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ON SOME STRUCTURAL PROPERTIES OF DISSIPATIVE LINEAR DYNAMICAL SYSTEMS

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Abstract . The relations between system theoretic concepts of controllability and observability and notions of dissipativity and stored energy are investigated, mainly in the case of linear, time invariant dynamical systems.

1. INTRODUCTION

The investigation of "passive" physical structures is a traditional field of Electrical Engineering. Its origins arose from electrical network analysis, and a lot of results were obtained in connection with the well known problem of "physical realizability". Recently the introduction of system theoretic methods made possible several more steps in the formal explanation of dissipativity and energy concepts. In particular some papers by Willems [1,2], Anderson [3,4,5] and others provided a clean and rigorous set up for the study of input/output dissipativity and state model properties. In this context the determination of energy functions of linear systems has been connected with the solution of suitable optimum least square problems.

These mostly theoretical results have been followed by several applications: it will be sufficient to mention here the introduction of general synthesis techniques for passive m-ports, both reciprocal and non reciprocal, on the basis of the so called positive real lemma.

The purpose of this paper is to investigate how the system theoretic concepts of controllability and observability interply with dissipativity properties of linear time-invariant dynamical systems. While observability implies the rather perspicuous consequence that any energy function is strictly positive definite, the consequences of (non) controllability are more elusive and refer to the possibility of obtaining non dissipative systems described by positive real matrices.

Such properties are also discussed when considering the so called "generalized linear systems" which constitute the natural framework for modeling linear electrical networks.

2. THE CONCEPT OF DISSIPATIVITY

In this section we briefly recall from [1] how the concepts of energy and dissipativity can be introduced in a system theoretic context. The definitions we give and the properties we derive are very general and do not refer in any way to the linearity hypothesis. In the next sections we shall restrict ourselves to linear time-invariant systems. As we shall see, the dissipativity property reflects very strongly on the internal structure of these systems as well as in the external input/output map.

Defin.1 *Let Σ be a regular, continuous, time-invariant systems. A supply function on the cartesian product of input and output alphabets* U x Y *is a real valued scalar function*

$$(2.1)\quad w : U \times Y \to \mathbb{R} : (\alpha,\beta) \mapsto w(\alpha,\beta)$$

such that

$$(2.2)\quad \int_{t_0}^{t_1} |w(u(t),y(t))|\, dt < \infty$$

for any finite time interval $[t_0,t_1]$, *any input* u *and initial state* x_0 *at time* t_0.

(y denotes the output which corresponds to x_0 and u_0). Clearly for any given Σ there is an infinite set of possible supply functions. From now on we will assume that a specific w has been chosen as supply function, and we will denote by (Σ,w) the system Σ with the supply w.

Needless to say, we shall find it convenient to choose a meaningful supply w. For instance, when dealing with electrical networks U and Y denote the spaces of instantaneous port currents i and port voltages v, and w(i,v) is just

$$(2.3)\quad w(i,v) = i^T v$$

which denotes the power flowing into the network.

The following definition formalizes the intuitive concept of dissipative system we think of as a physical structure which accepts some work at the device terminals, accumulates the work as "internal energy" and gives back (partially or totally) the previously stored energy.

Defin.2 *A dynamical system* (Σ,w) *is dissipative (with respect to a supply function* w) *if there* *exists a non negative real function on the state set* X

(2.4) $S : X \to \mathbb{R}_+$

such that, for any time interval $[t_0, t_1]$, *for any initial state* x_0 *and for any input* u

$$(2.5) \quad \int_{t_0}^{t_1} w(u(t),y(t))dt \geq S(x_1) - S(x_0)$$

where y *denotes the output which corresponds to the input* u *and to the initial state* $x(t_0) = x_0$. *Every function* S *satisfying condition (2.5) is called an "energy function".*

When the initial state x_0 and the terminal state x_1 coincide, we write the left hand side in (2.5) as

$$\oint w(t)dt$$

so that (2.5) becomes

$$(2.6) \quad \oint w(t)dt \geq 0$$

Dissipativity checking constitutes in general a very hard task. In this regard the "available storage" and the "required supply" are fundamental state functions which provide some necessary and sufficient conditions. As a matter of fact, they are also the cornerstones of our subsequent analysis of dissipative linear systems.

The available storage S_a can be defined for every system - independently on its dissipativity properties: intuitively $S_a(x)$ represents a measure of the maximum work the system (Σ,w) can deliver to some external device when starting from the state x.

Defin.3 *Let* (Σ,w) *be a dynamical system with a supply function* w. *The available storage* S_a *is the map* (°)

$$(2.7) \quad S_a : X \to \mathbb{R}_+^e : x_0 \mapsto S_a(x_0) = \sup_{\substack{x_0 \to \\ t_1 \geq t_0}} - \int_{t_0}^{t_1} w(t)dt$$

The supremum is evaluated along all trajectories starting from the state x_0 *at time* t_0.

The following Theorem is important in two respects: first of all it shows how the available storage is related to the dissipativity, next it proves that the set of the energy functions (if not empty) has a minimum element.

Theor.1 [1] *A system* (Σ,w) *is dissipative if and only if its available storage is everywhere finite*

$$(2.8) \quad 0 \leq S_a < +\infty$$

In this case the function S_a *is an energy function and the inequality*

$$(2.9) \quad S_a \leq S$$

holds for any energy function S.

When the state set X is reachable from some state $\bar{x}$ the check of finiteness involved in Theorem 1 can be restricted to the trajectories starting from $\bar{x}$.

(°) $\mathbb{R}_+^e$ denotes the set of non negative real numbers extended with the symbol $+\infty$.

Theor.2 [1] *Let the state set* X *in* (Σ,w) *be completely reachable from* $\bar{x}$ *in* X. *Then* (Σ,w) *is dissipative if and only if there is a constant* K *such that*

$$(2.10) \quad \inf_{\substack{\bar{x} \to x \\ t_1 \geq t_0}} \int_{t_0}^{t_1} w(t)dt \geq K \quad , \forall\, x \in X$$

and a possible energy function is given by

$$S(x) = S_a(\bar{x}) + \inf_{\substack{\bar{x} \to x \\ t_1 \geq t_0}} \int_{t_0}^{t_1} w(t)dt$$

In the sequel we shall assume that the available storage S_a of every system (Σ,w) we consider vanishes in some state $x^* \in X$. (As long as we are interested in linear dissipative systems, this is by no means an effective restriction, since the zero-state available storage is either zero or infinity). Hence the integral supply the system needs when starting from x^* is non negative.

This makes the "required supply" we introduce below a good candidate for energy function.

Defin.4 *Let* x^* *in* X *satisfy* $S_a(x^*) = 0$. *The required supply function (from the state* x^**) is a map*

$$(2.11) \quad S_{r/x^*} : X \to \mathbb{R}_+^e$$

defined as follows

$$S_{r/x^*}(x) = \begin{cases} \infty & \text{if } x \text{ is not reachable from } x^* \\ \inf\limits_{x^* \to x} \int_{t_0}^{t_1} w(t)dt & \text{if } x \text{ is reachable from } x^* \end{cases}$$

$\inf\limits_{x^* \to x} \int_{t_0}^{t_1} w(t)dt$ is evaluated along all trajectories from x^* to x and it exists as a real number if and only if x can be reached from x^*. The way S_{r/x^*} is related to the set of energy functions is shown by the dissipation inequality. It provides upper and lower bounds for the family of energy functions which vanish at the reference state x^*.

Theor.3 (Dissipation Inequality [1]) *Let* (Σ,w) *be a dissipative dynamical system and let* x^* *satisfy* $S_a(x^*) = 0$. *Then for any energy function* S *which satisfies* $S(x^*) = 0$ *one has*

$$(2.12) \quad S_a(x) \leq S(x) \leq S_{r/x^*}(x)$$

Moreover, if X *is completely reachable from* x^*, *then* $S_{r/x^*}(x)$ *is an energy function.*

It is interesting to point out that the set of all possible energy functions of a dissipative dynamical system (Σ,w) is a convex set. In particular, if X is reachable from x^* and if $0 \leq \beta \leq 1$, then $\beta S_a + (1-\beta)S_{r/x^*}$ is still an energy function.

Defin.5 *Let* (Σ,w) *be a dissipative dynamical system, and let* S *be one of its energy functions.* (Σ,w) *is lossless (with respecy to S) if for any state* x_0 *in* X *and any input driving* Σ *from* x_0 *at time* t_0 *to* x_1 *at time* t_1 *one has*

$$(2.13) \quad \int_{t_0}^{t_1} w(t)dt = S(x_1) - S(x_0)$$

In other words, in a lossless system the increase of the stored energy along any system trajectory is equal to the supply the system gets by describing the trajectory. If the state set is connected, the lossless property does not depend particular energy function one considers. In fact the energy function is unique as far as the energy functions have to vanish in x .

Theor.4 [1] *Let* (Σ,w) *be a dissipative dynamical system and assume the state set X to be connected. If* (Σ,w) *is lossless (with respect to the energy function S), then*

i) $S_a(x) = S(x) = S_{r/x^*}(x)$ *for any* x *in* X

ii) $S(x) = \int_0^{t_1} w(t)dt = -\int_0^{t_2} w(t)dt$ *for any* x *in* X
$\qquad\quad x^*\to x \qquad\qquad x\to x^*$

and for any trajectory from x^* *to* x *(from* x *to* x^**).*

3. DISSIPATIVITY IN LINEAR SYSTEMS

Consider now a linear time-invariant system $\Sigma = (A,B,C,D_0)$ of dimension n

$$(3.1)\qquad \begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + D_0u\end{aligned}$$

with m inputs and m outputs, and introduce the standard supply function

$$(3.2)\qquad w = u^Ty$$

We are now in a position to explicitly determine the structure of an energy function.

Lemma 1 . *Let* $\Sigma = (A,B,C,D_0)$ *be dissipative. Then the available storage is a non negative definite quadratic form.*

The proof relies on the property that a map $f : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form if and only if i) $f(\lambda x) = \lambda^2 f(x)$ for any real λ, ii) $f(x_1) + f(x_2) = \frac{1}{2}f(x_1+x_2) + \frac{1}{2}f(x_1-x_2)$. In a linear system, S_a satisfies conditions i) and ii) whenever it is finite.

In general the family of quadratic energy functions is not restricted to the available storage. In any case, its members are strongly related to the structure of matrices A,B,C,D_0, and it is possible to characterize dissipativity directly in terms of these matrices. In this respect, Theorem 5 provides the fundamental facts.

Theor.5 [1,3] *Let* $\Sigma = (A,B,C,D_0)$ *be an n-dimensional linear system. Then the following propositions are equivalent:*

o) Σ *is dissipative*

i) *the set* $\mathcal{S}_1$ *of non negative definite*[(o)] $n \times n$ *matrices* Π *such that* $S(x) = x^T\frac{\Pi}{2}x$ *is an energy function is non empty*

ii) *the set* $\mathcal{S}_2$ *of non negative definite solutions of the inequality*

$$(3.3)\qquad \begin{bmatrix} D_0 + D_0^T & C - B^T\Pi \\ C^T - \Pi B & -A^T\Pi - \Pi A \end{bmatrix} \geq 0$$

is non empty

iii) *the set* $\mathcal{S}_3$ *of the triples* (Π,H,J), $\Pi \geq 0$, *which satisfy the equations*

$$(3.4)\qquad \begin{aligned}\Pi A + A^T\Pi &= -H^TH\\ \Pi B &= C^T - H^TJ\\ J^TJ &= D_0 + D_0^T\end{aligned}$$

is non empty.

iv) *if* $D_0 + D_0^T$ *is a unit, the set* $\mathcal{S}_4$ *of non negative definite solutions of*

$$(3.5)\qquad \Pi A + A^T\Pi + (\Pi B - C^T)(D_0 + D_0^T)^{-1}(B^T\Pi - C) \geq 0$$

is non empty.

If $D_0 + D_0^T$ *is not an unit, the set* $\mathcal{S}_4' = \lim_{\varepsilon\to 0+} \mathcal{S}_{4\varepsilon}$ *is not empty:* $\mathcal{S}_{4\varepsilon}$ *denotes the set of non negative definite solutions of*

$$(3.6)\qquad \Pi A + A^T\Pi + (\Pi B - C^T)(D_0 + D_0^T + \varepsilon I_m)^{-1}$$
$$(B^T\Pi - C) \leq 0$$

The sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ *and* $\mathcal{S}_4$ *(or* $\mathcal{S}'_4$*) coincide.*

Proof. o)$\Leftrightarrow$i). If Σ is dissipative, by Lemma 1 a quadratic energy function does exist, namely S_a.

i)$\Leftrightarrow$ii). Let $S(x) = \frac{1}{2}x^T\Pi x$ be any quadratic energy function. Then for any $x(t_0) \in X$ and for any input u we have

$$\int_{t_0}^{t_0+\varepsilon} u^Tydt \geq x(t_0+\varepsilon)^T\frac{\Pi}{2}x(t_0+\varepsilon) - x(t_0)^T\frac{\Pi}{2}x(t_0)$$

Taking the limit as $\varepsilon\to 0$, we obtain

$$\begin{bmatrix} u^T(t_0) & x^T(t_0)\end{bmatrix} \begin{bmatrix} D_0+D_0^T & -B^T\Pi + C \\ C^T - \Pi B & -\Pi A - A^T\Pi\end{bmatrix} \begin{bmatrix} u(t_0) \\ x(t_0)\end{bmatrix} \geq 0$$

Since $u(t_0)$ and $x(t_0)$ are arbitrary, the matrix Π satisfies the inequality (3.3) and $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

Conversely, if $\Pi \geq 0$ is a solution of (3.3), then

$$(3.7)\qquad u^Ty \geq \frac{d}{dt}\, x^T\frac{\Pi}{2}x$$

Integration of (3.7) shows that the quadratic form $x^T\frac{\Pi}{2}x$ is an energy function and $\mathcal{S}_1 = \mathcal{S}_2$:

$$\int_{t_0}^{t_1} u^Tydt \geq x^T\frac{\Pi}{2}x\Big|_{t_0}^{t_1}$$

ii)$\Rightarrow$iii). If $\Pi \geq 0$ is a solution of (3.3), the following factorization holds

$$\begin{bmatrix} D_0+D_0^T & C - B^T\Pi \\ C^T - \Pi B & -\Pi A - A^T\Pi\end{bmatrix} = \begin{bmatrix} J^T \\ H^T\end{bmatrix}\begin{bmatrix} J & H\end{bmatrix}$$

because the left hand side is non negative definite.

Implication iii)$\Rightarrow$ii) and equality $\mathcal{S}_2 = \mathcal{S}_3$ are clear.

iii)$\Rightarrow$iv). Let now $D_0 + D_0^T$ have full rank and let (Π,H,J) satisfy (3.4). Since the spectrum of $J(J^TJ)^{-1}J^T$ contains just the eigenvalues 0 and 1, $1_m - J(J^TJ)^{-1}J$ is non negative definite and $H^TJ(J^TJ)^{-1}J^TH \leq H^TH$ implies iv):

$$\Pi A + A^T\Pi = -H^TH \leq -H^TJ(J^TJ)^{-1}J^TH =$$
$$= -(\Pi B - C^T)\,(D_0+D_0^T)^{-1}(B^T\Pi - C)$$

iv)$\Rightarrow$ii). Let $\Pi \geq 0$ satisfy inequality (3.5). Then there exists a matrix N such that

(o) For any square matrix M, $M\geq 0$ (resp. $M>0$) denotes that M is non negative (resp. positive) definite.

$$\Pi A+A^T\Pi+(\Pi B-C^T)(D_0+D_0^T)^{-1}(B^T\Pi-C) = -N^TN$$

holds and (3.4) is proved by introducing the matrices

$$H^T = \left[-(\Pi B-C^T)(D_0+D_0^T)^{-1/2} \;\middle|\; N^T\right]$$

$$J = \left[(D_0+D_0^T)^{1/2} \;\middle|\; 0\right]$$

As an easy consequence, when $D_0+D_0^T$ is a unit $\mathcal{S}_3$, $\mathcal{S}_4$ (and of course $\mathcal{S}_2$) coincide.

Finally let ii) hold and let $D_0+D_0^T$ be singular. For any $\varepsilon>0$, $D_0+D_0^T+1_m\varepsilon$ is full rank, and the system $(A,B,C,D_0+\frac{\varepsilon}{2}1_m)$ (which is dissipative whenever Σ is) gives a non empty set $\mathcal{S}_{2\varepsilon}$ of non negative solutions of the inequality

$$0\leq\begin{bmatrix} D_0+D_0^T+I_m\varepsilon & C-B^T\Pi \\ C^T-\Pi B & -A^T\Pi-\Pi A\end{bmatrix}=\begin{bmatrix}\varepsilon I_m & 0\\ 0 & 0\end{bmatrix}+$$

$$+\begin{bmatrix} D_0+D_0^T & C-B^T\Pi \\ C^T-\Pi B & -A^T\Pi-\Pi A\end{bmatrix}$$

Plainly

(3.8) $\mathcal{S}'_{2\varepsilon} = \mathcal{S}_{4\varepsilon}$

for any $\varepsilon>0$, and $\{\mathcal{S}_{2\varepsilon}\}$ is an ε-indexed family of sets which monotonically decreases as $\varepsilon\to0_+$. Since the non empty set $\mathcal{S}_2$ is the intersection of the family

$$\mathcal{S}_2 = \lim_{\varepsilon\to0_+}\mathcal{S}_{2\varepsilon} = \bigcap_{\varepsilon>0}\mathcal{S}_{2\varepsilon}$$

we get from (3.8)

$$\mathcal{S}'_4 = \lim_{\varepsilon\to0_+}\mathcal{S}_{4\varepsilon} = \lim_{\varepsilon\to0_+}\mathcal{S}_{2\varepsilon} = \mathcal{S}_2$$

This gives also the converse iv)$\Rightarrow$ii). □

The equivalence o)$\Leftrightarrow$iii) is usually called the positive real lemma [3].

It is interesting at this point to investigate how controllability and observability are related to some simple properties of the set of non negative solutions of (3.3). The proof of the following Lemma is identical with that of Lemma 1.

Lemma 2 . *Let $\Sigma=(A,B,C,D_0)$ be a dissipative and controllable linear system. The required supply (from zero state) is a non negative quadratic function:*

(3.9) $S_{r/0}(x) = x^T\frac{\Pi_r}{2}x \quad , \quad \Pi_r\geq0$

Since non negative solutions of (3.3) biuniquely correspond with quadratic energy functions, the dissipation inequality leads to the following

Theor.6 *Let $\Sigma=(A,B,C,D_0)$ be dissipative and controllable. Then the set of non negative solutions of (3.3) has Π_r as l.u.b.*

The observability assumption is completely equivalent to stipulate that the origin is a strong minimum point of any energy function.

Theor.7 *Let $\Sigma=(A,B,C,D_0)$ be dissipative. Then Σ is observable if and only if*

$$S(x) = 0 \Rightarrow x = 0$$

for any energy function S.

Proof. Let Σ be observable and assume $S(x_0)=0$ for some $x_0\neq0$. Then for any input u and any instant $t_1\geq0$ one has

$$\int_0^{t_1}\underset{x_0\to}{u^Ty}\,dt = \int_0^{t_1}\underset{x_0\to}{(u^TCx+u^TD_0u)}\,dt\geq0$$

Hence the ε indexed family of inputs

$$u_\varepsilon = \varepsilon C\exp(At)x_0 \qquad \varepsilon\in\mathbb{R}$$

gives

$$0\leq\int_0^{t_1}\Big[u^TC(\exp(A^Tt)x_0 + $$

$$+\int_0^t\exp(A(t-\sigma))Bu_\varepsilon(\sigma)d\sigma) + u^TD_0u\Big]\,dt =$$

(3.10) $$= \varepsilon\int_0^{t_1}\Big[x_0^T\exp(A^Tt)C^TC\exp(At)x_0\Big]dt +$$

$$+\varepsilon^2\int_0^{t_1}\Big[u_1^TC\int_0^t(\exp(A(t-\sigma))Bu_1(\sigma))d\sigma +$$

$$+ u_1^TD_0u_1\Big]dt$$

When t_1 is fixed, the integrals in the right hand side of (3.10) are constant terms, which we will denote by m and n respectively. Hence

$$0\leq\varepsilon m+\varepsilon^2 n \qquad \forall\varepsilon\in\mathbb{R}$$

This implies $n\geq0$ and

$$m=0=x_0^T\int_0^{t_1}\exp(A^Tt)C^TC\exp(At)dt\; x_0$$

Consequentlt Σ is non observable, contrary to the assumptions.

The converse holds too. In fact, if $x_0\neq0$ is indistinguishable from the zero state, the available storage is zero in x_0. Hence an energy function exists which annihilates in (at least) two points. □

Finally, we shall establish a dissipativity condition which holds when the system (Σ,w) starts from the zero initial state. This condition is fundamental in the dissipativity analysis of transfer functions.

We emphasize that controllability is needed in proving the equivalence between dissipativity and inequality (3.11).

Theor.8 *Let $\Sigma=(A,B,C,D_0)$ be dissipative and assume $x(0)=0$. Then for any input* u *and for any time* $t_1\geq0$

(3.11) $$\int_0^{t_1}\underset{0\to}{u^Ty}\,dt\geq0$$

Conversely, let (2.11) hold and let Σ be controllable. Then Σ is dissipative.

Proof. Suppose first that Σ is dissipative and assume $x(0)=0$. When replacing an input u by the input ku ($k\in\mathbb{R}$), the corresponding output y becomes ky. Consequently if some input u gives

$$\int_{0}^{t_1} u^T(t)y(t)dt = h < 0 \qquad (0\to)$$

the input ku gives

$$\int_{0}^{t_1} ku^T(t)ky(t)dt = k^2h < 0 \qquad (0\to)$$

This would imply that the available storage in the zero state is infinite, contrary to the assumption of dissipativity.

Conversely, if Σ is controllable and (3.11) holds, Theorem 2 can be applied. In fact the state space X is reachable from the zero state and

$$\inf_{\substack{0\to \\ t_1\geq 0}} \int_{0}^{t_1} u^T y dt \geq 0$$

□

Remark 1. When Σ is non controllable, in general the second part of Theorem 8 does not hold. For instance the electrical network of fig. 1, which includes a driven generator, does not constitute a dissipative dynamical system. Observe that (3.11) still holds and the state space is not completely controllable.

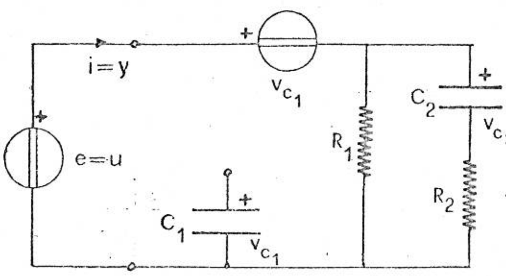

fig. 1

Remark 2. The dissipativity analysis in the Laplace transform domain is based on the result of Theorem 8. In fact the (matrix) transfer function of a system $\Sigma = (A,B,C,D_0)$ is positive real if and only if the inputs and outputs it relates satisfy condition (3.11) in the time domain [3,6].

It is worthwhile to point out that the positive real property of the transfer function does not guarantee the dissipativity of Σ if Σ is not controllable.

4. LOSSLESS LINEAR SYSTEMS

The relations we considered in Sec.3 have some interesting particularizations when $\Sigma = (A,B,C,D_0)$ is lossless with respect to some energy function S.

Lemma 3 . *Let* $\Sigma = (A,B,C,D_0)$ *be lossless with respect to an energy function* S. *Then* Σ *is lossless with respect to the available storage* S_a.

Proof. Assume that x_0 and x_1 in X satisfy

$$\int_{x_0\to x_1} w(t)dt = S_a(x_1) - S_a(x_0) + \varepsilon$$

for some $\varepsilon>0$.

Then, by the definition of available storage

$$S(x_1) - S(x_0) = -\inf_{x_1\to z}(S(z)-S(x_1)) + \inf_{x_0\to v}(S(v)-S(x_0)) + \varepsilon$$

Hence

$$\inf_{x_1\to z} S(z) = \inf_{x_0\to v} S(v) + \varepsilon$$

which implies the existence of some state v_0, reachable from x_0, such that

(4.1) $S(z) > S(v_0) + \frac{1}{2}\varepsilon$

for any state z reachable from x_1.

Let $x(\cdot)$ and $\bar{x}(\cdot)$ be two different trajectories of Σ both starting at time $t=0$ from the state $x(0)=x_0$. Then for any pair t_1,t_2 with $0\leq t_1<t_2$ there exists a trajectory connecting $x(t_1)$ and $\bar{x}(t_2)$.

Consider now two trajectories of Σ: the first one starts at time $t=0$ from x_0, meets x_1 and ends in some state z at time t_z; the second one starts at time $t=0$ from x_0, meets v_0 at some time t_{v_0} and ends in v_1 at some time $t_{v_1}>t_z$. Suppose that in the time interval $[t_{v_0},t_{v_1}]$ the second trajectory is given by a free evolution of Σ (i.e. $u(t)=0$ for $t\in[t_{v_0},t_{v_1}]$).

The assumptions we made imply

(4.2) i) $S(v_1) = S(v_0) + \int_{t_{v_0}}^{t_{v_1}} w dt = S(v_0)$

ii) there exists a trajectory connecting z at time t_z with v_1 at time t_{v_1}.

Then reachability of v_1 from x_1 gives

$$S(v_1) > S(v_0) + \tfrac{1}{2}\varepsilon$$

which contradicts (4.2). □

As a consequence of Lemma 3, whenever Σ is lossless we assume that it is lossless with respect to a quadratic energy function (e.g. the available storage).

Theor.9 *The following propositions are equivalent:*

i) $\Sigma = (A,B,C,D_0)$ *is lossless dissipative (with respect to some energy function)*

ii) *The set* $\mathcal{S}_1$ *of non negative definite nxn matrices* Π *such that* Σ *is lossless with respect to* $S(x) = \frac{1}{2}x^T\Pi x$ *is non empty*

ii) *The set* $\mathcal{S}_2$ *of non negative definite matrices* Π *which satisfy the equation*

$$\begin{bmatrix} D_0+D_0^T & C-B^T\Pi \\ C^T-\Pi B & -A^T\Pi-\Pi A \end{bmatrix} = 0 \qquad (4.3)$$

is non empty.

iii) *The set* $\mathcal{S}_3$ *of non negative definite solutions* Π of the system

$$\begin{aligned} \Pi A + A^T\Pi &= 0 \\ \Pi B &= C^T \\ 0 &= D_0 + D_0^T \end{aligned} \qquad (4.4)$$

is non empty.

The sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ *coincide.*

The equivalence o)⇔i) follows from Lemma 3 and the remaining equivalences are proved as in Theorem 6.

Remark. By Theorem 4, if Σ is controllable the set $\mathscr{S}_1$ contains exactly one matrix.

The following Theorem is the lossless counterpart of Theorem 8.

Theor.10 *If Σ is a lossless linear system, then each closed trajectory in X satisfies*

$$(4.5)\quad \oint w(t)dt = 0$$

If Σ is controllable and dissipative, and if

$$\oint w(t)dt = 0$$

holds for any closed trajectory in X, then Σ is lossless.

Proof. The first part is obvious. To prove the second part, we resort to the connectedness of X. If Σ is not lossless with respect to some energy function S, along some trajectory in X the strong inequality

$$\int_{\substack{t_0 \\ x_0 \to x_1}}^{t_1} w(t)dt > S(x_1) - S(x_0)$$

holds. Taking now a trajectory from x_1 to x_0, dissipativity implies

$$\int_{\substack{t_1 \\ x_1 \to x_0}}^{t_2} w(t)dt \geq S(x_0) - S(x_1)$$

Concatenating the two trajectories one gets

$$\oint w(t)dt > 0$$

contrary to the assumption. □

Remark. Condition (4.5) refers to the set of all closed trajectories in X. It is easy to see that one could restrict (4.5) to the set of trajectories which have their origin and their end in the zero state. In fact, suppose $\vec{\gamma}$ to be a cycle which "starts" and "ends" in x_0. By the connectedness of X, there is a trajectory $\vec{\lambda}_1$ from 0 to x_0 as well as a trajectory $\vec{\lambda}_2$ from x_0 to 0. Clearly

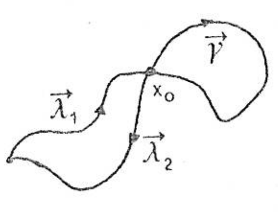

fig. 2

$$(4.6)\quad \oint_{\vec{\lambda}_1+\vec{\gamma}+\vec{\lambda}_2} w(t)dt = 0 \qquad \oint_{\vec{\lambda}_1+\vec{\lambda}_2} w(t)dt = 0$$

hold, so that along the $\vec{\gamma}$ trajectory we have

$$\oint_{\vec{\gamma}} w(t)dt = 0$$

5. DISSIPATIVITY IN GENERALIZED LINEAR SYSTEMS

It is known that the model of linear system we considered in Sec. 3 in general is not sufficient for modeling a linear time-invariant electrical network [3].

This drawback can be overcome by extending the class of dynamical systems up to include the so called "generalized linear systems (GLS)":

$$(5.1)\quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \sum_{0}^{d} D_i \frac{d^i u}{dt^i} \end{aligned}$$

Dissipativity definitions and theorems require some restrictions on the input admissible functions when GLSs are considered. These restrictions could be justified from an heuristic point of view: if we pursue in adopting $u^T y$ as supply function we should avoid the possibility of impulses in the output function y, which would imply a non infinitesimal flow of energy through the system gates during an infinitesimal time interval.

Consequently *when dealing with supply functions* we will restrict the set of possible inputs of (5.1) to the d-1 times continuously differentiable functions which fulfill the "initial" conditions

$$(5.2)\quad \begin{aligned} u(0) &= u(0_-) = u(0_+) \\ u'(0) &= u'(0_-) = u'(0_+) \\ &\cdots\cdots\cdots\cdots \\ u^{(d-1)}(0) &= u^{(d-1)}(0_-) = u^{(d-1)}(0_+) \end{aligned}$$

In this way no impulses are included in the output functions and the supply integral makes sense. The stored energy at time t = 0 is assumed to depend on the state x(0) as well as on the values $u(0_-)\ldots u^{(d-1)}(0_-)$, all of which determine the behaviour of the GLS on the closed interval $[0,+\infty]$ in the sense that the impulse content at time zero depends on the jumps $u^{(i)}(0_+) - u^{(i)}(0_-)$, i = 0,1,... d-1.

We therefore give the following Definition.

Defin.6 *A GLS $\Sigma = (A,B,C,D_0,D_1,\ldots D_d)$ with m inputs and m outputs is dissipative with respect to the supply function $w = u^T y$ if there exists a function* S

$$S : \mathbb{R}^n \times \underbrace{\mathbb{R}^m \times \cdots\cdots \times \mathbb{R}^m}_{d\ \text{times}} \to \mathbb{R}_+$$

$$S : (x(0_-), u(0_-), \ldots u^{(d-1)}(0_-)) \mapsto$$

$$\mapsto S(x(0_-), \ldots u^{(d-1)}(0_-))$$

such that for any $t_1 \geq 0$ and any d-1 times continuously differentiable input u which satisfies conditions (5.2) one has

$$(5.3)\quad \int_0^{t_1} u^T y dt \geq S(x(t_1), u(t_1)\ldots u^{(d-1)}(t_1)) - S(x(0), u(0_-), \ldots u^{(d-1)}(0_-))$$

Starting from its d-th derivative $u^{(d)}$, a function u can be reconstructed when $u(0), u'(0), \ldots u^{(d-1)}(0)$ are known. Assuming that the input u of the GLS $\Sigma = (A,B,C,D_0,D_1,\ldots D_d)$ satisfies (5.2), the standard linear system $\overline{\Sigma} = (F,G,H,J)$

$$(5.4)\quad \begin{aligned} \dot{z} &= Fz + Gv \\ y &= Hz + Jv \end{aligned}$$

gives the same output as the original GLS does if we assume $v(t) = u^{(d)}(t)$ and

$$G^T = [0\ 0\ 0\ \dots\ I_m] \qquad H = [C\ D_0\ D_1\ \dots\ D_{d-1}]$$

$$(5.5)\quad F = \begin{bmatrix} A & B & 0 & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ & & & \ddots & \\ & & & & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \qquad z(0) = \begin{bmatrix} x(0) \\ u(0\_) \\ \dots \\ u^{(d-1)}(0\_) \end{bmatrix}$$

Therefore the supply $w = u^T y$ of the GLS Σ can be viewed in the linear system $\bar{\Sigma}$ as a function of the state z and of the input v

$$(4.6)\quad w = z^T \begin{bmatrix} 0 & 0 & \dots & 0 \\ C & D_0 & \dots & D_{d-1} \\ 0 & 0 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & 0 \end{bmatrix} z + z^T \begin{bmatrix} 0 \\ D_d \\ 0 \\ \dots \\ 0 \end{bmatrix} v$$

The definition of available storage extends to a GLS in a natural way:

Defin.7 *The available storage of a GLS* $\Sigma = (A,B,C,D_0,D_1,\dots D_d)$ *with supply function* $w = u^T y$ *is the map*

$$S_a : \mathbb{R}^n \times \mathbb{R}^m \times \dots \mathbb{R}^m \to \mathbb{R}^e_+ :$$

$$(5.7)\quad : (x(0), u(0\_), \dots u^{(d-1)}(0\_)) =$$

$$= z(0) \mapsto \sup_{\substack{u \in \tilde{\mathcal{U}} \\ t_1 \geq 0}} - \int_0^{t_1} u^T y\, dt$$

where $\tilde{\mathcal{U}}$ *denotes the set of d-1 times continuosly differentiable input functions which satisfy condition (5.2).*

Following the pattern of the proofs given for standard dynamical systems, it is easy to show that S_a is everywhere finite if and only if Σ is dissipative. Moreover when Σ is dissipative S_a is an energy function expressed by a non negative definite quadratic form:

$$(5.8)\quad S_a(z) = z^T \frac{\Pi}{2} z$$

The interest of the following Lemma is in that it restricts to the first input derivative the dependence of y on u in any dissipative GLS.

Lemma 4 . *A dissipative GLS* $\Sigma = (A,B,C,D_0,D_1,\dots D_d)$ *satisfies the following conditions:*

$$(5.9)\quad D_2 = D_3 = \dots = D_d = 0$$

Proof. Since S_a is an energy function, (5.8) gives

$$(5.10)\quad \int_0^{\varepsilon} u^T y\, dt \geq \geq [x^T(t) u^T(t) \dots u^{(d-1)^T}(t)] \frac{\Pi}{2} \begin{bmatrix} x(t) \\ u(t) \\ \dots \\ u^{(d-1)}(t) \end{bmatrix} \Bigg|_0^{\varepsilon}$$

for every input u which satisfies conditions (5.2).

Let $d > 1$ and $D_d \neq 0$. Taking in (5.10) the limit as ε goes to zero and partitioning Π conformably with the block partitions of F,G,etc., from (5.6) we have

$$(5.11)\quad z^T(0) \left\{ \begin{bmatrix} 0 & 0 & \dots & 0 \\ C & D_0 & \dots & D_{d-1} \\ \dots & & & \\ 0 & 0 & \dots & 0 \end{bmatrix} - \Pi F \right\} z(0) \geq \geq z^T(0) \begin{bmatrix} \Pi_{d+1,1} \\ \Pi_{d+1,2} - D_d \\ \dots \\ \Pi_{d+1,d+1} \end{bmatrix} v(0)$$

The above inequality has to be satisfied for every choice of v(0) and z(0). Hence

$$\Pi_{d+1,1} = 0,\ \Pi_{d+1,2} = D_d,\ \Pi_{d+1,3} = \Pi_{d+1,4} = \dots = \Pi_{d+1,d+1} = 0$$

and Π has the following structure

$$(5.12)\quad \Pi = \left[\begin{array}{c|c} & 0 \\ & D_d^T \\ * & 0 \\ & \vdots \\ \hline 0\ \ 0\ \ 0\ \dots & 0 \end{array} \right] \quad \text{d+1 blocks}$$

which agrees with condition $\Pi \geq 0$ if $D_d = 0$.

The only alternate possibility is to assume $d = 1$. This implies that $D_d = D_1$ has diagonal position in (5.12) and the resulting structure of Π is compatible with the assumption $\Pi \geq 0$. □

When $d = 1$, (5.11) simplifies as follows

$$(5.13)\quad z^T(0) \left\{ \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix} - \Pi F \right\} z(0) \geq z^T(0) \begin{bmatrix} \Pi_{21} \\ \Pi_{22} - D_1 \end{bmatrix} v(0)$$

and we get the following conditions on the blocks of Π

$$\Pi_{21} = \Pi_{12} = 0$$

$$(5.14)\quad \Pi_{22} = D_1 \geq 0$$

$$\Pi_{11} \geq 0$$

After replacing F with its explicit formula (5.5), we have

$$(5.15)\quad \begin{bmatrix} -\Pi_{11}A - A^T\Pi_{11} & C^T - \Pi_{11}B \\ C - B^T\Pi_{11} & D_0 + D_0^T \end{bmatrix} \geq 0$$

This shows that the relation (3.3) which characterizes a dissipative standard linear system is still a necessary dissipativity condition for a GLS. (5.14) and (5.15) therefore proved the necessity part of the following

Theor.11 *A GLS* $\Sigma = (A,B,C,D_0,D_1,\dots D_d)$ *is dissipative if and only if*

i) $D_2 = D_3 = \dots = D_d = 0$

ii) $D_1 \geq 0$

iii) *the linear system* $\Sigma' = (A,B,C,D_0)$ *is dissipative.*

For the sufficiency part, observe that dissipativity of Σ' is equivalent to the non negative definiteness of

$$\begin{bmatrix} D_0 + D_0^T & C - B^T \Pi \\ C^T - \Pi B & -\Pi A - A^T \Pi \end{bmatrix}$$

This implies

$$(5.16) \quad u^T D_0 u + u^T C x \geq x^T \Pi A x + x^T \Pi B u$$

for any $u \in \mathbb{R}^m$ and for any $x \in \mathbb{R}^n$. From the GLS equations

$$\dot{x} = Ax + Bu$$

$$y = Cx + D_0 u + D_1 \dot{u}$$

and from (5.16) one gets

$$u^T (y - D_1 \dot{u}) \geq x^T \Pi \dot{x}$$

$$u^T y \geq \frac{d}{dt} \tfrac{1}{2}(u^T D_1 u + x^T \Pi x)$$

Hence

$$\int_0^{t_1} u^T y dt \geq \left. (u^T \frac{D_1}{2} u + x^T \frac{\Pi}{2} x) \right|_0^{t_1}$$

and we conclude that the function $S(x,u)$

$$S(x,u) = \tfrac{1}{2} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} \Pi & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix}$$

is an energy function of Σ. □

It is interesting to point out that, even if $\Sigma' = (A,B,C,D_0)$ is observable, the dissipative GLS $\Sigma = (A,B,C,D_0,D_1)$ exhibits a non strictly positive definite energy function whenever the matrix D_1 is non invertible. As a matter of fact, we should think at

$$\bar{x} = u(0_+) - u(0_-)$$

as to a supplementary state vector, since the impulse content of y at time $t = 0$ depends on the jumps of the input and input derivatives. If D_1 does not have full rank, it is easy to check that $\bar{x}$ cannot be recovered from the output impulse content and we could label this situation as a non observable one.

Hence, provided that a reasonable extension of observability have been introduced, Theorem 7 holds for GLS too.

Finally, the inequality

$$(5.17) \quad \int_{0\ 0\to}^{t} u^T y dt \geq 0$$

for any input u satisfying (5.2), is still a necessary (and, when $\Sigma' = (A,B,C,D_0)$ is controllable, sufficient) dissipativity condition. In terms of Laplace transform (5.17) is now the time domain exact counterpart of the positive real property of transfer functions in the classical Network Theory setup, because the presence of poles at infinity is allowed in transfer functions of GLS.

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