2.6}$$

is an immediate consequence of our choice of K_1, and shows that $h(x_o, t_1)$ is bounded from below by $-x_o^T \Pi_d x_o$ for any $t_1 \geq 0$.

From the S_d definition, given $\varepsilon > 0$, there exists T_ε such that whenever $t_1 > T$ and the input is suitably chosen we have

$$\int_0^{t_1} 2u^T y \, dt \leq -x_o^T \Pi_d x_o + \varepsilon \quad (x_o \to)$$

and consequently

$$\begin{aligned} -x_o^T \Pi_d x_o &\leq \int_0^{t_1} 2u^T y \, dt - x^T(t_1) K_1 x(t_1) \leq -x_o^T \Pi_d x_o - x^T(t_1) K_1 x(t_1) + \varepsilon \\ &\leq -x_o^T \Pi_d X_o + \varepsilon \end{aligned}$$

Taking the limit as $t_1 \to +\infty$ in

$$-x_o^T \Pi_d x_o \leq h(x_o, t_1) \leq -x_o^T \Pi_d x_o + \varepsilon$$

then

$$\lim_{t_1 \to +\infty} h(x_o, t_1) = -x_o^T \Pi_d x_o \tag{2.7}$$

Next substitute $y = Cx + D_o u$ in (2.5). Thus we reduce to the optimal control problem

$$h(x_o, t_1) = \inf_u \int_0^{t_1} \begin{bmatrix} u^T x^T \end{bmatrix} \begin{bmatrix} R & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} dt \tag{2.8}$$

As it is known, problem (2.8) is equivalent to the following standard form:

$$\inf_{\hat{u}} \int_0^{t_1} \begin{bmatrix} \hat{u}^T x^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -C^T R^{-1} C \end{bmatrix} \begin{bmatrix} \hat{u} \\ x \end{bmatrix} dt \qquad (2.9)$$

$$\dot{x} = (A - BR^{-1}C)x + B\,\hat{u} = \hat{A}x + B\hat{u}$$

From (2.6) we get that problem (2.9) admits a finite solution for any $t_1 \geq 0$. Therefore the solution $\tilde{\Pi}(t,-K_1,t_1)$ of the Riccati equation (2.4) exists for any interval $[0,t_1]$, and we can express $h(x_o,t_1)$ in the form

$$h(x_o,t_1) = x_o^T \Pi(0,-K_1,t_1)x \qquad (2.10)$$

Finally we observe that

$$\lim_{t_1 \to +\infty} x_o^T \Pi(0,-K_1,t_1)x_o = -x_o^T \Pi_d x_o \qquad (2.11)$$

holds for any $x_o \in X$. This implies our conclusion

$$\lim_{t_1 \to +\infty} \Pi(0,-K_1,t_1) = \lim_{t \to -\infty} \Pi(t,-K_1,0) = -\Pi_d \qquad (2.12)$$

□

Coroll.1 Π_d <u>is the minimal nonnegative definite solution of the algebraic Riccati equation (2.2).</u>

□

This is proven by introducing $\Pi(t,-K_1,0)$ in (2.4) and taking the limits as $t \to -\infty$.

<u>Remark</u>. A triple of matrices which satisfies the P.R. lemma is easily obtained from Π_d. This requires only to factorize R

$$J^T J = R\,, \qquad J \in \mathbb{R}^{nxn} \qquad (2.13)$$

and construct H as follows

$$H = (J^T)^{-1}(B^T \Pi_d - C) \qquad (2.14)$$

If $D_o + D_o^T$ is not full rank, the Π_d matrix is no longer directly derived from an algebraic Riccati equation.

Denote by D_ε the matrix $D_o + \frac{\varepsilon}{2}\,\mathbb{I}$, $\varepsilon > 0$, and by $S_{d,\varepsilon}$ the available storage of the dissipative linear system $\Sigma_\varepsilon = (A,B,C,D_\varepsilon)$. $S_{d,\varepsilon}$ is a non negative quadratic form

$$S_{d,\varepsilon}(x) = \frac{1}{2} x^T \Pi_{d,\varepsilon} x \tag{2.15}$$

and $\Pi_{d,\varepsilon}$ is derived from a Riccati equation, because of the full rank of D_ε. Now, for any $\varepsilon_2 > \varepsilon_1 > 0$ we have

$$S_d(x) = \sup_{\substack{u \\ t_1 \geq 0}} - \int_0^{t_1} \underset{x\to}{u^T(Cx+D_o u)}\,dt \geq \sup_{\substack{u \\ t_1 \geq 0}} - \int_0^{t_1} \underset{x\to}{u^T(x+D_{\varepsilon_1} u)}\,dt =$$

$$= S_{d,\varepsilon_1}(x) \geq \sup_{\substack{u \\ t_1 \geq 0}} - \int_0^{t_1} \underset{x\to}{u^T(Cx+D_{\varepsilon_2} u)}\,dt = S_{d,\varepsilon_2}(x)$$

so that $S_{d,\varepsilon}(x)$ is increasing (for x fixed) as ε decreases. Since $S_{d,\varepsilon}(x)$ is bounded from above by $S_d(x)$, a matrix $\bar{\Pi} \geq 0$ exists which satisfies

$$S_d(x) \geq \lim_{\varepsilon\to 0^+} S_{d,\varepsilon}(x) = \lim_{\varepsilon\to 0^+} \frac{1}{2} \Pi_{d,\varepsilon} x = \frac{1}{2} x^T \bar{\Pi} x \tag{2.17}$$

If a strict inequality holds in (2.17), the strict inequality

$$-\int_0^{\bar{t}_1} \underset{x\to}{u^T(Cx+D_o u)}\,dt > \lim_{\varepsilon\to 0^+} S_{d,\varepsilon}(x)$$

would also hold for some u and some $\bar{t}_1 \geq 0$. Hence an interval $(0,\delta]$ would exist and a positif k such that

$$-\int_0^{\bar{t}_1} \underset{x\to}{u^T(Cx+D_o u)}\,dt > S_{d,\varepsilon}(x)+k$$

for any ε in $(0,\delta]$. Take $\varepsilon > 0$ small enough to get

$$\int_0^{\bar{t}_1} u^T \frac{\varepsilon}{2} u\,dt < k \qquad \varepsilon < \delta$$

We have

$$-\int_0^{\bar{t}_1} \underset{x\to}{u^T(Cx+D_\varepsilon u)}\,dt > S_{d,\varepsilon}(x)$$

whence

$$\sup_{\substack{u \\ t_1 \geq 0}} -\int_0^{t_1} u^T(Cx+D_\varepsilon u)dt > S_{d,\varepsilon}(x) \;,$$
$$x\to$$

a contradiction. This gives the equality

$$S_d(x) = \frac{1}{2}\, x^T \Pi_d x = \lim_{\varepsilon\to 0^+} \frac{1}{2}\, x^T \Pi_{d,\varepsilon} x \tag{2.18}$$

so that Π_d is computable (at least in principle) as

$$\Pi_d = \lim_{\varepsilon\to 0^+} \Pi_{d,\varepsilon}$$

We have therefore proved the following.

Theor.2 <u>Assume $\Sigma = (A,B,C,D_o)$ be dissipative and $R = D_o + D_o^T$ be singular. For any $\varepsilon>0$, denote by $S_{d,\varepsilon}$ the available storage of the system $\Sigma_\varepsilon = (A,B,C,\ D_o + \frac{\varepsilon}{2}\, I)$. Then the available storage of Σ is given by</u>

$$S_d = \lim_{\varepsilon\to 0^+} S_{d,\varepsilon} \tag{2.19}$$

<u>Remark</u>. Theorem 2 is important essentially from a theoretic point of view. In fact in electric network synthesis the so called Foster preamble always allows to deal with PR matrix transfer functions which fulfill the condition $R > 0$.

REQUIRED SUPPLY

Assume the dissipative system $\Sigma = (A,B,C,D_o)$ to be controllable. Then the required supply (from zero state) is finite everywhere, and a nonnegative definite matrix Π_r exists such that

$$S_r(x) = \frac{1}{2}\, x^T \Pi_r x \tag{2.20}$$

Our purpose here is to show that Π_r satisfies (1.3) as an equality and represents the limit behaviour of a suitable solution of (2.4).

We associate first with the system $\Sigma = (A,B,C,D_o)$ the functional

$$\chi(u,x_1) = -\int_{t_o}^{t_1} \begin{bmatrix} u^T & x^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} dt + x^T(t_o) K_o x(t_o) \tag{2.21}$$

The integration is taken along the state space trajectories which end in x_1. Under the assumption $R > 0$, the differential Riccati equation

$$\dot{K} = -A^T K - KA - Q + KBR^{-1}B^T K \tag{2.22}$$

with the initial condition $K(t_o) = K_o$ will be the tool we shall use in maximizing the functional χ.

Lemma 1 <u>If equation (2.23) admits an integral curve $\Pi(t,k_o,t_o)$ on the time interval $[t_o,t_1]$, then</u>

$$\sup \chi(u,x_1) = x_1^T \Pi(t_1,K_o,t_o)x_1 \tag{2.24}$$

<u>Conversely, let the upper bound of χ exist finite for any t_1 in $[t_o,T]$. Then there exists an integral curve of (2.23) on the same interval, with initial condition K_o</u>

The proof relies on the subtraction of

$$0 = \int_{t_o}^{t_1} \left[u^T x^T\right] \begin{bmatrix} 0 & B^T \\ B & -Q+\Pi BR^{-1}B^T\Pi \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} dt - x^T \Pi x \Big|_{t_o}^{t_1} \tag{2.25}$$

from (2.22), which gives

$$\chi(u,x_1) = -\int_{t_o \to x_1}^{t_1} \left[u^T x^T\right] \begin{bmatrix} R & B^T\Pi \\ \Pi B & \Pi BR^{-1}B^T\Pi \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} dt + x_1^T \Pi(t_1,K_o,t_o)x_1 =$$

$$= -\int_{t_o}^{t_1} (u + R^{-1}B^T\Pi x)^T_{\to x_1} R(u + R^{-1}B^T\Pi x)dt + x_1^T\Pi(t_1,K_o,t_o)x_1$$

For the details one can refer to the analogous minimization problem in least squares theory [B.1] □

Theor.3 <u>Let $\Sigma = (A,B,C,D_o)$ be dissipative and controllable and let $R = D_o + D_o^T$ be non singular. Assume that the Riccati equation</u>

$$\frac{dK}{dt} = -\hat{A}^T K - K\hat{A} - \hat{Q} + KBR^{-1}B^T K$$

$$\hat{A} = A - BR^{-1}C, \quad \hat{Q} = -C^T R^{-1} C$$

<u>has to be solved with initial condition $K(0) = -K_o$ for some matrix $K_o \geq \Pi_r$. Then the solution $\Pi(t,-K_o,0)$ exists for any $t \geq 0$, and</u>

$$\lim_{t\to+\infty} -\Pi(t,-K_o,0) = -\Pi_r \tag{2.27}$$

proof. Let $g\colon X\times\mathbb{R}_+\to\mathbb{R}$ be the following map:

$$g(x_1,t_1) = \sup_u\left(-\int_{0\ \to x_1}^{t_1} 2u^Ty\,dt - x^T(0)K_o x(0)\right) \tag{2.28}$$

Since S_r is an energy function, the choice of K_o implies

$$-\int_{0\ \to x_1}^{t_1} 2u^Ty\,dt - x(0)^TK_ox(0) \le -\int_{0\ \to x_1}^{t_1} 2u^Ty\,dt - x(0)^T\Pi_r x(0) \le -x_1^T\Pi_r x_1 \tag{2.29}$$

Hence $g(x_1,t_1)$ is bounded from above by $-x_1^T\Pi_r x_1$, for any $t_1\ge 0$. From the definition of required supply, given $\varepsilon>0$, there exists T_ε such that if $t_1>T_\varepsilon$ and u is properly chosen then

$$-\int_{0\ \to x_1}^{t_1} 2u^Ty\,dt \ge -x_1^T\Pi_r x_1 - \varepsilon$$

Hence

$$-x_1^T\Pi_r x_1 \ge -\int_{0\ \ 0\to x_1}^{t_1} 2u^Ty\,dt - x^T(0)K_ox(0) \ge -x_1^T\Pi_r x_1 - \varepsilon$$

Since in

$$-x_1^T\Pi_r x_1 \ge g(x_1,t_1) \ge -x_1^T\Pi_r x_1 - \varepsilon$$

$\varepsilon>0$ is arbitrary, then

$$\lim_{t_1\to+\infty} g(x_1,t_1) = -x_1^T\Pi_r x_1 \tag{2.30}$$

Rewrite (2.28) in the following more suitable from:

$$g(x_1,t_1) = \sup_{\hat u}\left(-\int_0^{t_1} \begin{bmatrix}\hat u^T x^T\end{bmatrix}\begin{bmatrix}R & 0\\ 0 & -C^TR^{-1}C\end{bmatrix}\begin{bmatrix}\hat u\\ x\end{bmatrix}dt - x^T(0)K_ox(0)\right)$$

$$\dot x = (A-BR^{-1}C)x + B\hat u = \hat A x + B\hat u$$

Since the above upper bound is finite for any $t_1\ge 0$, the integral curve through the point $K(0) = -K_o$ of the differential equation

$$\dot K = -\hat A^TK - K\hat A - \hat Q + KBR^{-1}B^TK$$

exists for any $t_1\ge 0$. Furthermore it results

$$g(x_1,t_1) = x_1^T\Pi(t_1,-K_o,0)x_1 \tag{2.31}$$

Finally, taking the limit on both sides of (2.31) we have

$$-x_1^T \Pi_r x_1 = \lim_{t_1 \to +\infty} x_1^T \Pi(t_1, -K_o, 0) x_1 \tag{2.32}$$

thereby proving the thorem □

Coroll.2 Π_r is the maximum non negative definite solution of the algebraic Riccati equation (2.22)

Remark. The computation of Π_d and Π_r in general cannot be done direc - tly from limits (2.12) and (2.32). The alternate way, i.e. the solution of algebraic equation (2.2) is based on numerical techniques. If proper ly tuned, these techniques converge to Π_d or to Π_r.

III.3 Non real-reduced matrices and dissipative realizations

In this section we specialize our study to an interesting subclass of PR matrices, the so-called non real reduced matrices.
When a (dissipative) minimal realization is considered the inputs cor - responding to the available storage and the required supply are obtai - ned in closed form, and the energy functions set satisfies an interesting "gap" property.
As it is well known, a P.R. matrix W(s) fulfills the condition

$$W(j\omega) + W^T(-j\omega) \geq 0 \tag{3.1}$$

for all real ω such that $j\omega$ is not a pole of W(s). As a matter of fact it is easy to show that matrix (3.1) definition can be continuously extended to the imaginary poles of W(s) and to the point at infinity.
This in turn implies the following boundedness condition (see e.g. [F.1])

Theor.1 If W(s) is a P.R. matrix then there exists a real $k > 0$, such that

$$0 \leq W(j\omega) + W^T(-j\omega) \leq k\, \mathbb{1}_m \tag{3.2}$$

for all real ω. □

In general the left hand inequality (3.2) does not holds as strong inequality for all real ω: in other words, $\det[W(s)+W^T(-s)]$ possibly va nishes in some points along the imaginary axis (the point at infinity is considered an imaginary point). We want to exclude P.R. matrices $W(s)$ which exhibit the above mentioned behaviour. We therefore have the following.

Def.1 A P.R. matrix $W(s)$ is non real reduced if

$$W(j\omega) + W^T(-j\omega) > 0 \tag{3.3}$$

holds for all real ω, including the infinity point.

Remark i) (3.3) implies that $D_o + D_o^T$ is non singular ii) $W(s)$ is non real reduced iff $W^{-1}(s)$ is non real reduced.

Remark In same sense non real-reduced matrices are completely opposite to LPR matrices. This comes not only from relation (3.3), when compared with $W(j\omega) + W^T(-j\omega) = 0$ in the LPR, but also from the structure of the energy functions set. As we shall see, minimal realizations on non--real reduced matrices exhibit a positive definite difference $\Pi_r - \Pi_d$. This strongly contrasts with lossless minimal realizations, which have a unique energy function. It is interesting to point out that non real reduced matrices generalize the class of non minimal P.R. functions, which constitute the starting point of the Foster preamble in network synthesis.

Non real reduced matrices satisfy a lower bound condition which completes, in some sense, the more general upper bound condition we stated in Theorem 2.

Theor.2 Let $W(s)$ be a P.R. non real reduced matrix. Then there exists a real constant $h > 0$ such that

$$W(j\omega) + \check{W}^T(j\omega) \geq h\, \mathbb{1}_m$$

holds for all real ω, including the infinity point $\square$

Consider now a dissipative controllable system $\Sigma = (A,B,C,D_o)$ and assume $D_o + D_o^T = R$ to be non singular. Denote by Π_d and Π_r the minimum and the maximum non negative solutions of the Riccati equation

$$KA + A^TK + (KB - C^T)R^{-1}(B^TK - C) = 0 \tag{3.5}$$

and by A_d and A_r the following matrices

$$A_d \triangleq A - BR^{-1}C + BR^{-1}B^T\Pi_d \tag{3.6}$$

$$A_r \triangleq A - BR^{-1}C + BR^{-1}B^T\Pi_r \tag{3.7}$$

Theorem 3 and 4 connect the spectra of A_d and A_r with the "gap" $\Delta\Pi = \Pi_r - \Pi_d$ and with the non-real reduced property.

Theor.3 <u>Let</u> (A,B,C,D_o) <u>be controllable and dissipative and let</u> R <u>be full rank. Then the following facts are equivalent</u>

i) $\Delta\Pi > 0$

ii) $\mathrm{Re}\,\lambda(A_r) > 0$

iii) $\mathrm{Re}\,\lambda(A_d) < 0$

proof. i) $\to$ ii) and i) $\to$ iii)

Since Π_d and Π_r satisfy (3.5), we get

$$A_d^T\,\Delta\Pi + \Delta\Pi A_d = -\Delta\Pi BR^{-1}B^T\Delta\Pi$$
$$A_r^T\,\Delta\Pi + \Delta\Pi A_r = \Delta\Pi BR^{-1}B^T\,\Delta\Pi$$

whence

$$(\Delta\Pi)^{-1}A_d^T + A_d(\Delta\Pi)^{-1} = -BR^{-1}B^T$$
$$(\Delta\Pi)^{-1}A_r^T + A_r(\Delta\Pi)^{-1} = BR^{-1}B^T$$

The controllability of the pair (A,B) imply the observability of $(A_d^T, R^{-1/2}B^T)$ and $(A_r^T, R^{-1/2}B^T)$. Thus Lyapunov lemma applies to (3.10,11)

ii) $\to$ i) and iii) $\to$ i).

Suppose $\mathrm{Re}\,\lambda(A_d) < 0$. Then the equation

$$A_dX + XA_d^T = -BR^{-1}B^T \tag{3.12}$$

admits an unique solution:

$$K = \int_0^{+\infty} \exp(A_dt)BR^{-1}B^T\exp(A_d^Tt)\,dt > 0 \tag{3.13}$$

Letting $\Pi = K^{-1}$, we have,

$$\Pi A_d + A_d^T\Pi = -\Pi BR^{-1}B^T\Pi, \quad \Pi > 0 \tag{3.14}$$

If we are able show that $\bar{\Pi} = \Pi_d + \Pi$ is a solution of equation (3.5), we can conclude that $\Delta\Pi$ is positive definite, since $\Pi_r \geq \bar{\Pi} > \Pi_d$. Indeed (3.14) can be written

$$(\bar{\Pi} - \Pi_d)A_d + A_d^T(\bar{\Pi} - \Pi_d) = -(\bar{\Pi} - \Pi_d)BR^{-1}B^T(\bar{\Pi} - \Pi_d)$$

and consequently

$$\bar{\Pi}(A-BR^{-1}C) + \bar{\Pi}BR^{-1}B^T\Pi_d + (A-BR^{-1}C)^T\bar{\Pi} + \Pi_d BR^{-1}B^T\bar{\Pi} - \Pi_d A_d - A_d^T\Pi_d =$$

$$= \bar{\Pi}BR^{-1}B^T\bar{\Pi} - \Pi_d BR^{-1}B^T\Pi_d + \bar{\Pi}BR^{-1}B^T\Pi_d + \Pi_d BR^{-1}B^T\bar{\Pi}$$

Now $\bar{\Pi}$ is a solution of (3.5) if Π_d satisfies

$$-\Pi_d A_d - A_d^T\Pi_d + \Pi_d BR^{-1}B^T\Pi_d - C^T R^{-1}C = 0 \qquad (3.15)$$

Since (3.15) holds if and only if Π_d is a solution of (3.5), implication ii) $\to$ i) is proved. Implication iii) $\to$ i) requires an exactly similar reasoning. □

Theor.4 <u>Let $\Sigma = (A,B,C,D_o)$ be controllable and dissipative and let</u> $R > 0$. <u>If</u> $\Delta\Pi = \Pi_r - \Pi_d$ <u>is positive definite, then</u> $W(s) = D_o + C(sI-A)^{-1}B$ <u>is non real reduced. On the other hand, if</u> $W(s)$ <u>is non real reduced, all its minimal realizations exhibit a positive definite gap</u> $\Delta\Pi$.

proof. First suppose Σ to be dissipative and controllable and let $\Delta\Pi > 0$. If some ω_o exists such that

$$\text{rank}(W^T(-j\omega_o) + W(j\omega_o)) < m$$

then a vector $v = [\alpha_1+j\beta_1, \dots \alpha_m+j\beta_m]^T \neq 0$ in $\mathbb{C}^m$ also exists which satisfies

$$v^T\left[W^T(-j\omega_o) + W(j\omega_o)\right]v = 0$$

We define the function $G_T(\omega)$

$$G_T(\omega) = T\left(\frac{\sin(\omega-\omega_o)T}{(\omega-\omega_o)T} + \frac{\sin(\omega+\omega_o)T}{(\omega+\omega_o)T}\right) \qquad T > 0$$

$G_T(\omega)$ is the Fourier transform of the L^2 function (°)

$$g_T(t) = (\cos\omega_o t)\pi_{[-T,T]}(t)$$

Suppose the first column of B is different from 0 (this is not a restrictive assumption), and let

$$U_T(j\omega) = \begin{bmatrix} (\alpha_1+j\beta_1)e^{j2\pi\omega/\omega_o} \\ (\alpha_2+j\beta_2) \\ \text{-------------} \\ (\alpha_m+j\beta_m) \end{bmatrix} G_T(\omega) \tag{3.16}$$

Hence $U_T(j\omega)$ is the Fourier transform of the complex signal

$$u_T(t) = \begin{bmatrix} (\alpha_1+j\beta_1)g_T(t+2\pi/\omega_o) \\ (\alpha_2+j\beta_2)g_T(t) \\ \cdots\cdots\cdots\cdots \\ (\alpha_m+j\beta_m)g_T(t) \end{bmatrix} \tag{3.17}$$

When applying the complex input (3.17) to the system Σ, the corresponding output is constituted by L^2 functions, and Parseval theorem implies

$$\mathrm{Re}\int_{-T-\frac{\omega_o}{2\pi}}^{+\infty} \check{u}_T^T(t)y(t)dt\Big|_{0\rightarrow} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \check{U}_T^T(j\omega)(W^T(-j\omega)+W(j\omega))U_T(j\omega)d\omega \tag{3.18}$$

In some instant τ, $-T-\omega_o/2\pi < \tau \leq -T$, the system Σ reaches a non-zero state $x_o \neq 0$, as it is easy to show. We therefore have

$$\mathrm{Re}\int_{-T-\frac{\omega_o}{2\pi}}^{\tau} \check{u}_T^T(t)y(t)dt\Big|_{0\rightarrow x_o} = \int_{-T-\frac{\omega_o}{2\pi}}^{\tau} \mathrm{Re}\, u_T^T(t)\mathrm{Re}\, y(t)dt\Big|_{0\rightarrow \mathrm{Re}\, x_o} + \int_{-T-\frac{\omega_o}{2\pi}}^{\tau} \mathrm{Im}\, u_T^T(t)\mathrm{Im}\, y(t)dt\Big|_{0\rightarrow \mathrm{Im}\, x_o}$$

$$\leq (\mathrm{Re}\, x_o^T)\frac{\Pi_r}{2}(\mathrm{Re}\, x_o) + (\mathrm{Im}\, x_o^T)\frac{\Pi_r}{2}(\mathrm{Im}\, x_o) = \check{x}_o^T\frac{\Pi_r}{2}x_o \tag{3.19}$$

On the other hand

(°) $\pi_{[-T,T]}(t)$ denotes the unit rectangular impulse on the time interval $[-T,T]$.

$$\mathrm{Re} \int_{\tau}^{+\infty} \overset{\vee}{u}{}_T^T(t)y(t)\,dt \underset{x_o \to}{\geq} -x_o^T \frac{\Pi_d}{2} x_o$$

is granted by the definition of available storage.
Then we obtain the inequality

$$0 < x_o^T \frac{\Pi_r - \Pi_d}{2} x_o \leq \frac{1}{4\pi} \int_{-\infty}^{+\infty} \overset{\vee}{U}{}_T^T(j\omega)\,(W^T(-j\omega) + W(j\omega))U(j\omega)\,d\omega \tag{3.20}$$

If the parameter T diverges, the x_o state is constantly reached in the time interval $[-T-\omega_o/2\pi, -T]$, so that (3.20) holds for any $T > \omega_o/2\pi$.

Now, when T goes to infinity, $U_T(j\omega)$ converges to an impulse distribution:

$$U_\infty(j\omega) = \begin{bmatrix} \alpha_1 + j\beta_1 \\ \alpha_2 + j\beta_2 \\ \vdots \\ \alpha_m + j\beta_m \end{bmatrix} (\delta(\omega - \omega_o) + \delta(\omega + \omega_o))$$

and the integral (3.20) converges to $\overset{\vee}{v}{}^T(W^T(-j\omega_o) + W(j\omega_o))v = 0$. This gives a contradiction.

The full rank of $W(0) + W^T(0)$ is identical identically proved: the only difference is that $g_T(t)$ has to be selected in a different way, that is $g_T(t) = \pi_{[-T,T]}(t)$.

Suppose now W(s) be P.R. non real reduced, and assume $\Sigma = (A,B,C,D_o)$ to be a minimal (hence dissipative) realization of W(s). Then there are two positive constant values h and k such that

$$0 < h\mathbb{1}_m \leq W(j\omega) + W^T(-j\omega) \leq k\mathbb{1}_m$$

and available storage $x^T \frac{\Pi_d}{2} x$ and required $x^T \frac{\Pi_r}{2} x$ supply are both positive definite quadratic forms.

Assume the input u drives Σ from the zero state at time zero to some state $x \neq 0$ at time T. We have

$$0 < x^T \frac{\Pi_r}{2} x \leq \int_0^T \underset{0 \to x}{u^T y}\,dt = \frac{1}{4} \int_{-\infty}^{+\infty} \overset{\vee}{U}{}^T(j\omega)(W(j\omega) + W^T(-j\omega))U(j\omega)\,d\omega \leq$$

$$\leq \frac{k}{4\pi} \int_{-\infty}^{+\infty} \overset{\vee}{U}{}^T(j\omega)U(j\omega)\,d\omega = \frac{k}{2} \int_0^T u^T u\,dt$$

whence

$$\int_0^T u^T u \, dt \geq k^{-1} x^T \Pi_r x \qquad (3.21)$$
$$0 \to x$$

Introduce next the system $\bar{\Sigma} = (A,B,C,D_o - \frac{h}{2} I_m)$. $\bar{\Sigma}$ is still minimal and dissipative, and the corresponding available storage $x^T\left(\bar{\Pi}_d/2\right)x$ and required supply $x^T\left(\bar{\Pi}_r/2\right)x$ still are positive definite quadratic forms.

Thus the supply integral of Σ decomposes in the supply integral of $\bar{\Sigma}$ and in the term we considered in (3.21).

$$\int_{0 \atop 0 \to x}^{T} u^T y \, dt = \int_{0 \atop 0 \to x}^{T} u^T \bar{y} \, dt + \frac{h}{2} \int_{0 \atop 0 \to x}^{T} u^T u \, dt$$

In the inequality

$$\int_{0 \atop 0 \to x}^{T} u^T y \, dt \geq x^T \frac{\bar{\Pi}_d}{2} x + hk^{-1} x^T \frac{\Pi_r}{2} x \qquad (3.22)$$

we take the g.l.b. along all trajectories from 0 to x. From positive definiteness of Π_r we get

$$x^T \frac{\Pi_r}{2} x = \inf_{u,T} \int_{0 \atop 0 \to x}^{T} u^T y \, dt \geq x^T \left(\frac{\bar{\Pi}_d}{2} + hk^{-1} \frac{\Pi_r}{2}\right) x > x^T \frac{\bar{\Pi}_d}{2} x \qquad (3.23)$$

On the other hand $\bar{\Pi}_d \geq \Pi_d$. In fact for any x we have

$$x^T \frac{\Pi_d}{2} x = \sup_{T,u} \left(-\int_{0 \atop x \to}^{T} u^T y \, dt\right) = \sup_{T,u} \left(-\int_{0 \atop x \to}^{T} u^T \bar{y} \, dt - \frac{h}{2} \int_{0 \atop x \to}^{T} u^T u \, dt\right) \leq$$

$$\leq \sup_{T,u} \left(-\int_{0 \atop x \to}^{T} u^T \bar{y} \, dt\right) = x^T \frac{\bar{\Pi}_d}{2} x \qquad (3.24)$$

Hence the inequality chain

$$\Pi_r \geq \frac{\bar{\Pi}_d}{2} + \frac{hk^{-1}}{2} \frac{\Pi_r}{2} > \bar{\Pi}_d \geq \Pi_d$$

implies $\Pi_r - \Pi_d > 0$, and this concludes the proof. □

Recall that the dissipation function d(x,u) associated with an energy function $S(x) = x^T \frac{\Pi}{2} x$ satisfies

$$d(x,u) = \frac{1}{2} \left\| u - R^{-1}(B^T\Pi - C)x \right\|_R^2$$

along the trajectories of the equation $\dot{x} = Ax + Bu$.
If the input u is obtained from the state x in the feedback form $u = R^{-1}(B^T\Pi - C)x$, then

$$\int_{t_o}^{t_1} \underset{x_o \to x_1}{w(t)dt} = \int_{t_o}^{t_1} \underset{x_o \to x_1}{d(x,u)dt} + x^T \frac{\Pi}{2} x \Big|_{t_o}^{t_1} = x^T \frac{\Pi}{2} x \Big|_{t_o}^{t_1}$$

and the state of Σ evolves along the solution of the following linear differential equation

$$\dot{x} = (A + BR^{-1}B^T\Pi - BR^{-1}C)x \tag{3.25}$$

Suppose now Σ fulfill the condition $\Delta\Pi = \Pi_r - \Pi_d > 0$, so that $\mathrm{Re}\lambda(A_r) > 0$ and $\mathrm{Re}\lambda(A_d) < 0$ by theorem 3. The solution of

$$\dot{x} = A_d x \tag{3.26}$$

is infinitesimal as $t \to +\infty$, whereas the solution of

$$\dot{x} = A_r x \tag{3.27}$$

is infinitesimal as $t \to -\infty$, whatever the initial state may be.
We therefore have the following

Theor.5 Assume that the controllable dissipative system $\Sigma = (A,B,C,D_o)$ satisfy the condition $\Delta\Pi > 0$. Then for any initial state x_o at time $t = t_o$ the input $u_d = R^{-1}(B^T\Pi_d - C)x$ extracts the available storage on the time interval $[t_o, +\infty)$.
Assume that $x(t)$ is the solution of the equation (3.27) with the initial condition $x(t_o) = x_o$. Then the inputs $u_r^{(n)} = R^{-1}(B^T\Pi_r - C)x$ give trajectories from the state $x(t_o - n)$ to the state x_o on the time intervals $[t_o - n, t_o]$ respectively, and $S_r(x_o)$ is the limit value of the corresponding sequences of supplies:

$$S_r(x_o) = \lim_{n \to +\infty} \int_{t_o - n}^{t_n} \underset{x(t_o - n) \to x_o}{u_r^{(n)T} y \, dt}$$

III.4. P.R. lemma and lossless linear systems

Conditions of sec. 1 specialize very simply when $\Sigma = (A,B,C,D_o)$ is a lossless system.

Theor.1 Let $\Sigma = (A,B,C,D_o)$. The following propositions are equivalent:

o) Σ is lossless dissipative

i) The set of non negative definite matrices Π which satisfy the equality

$$\int_{t_o}^{t_1} w(t)dt = x^T \frac{\Pi}{2} x \Big|_{t_o}^{t_1}$$

is not empty

ii) The set of non negative definite solutions of the equation

$$\begin{bmatrix} D_o + D_o^T & C - B^T K \\ C^T - KB & -A^T K - KA \end{bmatrix} = 0 \tag{4.1}$$

is not empty.

iii) The set of non negative definite matrices Π which satisfy the following equations

$$\begin{aligned} \Pi A - A^T \Pi &= 0 \\ \Pi B &= C^T \\ 0 &= D_o + D_o^T \end{aligned} \tag{4.2}$$

is not empty.

The sets $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ are equal.

The proof is quite similar with that of the corresponding statement in sec. 1. □

Remark. If Σ is a controllable lossless system, then $\mathscr{S}_1(\mathscr{S}_2,\mathscr{S}_3)$ contains only one solution.

It is natural to ask whether the solution of (4.2) requires a less severe computational task than solving a Riccati equation. The answer is positive, as stated by the following theorem.

Theor.2 Let $\Sigma = (A,B,C,D_o)$ be lossless and controllable. The unique non negative definite solution of (4.2) is linearly obtained from the system matrices.

proof. The matrix Π satisfies the relations

$$\Pi B = C^T$$

$$\Pi AB = -A^T\Pi B = -A^T C^T \tag{4.3}$$

$$\Pi A^2 B = -A^T \Pi AB = (A^T)^2 C^T$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Pi A^{n-1}B = (-1)^{n-1}(A^T)^{n-1}C^T$$

Let $\mathcal{K}$ denote the controllability matrix and $\hat{\mathcal{O}}$ the matrix

$$\hat{\mathcal{O}} = \left[C^T \mid -A^T C^T \mid \ldots \ (-1)^{n-1}(A^T)^{n-1}C^T\right]$$

then

$$\Pi\mathcal{K} = \hat{\mathcal{O}} \tag{4.4}$$

so that the unique solution Π is easily derived as

$$\Pi = \hat{\mathcal{O}}\mathcal{K}^T(\mathcal{K}\mathcal{K}^T)^{-1} \tag{4.5}$$

Remark. If Σ is a controllable linear system and dissipativity and lossless property are a priori not known, a possible check for testing these properties consists in

i) constructing Π from (3.5)

ii) verifying if Π is non negative definite

iii) verifing if $D_o + D_o^T$ is the zero matrix

III.5 Spectral factorization

Topics we will here briefly consider have several important extensions and consequences (e.g. in filtering theory). We will confine ourselves to sketch in what sense the P.R. lemma can be interpreted as an "algebraic" counterpart of the spectral factorization in the s domain. Our main purpose is to show how a very wide class of problems can be analyzed and eventually solved with the techniques we discussed above.

Recall that the dissipativity of a linear system $\Sigma = (A,B,C,D_o)$ is equivalent to the existence of a quadratic non negative function $S: X \to \mathbb{R}_+$ which fulfills the following inequality

$$S(x_1) - S(x_2) = x^T \frac{\Pi}{2} x \Big|_{t_o}^{t_1} \leq \int_{t_o}^{t_1} u^T (Cx + D_o u)\, dt \quad x_o \to x_1 \tag{5.1}$$

In sec. 1 we considered a set of propositions equivalent to (5.1). It should be clear that, under the controllability hypothesis, this set includes also the proposition "W(s) is a P.R. matrix". As a matter of fact, the spectral factorization of the matrix $W(s) + W^T(-s)$ could be another possible element. However, we prefere to derive this equivalence in connection with an extended notion of dissipativity, which do not require the non negative definiteness of S. This will allow us a more complete picture of several facts connected with the concept of "system energy".

Def.1 Let $\Sigma = (A,B,C,D_o)$ and let $w = u^T y$ be the supply function of Σ. A quadratic function $S: X \to \mathbb{R}: x \to x^T \frac{\Pi}{2} x$ $(\Pi = \Pi^T)$ is a "signed energy" if, for any pair of states x_o and x_1

$$\int_{t_o}^{t_1} w\, dt \geq S(x_1) - S(x_o) \quad x_o \to x_1 \tag{5.2}$$

Theor.1 The following propositions are equivalent

i) Σ has asigned energy function

ii) There exists a symmetric solution of the matrix inequality

$$\begin{bmatrix} D_o + D_o^T & C - B^T K \\ C^T - KB & -A^T K - KA \end{bmatrix} \geq 0 \tag{5.3}$$

iii) There exists a triple of matrices (Π, H, J), $\Pi = \Pi^T$ which satisfies the generalized P.R. lemma

$$\begin{aligned} \Pi A + A^T \Pi &= -H^T H \\ \Pi B &= C^T - H^T J \\ J^T J &= D_o + D_o^T \end{aligned} \tag{5.4}$$

The proof follows the pattern of the analogous statement in sec. 1. □

Theor.2 Let $\Sigma = (A,B,C,D_o)$ admit a signed energy function and let $W(s) = D_o + C(sI-A)^{-1}B$. Then

i) $W(j\omega) + W^T(-j\omega)$ is non negative definite for any real ω such that $j\omega$ is not a pole of $W(s)$.

ii) There exists a real rational matrix $\Phi(s)$ which is a "spectral factor" of $W(s) + W^T(-s)$:

$$W^T(-s) + W(s) = \quad ^T(-s) \quad (s) \tag{5.5}$$

If Σ is controllable, each one of i) and ii) conditions guarantees that Σ has a signed energy function.

proof. If Σ admits a signed energy function S, then there exists a solution Π of (5.3). Hence for any real ω

$$\begin{bmatrix} D_o + D_o^T & C - B^T\Pi \\ C^T - \Pi B & (-j\omega \mathbb{1} - A^T)\Pi + \Pi(j\omega \mathbb{1} - A) \end{bmatrix} \geq 0$$

implies $W^T(-j\omega) + W(j\omega) \geq 0$. To see this it is sufficient to multiply the inequality above by $|\mathbb{1}|B^T(-j\omega\mathbb{1}-A^T)^{-1}|$ on the left and by

$$\begin{bmatrix} \mathbb{1} \\ (j\ \mathbb{1}-A)^{-1}B \end{bmatrix}$$

on the right.

Next consider the following form of (5.3):

$$\begin{bmatrix} D_o + D_o^T & C - B^T\Pi \\ C^T - \Pi B & (-s\mathbb{1} - A^T)\Pi + \Pi(s\mathbb{1} - A) \end{bmatrix} = \begin{bmatrix} J^T \\ H^T \end{bmatrix} \begin{bmatrix} J & H \end{bmatrix} \tag{5.6}$$

If one multiplies (5.6) by

$$\begin{bmatrix} 1 & B^T(-s1-A^T)^{-1} \end{bmatrix}$$

on the left and by

$$\begin{bmatrix} \mathbb{1} \\ (s\mathbb{1}-A)^{-1}B \end{bmatrix}$$

on the right, one gets

$$W^T(-s) + W(s) = \left[J + H(-s\mathbb{1}-A)^{-1}B\right]^T \left[J + H(s\mathbb{1}-A)^{-1}B\right]$$

This shows that

$$\Phi(s) = J + H(s\mathbb{1}-A)^{-1}B \tag{5.7}$$

is a spectral factor.

Conversely, let $W^{-T}(-j\omega) + W(j\omega)$ be non negative definite for any real ω and let Σ be controllable. Consider an input u with support on the time interval $[0, t_1]$ and assume that the corresponding trajectory in X satisfies $x(0) = x(t_1) = 0$. The corresponding integral supply is non negative:

$$\int_{\substack{0 \\ 0\to 0}}^{t_1} w(t)dt \geq 0 \tag{5.8}$$

In fact u and y are both L^2 functions, so Parseval theorem applies:

$$\int_{\substack{0 \\ 0\to 0}}^{t_1} w(t)dt = \int_{-\infty}^{+\infty} u^T y\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \check{U}^T(j\omega)Y(j\omega)d\omega =$$

$$= \frac{1}{4\pi}\int_{-\infty}^{+\infty} \check{U}^T(j\omega)\left[W(j\omega) + \check{W}^T(j\omega)\right]U(j\omega)d\omega \geq 0$$

As a consequence of (5.8), the function S defined by

$$S(x) = \inf_{\substack{u \\ t_1 \geq t_o}} \int_{\substack{t_o \\ 0\to x}}^{t_1} w(t)dt \tag{5.9}$$

is finite for any x and corresponds to the required supply in the theory of standard dissipative systems. In fact, assuming $S(x) = -\infty$ for some x in X, for any trajectory $\vec{\gamma}_1$ from x to the zero state, we could find a trajectory $\vec{\gamma}_2$ from the zero state to x such that

$$\int_{\vec{\gamma}_2} w(t)dt < \int_{\vec{\gamma}_1} w(t)dt$$

This would imply

$$\oint_{\vec{\gamma}_1 + \vec{\gamma}_2} w(t)dt < 0$$

contrary to (5.8).

Note that the S function we defined in (5.9) is a quadratic function on X, not necessarily non negative definite (whenever it is, Σ is dissipative). Since the existence of a spectral factor $\Psi(s)$ implies

$$W^T(-j\omega)+W(j\omega) = \Psi^T(-j\omega)\Psi(j\omega) = \check{\Psi}^T(j\omega)\Psi(j\omega) \geq 0$$

clearly ii)$\Rightarrow$i), so that our proof will be complete when showing that S is a signed energy function. This is a direct consequence of the following relations chain

$$\int_{t_o}^{t_1} \underset{x_o\to x_1}{w(t)dt} = \inf_u \int^{t_o} \underset{0\to x_o}{w(t)dt} + \int_{t_o}^{t_1} \underset{x_o\to x_1}{w(t)dt} - \inf_u \int \underset{0\to x_o}{w(t)dt} \geq$$

$$\geq \inf_u \int \underset{0\to x_1}{w(t)dt} - \inf_u \int \underset{0\to x_o}{w(t)dt} = S(x_1)-S(x_o)$$

Remark 1. Parseval theorem application in Theorem 2 does not extend to the more general situation $\int_0^{t1} \underset{0\to}{w(t)dt} \geq 0$. [This would prove that dissipativity does not require analiticity of W(s) in Re(s) >0!]. To be more precise, if the $[0,t_1]$ interval contains the support of u and $x(t_1)$ is different from zero, in general the support of y is $[0,+\infty)$. Since Parseval theorem holds in L^2 spaces, we have to introduce an extra hypothesis on the analiticity of W(s) in Re s >0, which guarantees that y is in L^2.

Remark 2. It is easy to show that if Σ is controllable and satisfies (5.8) for any cyclic trajectory which crosses the zero state then

$$\int_{\vec{\gamma}} w(t)dt \geq 0 \tag{5.10}$$

holds for every cyclic trajectory in X.

Condition (5.10) is summarized in the following proposition: "Σ is dissipative along any cyclic trajectory". Dissipativity along cyclic trajectories does not imply dissipativity. For instance, the minimal realization Σ = (1,-1,-1,0) of

$$W(s) = \frac{-1}{s-1} \tag{5.11}$$

is dissipative along any cyclical trajectory, because of the inequality

$$W(j\omega)+W^T(-j\omega) = \frac{2}{1+\omega^2} \geq 0$$

However Σ is not dissipative. This can be derived from non-PR property of $W(s)$, or more directly from dissipativity definition. In fact the zero state response to a unit step function is given by $(1-e^t)\delta_{-1}(t)$ so that the integral supply is

$$\int_0^t w_{0\to}(t)\,dt = \int_0^t (1-e^\tau)\,d\tau = t-e^t+1 \qquad (5.12)$$

As t goes to infinity (5.12) is negative.
It is interesting to remark that (5.11) is the impedance of the electric circuit in fig. 1, which includes a negative inductor. One could show that electrical linear networks including positive resistances and positive and/or negative inductors and/or capacitors fulfill condition (5.10).

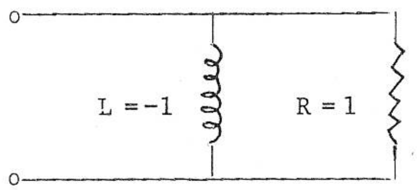


Fig. 1

Remark 3. Relation

$$\int_o^\infty u^T y_{0\to}\,dt = \frac{1}{2\pi}\int_o^\infty \check{U}^T(j\omega)(\check{W}^T(j\omega)+W(j\omega)U(j\omega)\,d\omega, \qquad u \in L^2$$

holding in dissipative systems, and relation

$$\int_o^{t_1} u^T y_{0\to 0}\,dt = \frac{1}{2\pi}\int_o^\infty \check{U}^T(j\omega)(\check{W}^T(j\omega)+W(j\omega))U(j\omega)\,d\omega$$

holding in systems which exhibit a signed energy function, both give in the ω domain a representation of the amount of work done by the input u.
It is rather intuitive to consider

$$\frac{1}{2\pi}\check{U}^T(j\omega)(\check{W}^T(j\omega)+W(j\omega))U(j\omega) = \left|\frac{1}{\sqrt{2\pi}}\Phi(j\omega)U(j\omega)\right|^2$$

as the "spectral density" of such a work. The spectral factorization theorem means that spectral density can be derived in a "rational way" from the input.

Theorem 2 contains implicitly an interesting result we tray to display

Def.2 Let $M(s)$ be a real rational $m \times m$ matrix. $M(s)$ is parahermitian if $M(s) = M^T(-s)$.

Coroll.1 A $m \times m$ real rational matrix $M(s)$ has a spectral factor if and only if $M(s)$ is parahermitian and non negative definite on the imaginary axis.

proof. Let $M(s)$ admit a spectral factor $\Phi(s)$

$$M(s) = \Phi^T(-s)\Phi(s)$$

Then

$$M^T(-s) = \Phi^T(-s)\Phi(s) = M(s)$$

$$M(j\omega) = \Phi^T(-j\omega)\Phi(j\omega) = \check{\Phi}^T(j\omega)\Phi(j\omega) \geq 0$$

Conversely let $M(s)$ be parahermitian and $M(j\omega)$ be nonnegative definite for any real ω. Then $M(s)/2 \triangleq W(s)$ satisfies

$$W(s) + W^T(-s) = M(s) \tag{5.13}$$

whence

$$W(j\omega) + W^T(-j\omega) = M(j\omega) \geq 0$$

for any real ω. This implies that $M(s)$ admits a spectral factor. In fact if $W(\infty) \neq \infty$, every controllable realization of $W(s)$ has a signed energy function, by Theorem 2. If $W(s)$ has a pole at $s = \infty$, then for some positive ν the matrix $\tilde{M}(s) = M(s)s^{-4\nu}$ is proper, parahermitian and nonnegative definite on the imaginary axis. Denoting by $\psi(s)$ a spectral factor of $M(s)$, the matrix $(s) = \Psi(s)s^{2\nu}$ is a spectral factor of $M(s)$. □

Def. 3 A matrix $U(s) \in R(s)^{m \times m}$ is paraunitary if

$$I_m = U^T(-s)U(s) \tag{5.14}$$

One checks that:

i) $U(s)$ is analytic on the imaginary axis, including the point $s = \infty$
ii) if $M(s)$ is parahermitian, so is $U(s)M(s)$
iii) if $M(s)$ and $M^{-1}(s)$ are analytic in $\mathrm{Re}\, s > 0$, then $U(s)$ is a constant orthogonal matrix.

Theorem 2 and Corollary 1 introduce to a wide class of problems, sometimes quite difficult.

The pole symmetry of parahermitian matrices suggests the first question. It concernes the existence of spectral factors whose poles belong to the half-plane Re $s \geq 0$ or to the half-plane Re $s \leq 0$. When the spectral factor is required to be analytic as well as its inverse in Re $s \geq 0$, this problem represents a critical point in the solution of the Wiener-Hopf equation.

Strictly related to the first question are the natural questions on the uniqueness of spectral factors which satisfy certain requirements and the effective construction of the spectral factors.

Finally, if M(s) is obtained as

$$M(s) = W(s) + W^T(-s) \tag{5.15}$$

and W(s) can be realized by a system $\Sigma = (A,B,C,D_o)$ which has a signed energy function, then the solution of (5.4) gives also a spectral factor. This partially answers the question on the relationships between spectral factors of M(s) and realizations of W(s). As a matter of fact, a more complete answer could be obtained since the spectral factorization is equivalent (under suitable assumptions) to the solution of (5.4). In other words the construction of signed energy functions for is equivalent to the construction of cyclic dissipative realizations.

In next section we will consider these problems in detail. However, our exposition will be restricted to situations which allow us to use the techniques we introduced in the previous sections.

III.6 Spectral factors structure

The structure of the spectral factors of a parahermitian matrix M(s) which satisfies the extra condition $M(j\omega) \geq 0$ for all real ω has been investigated by Youla.

Theor.1 [YOULA] Let $M(s) \in R(s)^{m\times m}$ be parahermitian. Assume that the normal rank of M(s) is $r \leq m$ and that $M(j\omega)$ is non negative definite for all real ω. Then there exists a spectral factor $\Psi_o(s) \in R(s)^{r\times m}$ which satisfies the following

i) $\Psi_o(s)$ is analytic in $Re(s) > 0$

ii) $\Psi_o(s)$ has full rank r at every point in $Re(s) > 0$.

$\Psi_o(s)$ is uniquely determined up a left multiplication by an arbitrary orthogonal, $r \times r$ matrix.

Spectral factors $\Psi_1(s) \in R(s)^{r \times m}$ which satisfy only hypothesis i) are given by the formula

$$\Psi_1(s) = U(s)\Psi_o(s)$$

as $U(s)$ runs over the set of $r \times r$ paraunitary matrices which are analytic in $Re\ s > 0$.

proof. Assume first that a spectral factor $\Psi_o(s)$ which satisfies i) and ii) does esist, and suppose $\Psi_1(s)$ is another spectral factor which satisfies i):

$$M(s) = \Psi_o^T(-s)\Psi_o(s) = \Psi_1^T(-s)\Psi_1(s) \tag{6.1}$$

The matrix $\Psi_o(s)\Psi_o^T(s)$ is invertible on $R(s)$, and the matrix

$$U(s) = \Psi_1(s)\Psi_o^T(s)(\Psi_o(s)\Psi_o^T(s))^{-1}$$

fulfills the condition

$$(U(s)\Psi_o(s) - \Psi_1(s))\Psi_o^T(s) = 0$$

Since the $R(s)$ rank of $\Psi_o^T(s)$ is r, we get

$$U(s)\Psi_o(s) = \Psi_1(s) \tag{6.2}$$

From (6.1) one proves that $U(s)$ is paraunitary: $U^T(-s)U(s) = 1_r$. Since $\Psi_o(s)\ \Psi_o^T(s)$ has rank r everywhere in $Re\ s > 0$, its inverse is analytic in $Re\ s > 0$. This shows analiticity of $U(s)$ in $Re\ s > 0$.

If $\Psi_1(s)$ is full rank on $Re\ s > 0$, then $U(s)$ is full rank too. Hence $U(s)$ is an orthogonal matrix, and the second part of the theorem is proved . The complete proof of the first part is very long and is based on Smith McMillan canonical form. Assuming $M(s)$ to be analytic along the imaginary axis and not singular at $s = \infty$, then a matrix $W(s)$ exists which fulfills the relations

$$M(s) = W(s) + W^T(-s)$$
$$\lim_{s\to\infty} W(s) = D_o \qquad (6.3)$$
$$D_o + D_o^T > 0$$

Let $\Sigma = (A,B,C,D_o)$ be a minimal realization of $W(s)$. Then the eigenvalues of A are in $\mathrm{Re}\ s \geq 0$, and dissipativity of Σ implies the existence of a matrix $\Pi_d > 0$ such that the triple $(\Pi_d, (J^T)^{-1}(B^T\Pi_d - C), J)$ considered in sec. 2 is a solution of the P.R. lemma. A spectral factor of $M(s)$ is given by

$$J + (J^T)^{-1}(B^T\Pi_d - C)(s\mathbb{1} - A)^{-1}B \qquad (6.4)$$

Clearly (6.4) is analytic in $\mathrm{Re}\ s > 0$, and its inverse

$$J^{-1} - R^{-1}(B^T\Pi_d - C)(s\mathbb{1} - A + BR^{-1}B^T\Pi_d - BR^{-1}C)BJ^{-1} \qquad (6.5)$$

is also analytic in the same region, by the structure of the spectrum of $A_d = A - BR^{-1}C + BR^{-1}B^T\Pi_d$ we will derive in the next lemma 1.

Hence $\Psi_o(s)$ is given by (6.4). □

Lemma 1 [ANDERSON] Let $\Sigma = (A,B,C,D_o)$ be dissipative and let $D_o + D_o^T = R$ be positive definite. Then the spectrum of

$$A_d = A - BR^{-1}C + BR^{-1}B^T\Pi_d \qquad (6.6)$$

belongs to $\mathrm{Re}(s) \leq 0$.

proof. Recall first that $-\Pi_d$ has been obtained from the solution $\Pi(t,0,0)$ of the equation

$$\dot{K} = -(A-BR^{-1}C)^T K - K(A-BR^{-1}C) + KBR^{-1}B^T K + C^T R^{-1} C \qquad (6.7)$$
$$K(0) = 0$$

taking the limit

$$\lim_{t\to-\infty} \Pi(t,0,0)$$

It is easy to verify that $\Pi(t,0,0)$ decreases at t decreases since

$$\inf_{u} \int_{t}^{0} 2u^T y \, dt = x^T \Pi(t,0,0) x$$
$x \to$

is decreasing too (for every x).
Π being the solution of the algebraic Riccati equation (2.2), $L(t) = \Pi_d + \Pi(t,0,0)$ is a solution of

$$\begin{aligned} \dot{L} &= -A_d^T L - LA_d + LBR^{-1}B^T L \\ L(0) &= \Pi_d \end{aligned} \tag{6.8}$$

on the interval $(-\infty, 0]$. Observe next that L(t) is positive definite in some left neighborhood $\mathscr{T}$ of t = 0. Then in $\mathscr{T}$, $N(t) = L^{-1}(t)$ satisfies the linear differential equation

$$\begin{aligned} \dot{N} &= A_d N + NA_d^T + BR^{-1}B^T \\ N(0) &= \Pi_d^{-1} \end{aligned} \tag{6.9}$$

Now the solution of (6.9) exists in all points of the interval $(-\infty, 0]$, so that, by the continuity of L(t) and N(t) the relation

$$L(t)N(t) = \mathbb{I} \tag{6.10}$$

extends from $\mathscr{T}$ to $(-\infty, 0]$. Since L(t) decreases as t decreases, and $\lim_{t\to-\infty} L(t) = 0$, the positive definiteness of L(t) and of N(t) extends to $(-\infty, 0]$. Clearly the maximum eigenvalue of L(t) is infinitesimal as $t \to -\infty$ and the minimum eigenvalue of N(t) goes to infinity as $t \to -\infty$.
Hence for any $v \in C^n$ we get

$$\lim_{t \to -\infty} \overset{\vee}{v}^T N(t) v = +\infty \tag{6.11}$$

Finally, assume that A_d^T has an eigenvalue λ with $\mathrm{Re}\lambda > 0$. Denoting by v an eigenvector, the function

$$n(t) = \overset{\vee}{v}^T (t) v \tag{6.12}$$

satisfies the following differential equation

$$\begin{aligned} \frac{dn}{dt} &= (\overset{\vee}{\lambda} + \lambda) n + \overset{\vee}{v}^T BR^{-1}B^T v = 2\mathrm{Re}\,\lambda\, n + \mathrm{cost} \\ n(0) &= \overset{\vee}{v}^T \Pi_d^{-1} v > 0 \end{aligned} \tag{6.13}$$

The solution of (6.13) is infinitesimal as $t \to -\infty$, contradictory (6.11). This proves the lemma. □

Coroll.1 [YOULA] Let $M(s) \in R(s)^{m\times m}$ satisfy the hypothesies of Theorem 1 and be analytic on the imaginary axis. Then the analiticity of $\Psi_o(s)$ extends to the imaginary axis.
Moreover, if $\mathrm{rank}(M(j\omega)) = r$ for any real ω, then rank $\Psi_o(s) = r$ in any point on $\mathrm{Re}\ s \geq 0$

proof. (in the case when $W(s)$ is invertible). In (6.3) $W(s)$ can be choosen P.R. and regular on the imaginary axis. This implies that the spectral factor (6.4) we get from a minimal realization is analytic on $\mathrm{Re}(s) \geq 0$. If rank $M(j\omega) = m$ for any real ω, then $W(s)$ in non real-reduced, so that its minimal realizations satisfy the condition $\mathrm{Re}\ \lambda(A_d) < 0$. Hence the rank of $\Psi_o(s)$ is m on $\mathrm{Re}(s) \geq 0$. □

Coroll.2 Let $M(s) \in R(s)^{m\times m}$ satisfy the hypothesies of Coroll.1 and let

$$M(s) = W(s) + W^T(-s)$$

with $W(s)$ P.R. and analytic along the imaginary axis. Then

i) the spectral factor $\Psi_o(s)$ of $M(s)$ and the matrix $W(s)$ exhibit the same McMillan degree.

ii) there exist minimal realizations of $\Psi_o(s)$ and $W(s)$ having common matrices A and B.

proof. The well known properties of McMillan degree give

$$\delta|M(s)| = \delta|\Psi_o^T(-s)\Psi_o(s)| \leq \delta|\Psi_o^T(-s)| + \delta|\Psi_o(s)| = 2\delta|\Psi_o(s)| \quad (6.14)$$

On the other hand the set of poles of $W(s)$ and $W^T(-s)$ are disjoint so that

$$\delta|M(s)| = \delta|W(s)| + \delta|W^T(-s)| = 2\delta|W(s)| \quad (6.15)$$

Assuming $M(\infty)$ to be invertible (6.4) gives a realization of $\Psi_o(s)$ which exhibits the same A and B matrices as a suitable minimal realization of $W(s)$ does.

Hence $\delta|W(s)| \geq \delta|\Psi_o(s)|$, and (6.14) proves both i) and ii).
We omit the proof in the case $M(\infty)$ is not invertible. One could refer to [A.1], pg. 249. □

Remark 1. In sec. III.5, theorem 2, we showed that if $\Sigma = (A,B,C,D_o)$ satisfies the generalized P.R. lemma then any solution (Π,H,J) of the lemma gives a spectral factor of $W(s)+W^T(-s)$. We annonced that - to some extent - the converse holds too: in fact, under suitable assumptions, the knowledge of spectral factors provides solutions of the PR lemma, as we will prove below.
Assume $\Sigma_W = (A,B,C,D_o)$ to be a minimal realization of a PR matrix $W(s)$ and let $\Sigma_{\Psi_o} = (F,G,H,J)$ be a minimal realization of the spectral factor $\Psi_o(s)$ we considered in theorem 1.
Decompose Σ_W as the direct sum of two subsystems $\Sigma_- = (A_-,B_-,C_-,D_o)$ and $\Sigma_I = (A_I,B_I,C_I,0)$. Σ_I is a lossless systems which realizes the imaginary poles of $W(s)$, whereas Σ_- is a dissipative realization of the remaining poles of $W(s)$. Since the transfer matrix $W_-(s)$ of Σ_- satisfies the condition

$$W_-(s) + W_-^T(-s) = \Psi_o^T(-s)\Psi_o(s)$$

by Corollary 2 there exists a minimal realization of $\Psi_o(s)$ having A_- and B_- as first two matrices. Such a realization $\Sigma'_{\Psi_o} = (A_-,B_-,H',J)$ is unique, and is determined by Σ_{Ψ_o}, A_- and B_-, in the sense that these matrices uniquely determine the change of basis from Σ_{Ψ_o} to Σ'_{Ψ_o}.
The linear equation in the unknown matrix K

$$KA_- + A_-^T K = -H'^T H' \tag{6.15}$$

has a unique solution $\Pi_- > 0$, which can be introduced in the factorization of $W(s)+W^T(-s)$:

$$\begin{aligned} \Psi_o^T(-s)\Psi_o(s) &= J^TJ + J^TH'(s\mathbb{1}-A_-)^{-1}B_- + B_-^T(-s\mathbb{1}-A_-^T)^{-1}H'^T + \\ &+ B_-^T(-s\mathbb{1}-A_-^T)^{-1}(\Pi_-(s\mathbb{1}-A_-) + (-s\mathbb{1}-A_-^T)\Pi_-)(s\mathbb{1}-A_-)^{-1}B_- = \\ &= J^TJ + (J^TH'+B_-^T\Pi_-)(s\mathbb{1}-A_-)^{-1}B_- + B_-^T(-s\mathbb{1}-A_-^T)^{-1}(\Pi_-B_- + H'^TJ) \end{aligned} \tag{6.16}$$

By equating (6.16) with

$$W^T(-s)+W(s) = (D_o+D_o^T)+C_-(s\mathbb{1}-A_-)^{-1}+B_-^T(-s\mathbb{1}-A_-)^{-1}C_-^T$$

one gets the triple (Π_-,H',J) which satisfies the following relations

$$\begin{aligned} &\Pi_-A_- + A_-^T\Pi_- = -H'^TH' \\ &\Pi_-B_- = C_-^T - H'^TJ \\ &D_o + D_o^T = J^TJ \end{aligned} \tag{6.17}$$

Since from sec. 4 we obtain easily a matrix Π_I which satisfies

$$\begin{aligned} &\Pi_IA_I + A_I^T\Pi_I = 0 \\ &\Pi_IB_I = C_I^T \end{aligned} \tag{6.18}$$

we conclude that the triple $\left(\begin{bmatrix}\Pi_- & 0 \\ 0 & \Pi_I\end{bmatrix}, \begin{bmatrix}H' & \vdots & 0\end{bmatrix}, J\right)$ is a solution of the PR lemma with respect to

$$\left(\begin{bmatrix}A_- & 0 \\ 0 & A_I\end{bmatrix}, \begin{bmatrix}B_- \\ B_I\end{bmatrix}, \begin{bmatrix}C_- & C_I\end{bmatrix}, D_o\right)$$

which realizes $W(s)$.

Clearly if an existence proof and/or a construction technique of $\Psi_o(a)$ are available which do not depend on the PR lemma, thein the above remark provides a proof of the PR lemma and/or a construction technique of its solutions which does not depend on the Riccati equation. A careful discussion of these problems can be found in [A.1] ore in [Y.1].

Remark 2. Let $\Sigma = (A,B,C,D_o)$ be a dissipative realization of $W(s)$, and let (Π,H,J) be a solution of the PR lemma with respect to Σ. The number r of the rows in H and in J matrices is also the number of the rows in the spectral factor $\Psi(s) = J+H(s\mathbb{1}-A)^{-1}B$ of the matrix $W(s)+W^T(-s)$. Since the rank of $\Psi(s)$ cannot be less than the rank of $W(s)+W^T(-s)$, one gets

$$r \geq \operatorname{rank}(W^T(-s)+W(s)) \overset{\Delta}{=} \rho \tag{6.19}$$

On the other hand, the factor $\Psi_o(s)$ in Theorem 1 has r rows, and remark 1 above shows that the solution of the PR lemma one gets from a minimal rea lization Σ and from the corresponding $\Psi_o(s)$ exhibits ρ-rows H and J matrices. We therefore have that every minimal realization of W(s) maps into a solution of the PR lemma in which the number of rows in H and J matrices attains the l.b.ρ. Denoting by Π_d the available storage matrix in $\Sigma = (A,B,C,D_o)$, one could also prove that

$$\rho = r_{min} = \text{rank} \begin{bmatrix} D_o+D_o^T & C-B^T\Pi_d \\ C^T-\Pi_d B & -\Pi_d A-A^T\Pi_d \end{bmatrix}$$

It is interesting to recall that ρ is the minimal number of resistor needed in the synthesis of the impedance matrix W(s).

REFERENCES

A.1 B.D.O. Anderson, S. Vongpanitlerd "Network Analysis and Synthesis" Prentice Hall, 1973

A.2 B.D.O. Anderson, J.B. More "LInear Optimal Control" Prentice Hall, 1971

B.1 R.W. Brockett "Finite Dimensional Linear Systems" J. Wiley, 1970 .

B.2 V. Belevitch "Classical Network Theory" Holden-Day, 1968

B.5 R. Bellman "Introduction to Matrix Analysis" Mc Graw, 1960

C.1 H. Cartan "Théorie Elémentaire des Fonctions Analytiques" Hermann, 1961

F.1 E. Fornasini "Appunti di Teoria delle Reti Elettriche" Cleup 1977

G.1 F.R. Gantmacher "The Theory of Matrices" Chelsea, 1959

K.2 D.L. Kleinman "On an Iterative Technique for Riccati Equation Computations" IEEE Trans. AC, Feb. 1968, pp. 114-115.

M.1 M. Marcus, H. Ming "A Survey of Matrix Theory and Matrix Inequalities" Allyn and Bacon, 1964

N.1 R.W. Newcomb "Linear Multiport Synthesis" Mc Graw, 1966

S.1 L. Schwartz "Cours d'Analyse" Hermann, 1967

T.1 E.C. Titchmarsh "Introduction to the Theory of Fourier Integrals" , Oxford, 1937.

W.1 J.C. Willems "Least Squares Stationary Optimum Control and the Algebraic Riccati Equation" IEEE Trans AC, Dec 1971, p. 621 - 634.

W.2 J.C. Willems "Dissipative Dynamical Systems" Arch. for Rat.Mech. and Anal., vol. 45, n. 5, 1972, pp. 321 - 393

W.3 M.R. Wohlers "Lumped and Distributed Passive Networks" Academic Press, 1969

Y.1 D.C. Youla "On the Factorization of Rational Matrices" IRE Trans IT, July 1961, pp. 685 - 698.