RICERCHE DI AUTOMATICA
Vol. 5, Nos. 2-3, December 1974
Printed in Italy
Short Paper

EVALUATION OF MASON'S FORMULA BY USING CONNECTION MATRICES

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Abstract. A procedure is presented for evaluating the transference of a signal flow graph. The procedure refers to Mason's formula and is based on a correspondence between addenda of numerator and denominator of this formula and suitable connection matrices.

1. Introduction.

The use of Mason's formula for evaluating the transference connecting a source node i to another node o of a signal flow graph, needs — as is well known — the preliminary identification of the i-o paths and of the loops of the graph; it is also necessary to ascertain if each loop is touching or not touching each other considered subpath, i. e. if it contains a node belonging also to another loop or to an i-o path [1, 2].

Mason's formula, in fact, may be presented in the form:

$$\frac{\sum_{1}^{p}{}_k \,[P_k (1-L_1) \dots (1-L_q)]^*}{[(1-L_1) \dots (1-L_q)]^*} \tag{1}$$

where P_k is the transference of the k-th path from i to o;
L_h is the loop transference of the h-th loop;
p is the number of different i-o paths of the graph;
q is the number of loops of the graph

and where the star $*$ denotes that the terms $(P_k L_i \dots L_j)$ or $(L_r \dots L_s)$ containing

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Manuscript received January 31, 1975.

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transferences of touching subgraphs (loops or i-o path) are to be eliminated from the expanded products.

Current literature presents some methods for evaluating i-o paths, loops and touching conditions. For instance the ones suggested by Mariani and Tassinari (1965) [3], by Bellert and Woźniacki (1968) [4] (both based on the use of structural numbers) and by the authors of this note (1957) [5].

Here a different procedure is presented for evaluating numerator and denominator terms of Mason's formula, that seems to be suitable both for manual and computer implementation.

2. Outline of the procedure.

To any signal flow graph G with n nodes, as is well known, a lattice connection (1) $n\times n$ matrix C can be associated. With reference to C, let us now construct the set:

$$\mathcal{C}=\{C_1', C_2', C_3', \dots\} \tag{2}$$

whose elements are lattice $n\times n$ matrices that meet the following conditions:

i) $C_j'\leq C$ (i. e. the elements equal to 1 in C_j' are equal to 1 also in C)

ii) each column and each row of C_j' contains at most one element equal to 1

iii) if the k-th row (column) of C_j', $k\neq i$, $(k\neq o)$ is zero, then the k-th column (row) is also zero; if the i-th row is zero, the o-th column is zero (and viceversa); the i-th column and the o-th row cannot be zero.

Each C_j' can be considered as the connection matrix of a suitable subgraph of G. Such subgraphs are formed by isolated nodes (i. e. by nodes without entering and outgoing arcs, corresponding to a null row-column pair of C_j'), by non touching loops and selfloops and by at most one, if any, i-o path, that does not touch loops and selfloops of the subgraph.

By neglecting isolated nodes, an 1-1 correspondence can therefore be set up between each C_j' matrix and each addendum of the numerator or of the denominator of Mason's formula.

Matrices with the i-th row and the o-th column equal to zero correspond to subgraphs with one source (node i) and one sink (node o), i. e. to subgraphs formed by one i-o path and by disjoint (nontouching) loops, if any; these matrices correspond therefore to the addenda of the numerator. Similarly C_j' matrices without zero rows and columns correspond to subgraphs formed only by nontouching loops and to the addenda of the denominator.

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(1) Let L be a lattice, and $L^{n\times n}$ be the set of $n\times n$ matrices whose elements belong to L. The $L^{n\times n}$ is endowed with a lattice structure in an obvious way: in particular, for any M, $N\in L^{n\times n}$

$$M\leq N \Leftrightarrow m_{ij}\leq n_{ij} \qquad i,j=1,2\dots n$$

Connection matrices are lattice matrices with elements in the boolean algebra $\{0,1\}$.

On the other hand, each element equal to 1 in C_j' corresponds to a branch of the subgraph, connecting the node corresponding to the row to the node corresponding to the column. Numerator and denominator addenda of Mason's formula can therefore be obtained by multiplying the transferences of the branches corresponding to the elements equal to 1 in each C_j'.

The generation of the set $\mathcal{C}$ may be easily implemented by a computer program, for instance by adopting procedures similar to the ones used for computing the determinant of a matrix. From each row (column) of C a set of rows (columns) can be obtained, that exhibit at most only one element equal to 1; if the considered row of C exhibits r elements equal to 1, the generated set is formed by $r+1$ elements, for it contains also a zero row. C_j' matrices are then to be formed by combining a row of the set generated by the first row of C, a row of the set generated by the second row of C and so on; for this combination procedure some simple rules have to be respected:
a row cannot be used that exhibit an element equal to 1 in the same position as a previously considered row;
if the k-th row is zero, the k-th column is zero too, except for $k=o$;
if the h-th column is zero, the h-th row is zero too, except for $h=i$.

The sign to be attributed to each addendum of Mason's formula can be evaluated taking into account the number of loops of the corresponding subgraph: if this number is even, the sign is positive; if it is odd the sign is negative.

The number of the loops in the subgraph can be evaluated in many ways. We suggest here the following:

i) *Counting directly the loops.* Each elements equal to 1 in C_j' corresponds to a branch of the subgraph and allows us to go from the node corresponding to its row to the node corresponding to the column. In such a way, the path of the subgraph (if any) can be followed from the source (node i) to the sink (node o). By starting from whichever of the remaining elements equal to 1, a loop of the subgraph can be followed till the same element is reached. In the same way all other loops of the subgraph can be followed and their number evaluated.

ii) *Ordering the C_j' matrices.* Let us consider a matrix $C_j' \in \mathcal{C}$, corresponding to a given subgraph with a certain number of nontouching loops and a path, if any. To each subgraph obtained by the one considered after the elimination of some loops, a matrix C_k' corresponds that satisfies $C_k' < C_j'$; in such a way the matrices of the set $\mathcal{C}$ can be ordered in many sequences $C_r', C_s', C_t' \dots$ with $C_r' < C_s' < C_t'$. If we consider, for instance, the matrices corresponding to numerator addenda, in each sequence we find firstly the matrix corresponding only to a path, then the one corresponding to the subgraph formed by the same path and by a loop; then the one corresponding to the path, to the previously considered loop and to another loop and so on. Clearly if the position of C_j' in an ordered sequence of « numerator » matrices is even, the number of loops is odd and viceversa. For the « denominator » matrices, to an even position corresponds an even number of loops.

iii) *Evaluating the eigenvalues of the matrix.* Let us consider the elements of C_j' matrices as belonging to the complex field and rearrange a matrix C_j' by ordering rows and columns in such a way that to directly connected nodes of the path (if any) and of each arbitrarily split loop correspond adjacent rows and columns of the rearranged matrix The latter can be obtained from C_j' via a similarity transformation, performed by permutating two rows and the corresponding columns at time. The eigenvalues of the rearranged matrix are therefore the same of C_j'. The rearranged matrix exhibits a block diagonal structure; each block is a companion submatrix with $a_{k,k+1}=1$ and with the element at the intersection of first column and last row equal to 0 for the path and equal to 1 for the loops; all other elements are equal to 0. The eigenvalues of such a matrix are therefore the solutions of the equation:

$$\lambda^p \prod_1^{\nu}{}_k (\lambda^{h_k}-1) = 0 \tag{3}$$

where: p is the number of nodes not belonging to the loops;

h_k is the number of nodes belonging to the k-th loop;

ν is the number of loops of the subgraph.

The number ν of the loops of the subgraph is therefore equal to the multiplicity of the eigenvalue $+1$ of C_j'. Such a multiplicity can be easily evaluated by computing the determinant of $\lambda \mathbb{I} - C_j'$: by letting $\lambda=\mu+1$, the lower degree of the obtained polynomial is equal to ν.

3. Conclusions.

A method has been suggested for evaluating the transference of a signal flow graph using Mason's formula. From the lattice connection matrix C of the graph a set of C_j' matrices is obtained that corresponds to subgraphs formed by nontouching loops and at most by an input-output path not touching the loops of the subgraph. There is, clearly, an one-to-one correspondence between each C_j' and an addendum of the numerator or of the denominator of Mason's formula. These addenda can be computed by multiplying the transferences of the branches that correspond to the elements equal to 1 in C_j'; the sign depends on the number of loops of the subgraph and can be evaluated via suitable procedures.

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