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A reachability criterion for linear time-invariant systems

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A new reachability criterion is presented for linear time-invariant systems that refers to input-state transferences. The criterion is based on the evaluation of the rank of a matrix formed with the coefficients of the numerator terms of such transferences.

The adopted approach and the simplicity of the proof make the suggested criterion very interesting from a didactic point of view. The application of the criterion presents also some advantages from a computational point of view, at least in some interesting cases.

A corresponding observability criterion can easily be derived by referring to state-output transferences.

1. Introduction

This paper deals with a reachability criterion for linear time-invariant systems. Our approach differs from the usual one (Kalman criterion) and seems to have interesting didactic implications.

In fact our approach makes use of the intuitive notion of reachability, that is, to the fact that the input can modify each state vector component independently from the other ones. The proof of the criterion is, therefore, very simple.

We may add, also, that the use of the suggested criterion (at least in some cases) may be more straightforward than that of the Kalman criterion. Furthermore, the adopted point of view can give us more insight into the causes of non-reachability. This approach can be useful, for instance, when designing an identification experiment in order to evaluate beforehand if the application of given inputs allows a complete identification of the system. The problem is very important, especially in the field of biological and environmental systems and in general when the identification experiment is dangerous or expensive and cannot be repeated. The identifiability problem, in these cases, has to be faced *a priori*, that is, when the values of the parameters to be determined are yet unknown.

The suggested procedure can also easily yield an observability criterion.

2. Outline of the procedure

Let us consider a linear, time-invariant, strictly proper system, with n-order state vector, and let

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} \end{cases} \qquad \mathbf{x}^{\mathrm{T}} = (x_1, x_2 \dots x_n) \tag{1}$$

be its equations.

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For the sake of simplicity of presentation we will refer, without loss of generality, only to the reachability problem and to the scalar input case ; the results can be easily extended to the observability problem and to the multi-input case.

If the state space is not completely reachable, a non-zero, constant vector $\mathbf{K}$ will exist such that

$$\mathbf{K}^{\mathrm{T}}\mathbf{x}=0, \quad \mathbf{K}^{\mathrm{T}}=(k_1, k_2, \ldots, k_n) \tag{2}$$

whatever the value of state vector $\mathbf{x}$ at any given time $t>0$, if we assume $\mathbf{x}(0)=0$. Hence eqn. (2) holds also if we consider $\mathbf{x}$ as a time function. The non-reachability condition can therefore be presented in terms of Laplace transforms :

$$\mathbf{K}^{\mathrm{T}}\mathbf{X}(s)=0, \quad \forall s \tag{2'}$$

On the other hand, by transforming the first equation of (1) we have

$$\mathbf{X}(s)=(sI-A)^{-1} \,.\, B \,.\, U(s)=\mathbf{W}(s) \,.\, U(s) \tag{3}$$

where

$$\mathbf{W}^{\mathrm{T}}(s)=(W_1(s),\ W_2(s),\ \ldots,\ W_n(s)) \tag{4}$$

is the input-state transference vector. The elements of such a vector are rational functions with the same denominator :

$$W_i(s)=\sum_{0}^{n-1}{}_j\, p_{ij}s^j/\det\,(sI-A) \tag{5}$$

Condition (2′) can therefore be rewritten in the form

$$\mathbf{K}^{\mathrm{T}}\mathbf{X}(s)=\mathbf{K}^{\mathrm{T}}\begin{pmatrix}\sum_j p_{1j}s^j\\ \vdots \\ \sum_j p_{nj}s^j\end{pmatrix} .\ U(s)/\det\,(sI-A)=0 \tag{2''}$$

Equation (2″) has to hold for each $U(s)$ and for each s ; therefore $\mathbf{K}$ exists if and only if the linear equation set

$$\sum_{0}^{n-1}{}_j\, k_j p_{ij}=0, \quad i=(1,2,\ldots,n) \tag{6}$$

exhibits a non-trivial solution.

In conclusion the complete reachability condition can be presented in the form

$$\mathrm{rank}\,(p_{ij})=n \tag{7}$$

where p_{ij} constants are defined by eqns. (3), (4) and (5).

The condition is obviously independent of the choice of the basis in the state space, even though $\mathbf{W}(s)$ depends on such a choice. Let us thus consider another basis and let

$$\hat{\mathbf{x}}=T\mathbf{x} \tag{8}$$

be the state vector referred to the new basis, T being an invertible matrix. The new transference vector :

$$\hat{\mathbf{W}}(s)=\hat{\mathbf{X}}(s)/U(s) \tag{9}$$

meets the condition

$$\hat{\mathbf{W}}(s) = T\mathbf{W}(s) \tag{10}$$

Therefore if matrix (p_{ij}) formed with the numerator coefficients of the transferences $W_i(s)$ has rank equal to n, matrix $(\hat{p}_{ij})$ of the numerator coefficients of the transferences $\hat{W}_i(s)$ will also have rank equal to n.

Coefficients p_{ij} of the numerators of transferences $W_i(s)$ in some cases can be easily evaluated by starting from the structure of eqn. (1) (cf. for example the case presented in the following section). In general one can use Mason's formula applied to the signal flow-graph corresponding to eqn. (1); the procedure can be easily implemented on a computer. Alternatively, we may write $\mathbf{W}(s)$ in the form

$$\mathbf{W}(s) = \frac{1}{\det(sI - A)} \operatorname{adj}(sI - A) \,.\, B = \frac{(M_0 + M_1 s + \dots M_{n-1} s^{n-1})B}{\det(sI - A)} \tag{11}$$

where the matrices M_j can be computed starting from A by means of the Faddeev procedure (cf. Gantmacher: *The Theory of Matrices*, Chelsea, New York, 1959, Vol. 1, p. 87). So we have

$$(p_{ij})_{i=1 \dots n} = M_j B, \quad j = 0, 1 \dots n-1 \tag{12}$$

The structure of the Faddeev procedure allows us to ascertain that, for the general case, the amount of computation necessary in order to evaluate rank (p_{ij}) is comparable with that required by the application of the Kalman reachability criterion. But our procedure supplies also input-state transferences, which can be useful for the study of different problems regarding the system's behaviour.

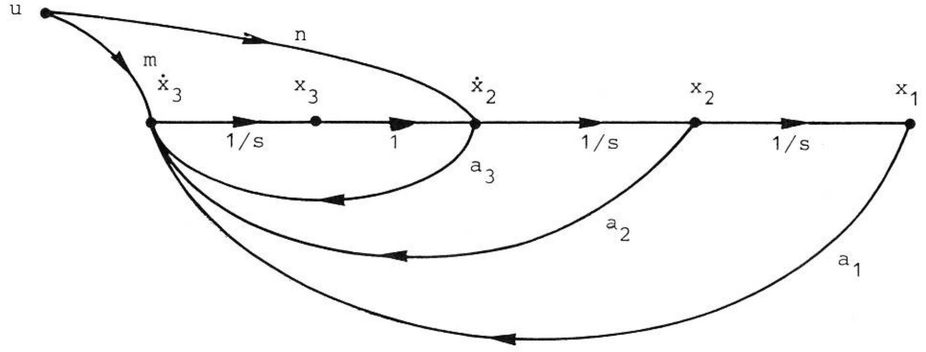
3. An example

Let us consider the system shown in the figure; its equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 + nu$$

$$\dot{x}_3 = a_1 x_1 + a_2 x_2 + a_3 x_3 + (a_3 n + m)u$$



From these equations or from the graph the following transferences

$$W_1 = \frac{m + ns}{s^3 - a_3 s^2 - a_2 s - a_1}$$

$$W_2 = \frac{ms + ns^2}{s^3 - a_3 s^2 - a_2 s - a_1}$$

$$W_3 = \frac{a_1 n + a_2 ns + (a_3 n + m)s^2}{s^3 - a_3 s^2 - a_2 s - a_1}$$

can be easily obtained.

Therefore, the rank has to be evaluated of the following matrix :

$$(p_{ji}) = \begin{pmatrix} m & n & 0 \\ 0 & m & n \\ a_1 n & a_2 n & a_3 n + m \end{pmatrix}$$

By the usual procedure on the basis of matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ n \\ a_3 n + m \end{pmatrix}$$

the reachability matrix

$$Q = \begin{pmatrix} 0 & n & a_3 n + m \\ n & a_3 n + m & a_2 n + a_3{}^2 n + a_3 m \\ a_3 n + m & a_2 n + a_3{}^2 n + a_3 m & a_1 n + 2a_1 a_3 n + a_2 m + a_3{}^3 n + a_3{}^2 m \end{pmatrix}$$

has to be formed.

The fact that rank (p_{ij}) and rank Q are equal can easily be ascertained ; the first column of Q is equal to (p_{i3}) ; the second column of Q is equal to $(p_{i2}) + a_3(p_{i3})$ and the third column of Q is equal to $(p_{i1}) + a_3(p_{i2}) + (a_2 + a_3{}^2)(p_{i3})$.

4. Concluding remarks

As has been said in the introduction, the criterion we propose is based on the intuitive notion of reachability and its proof is very simple ; the criterion seems therefore to be useful from a didactic point of view. On the other hand, the example we considered allows us to ascertain that, at least in some cases, the computation of matrix (p_{ij}) is not so cumbersome as that of matrix Q ; as a consequence, in these cases, the functions of signal flow-graph coefficients, corresponding to each p_{ij}, are simpler than those corresponding to the elements of Q. Therefore the evaluation of the effects of coefficient variations on the structural properties of the system is easier considering rank (p_{ij}) than considering rank Q. We may note, furthermore, that the system of the figure has not been built *ad hoc*, in order to provide an easy application of the suggested procedure ; on the contrary, its signal flow-graph has a very ordinary structure. This suggests that the procedure can be easily adopted for systems of higher order than that of the figure that exhibit the same flow-graph structure ; the same considerations can be made with reference to other simple graph structures with a companion type matrix A.

If the matrix A structure does not exhibit particular characteristics, the numerators of the input-state transferences W_i are to be computed, for instance by resorting to Mason's formula, as previously said.

The signal flow-graph corresponding to system equations is very significant for the study of system properties ; not only for the computation of Mason's formula but also from a more general point of view, if the state variables we consider have a precise physical meaning. In fact, space state may not be completely reachable if :

(1 *a*) no graph paths connect the input to a state vector component ;

(1 *b*) the sum of the path transferences connecting the input to a state vector component is zero ;

(2) such a sum is non-zero but it is linearly dependent on other input-state transferences.

This classification of non-reachability causes seems to be interesting for some problems, for instance, for the design of identification experiments, as previously said.

Generally speaking, the problem of the reachability of a single state-vector component (if intrinsically meaningful) can be considered as an interesting subproblem of the state reachability, although current literature does not emphasize it. Obviously such a subproblem can also be studied by conventional methods (with reference to properties of reachability matrix Q) ; the approach adopted here, however, is connected in a more direct way to the very nature of the question. The problem, indeed, is one of ascertaining if situation (1) or (2) holds. In this context the distinction between situations (1 *a*) and (1 *b*) is also interesting. In fact in case (1 *a*) (if the realization considered is physically meaningful) no action is possible on x_i by the input ; in case (1 *b*) a different combination of parameter values can make the x_i component reachable. In the design of identification experiments, if the graph of the experimental model corresponds to situation (1 *a*), the system is not identifiable ; otherwise the experiment is worthwhile because parameter values are not yet known and an *a priori* distinction between non-identifiability (1 *b*) or (2) and identifiability is impossible.