Data-Driven Distributed Output Synchronization of Heterogeneous Discrete-Time Multi-Agent Systems

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Abstract— **In this paper, we assume that an autonomous exosystem generates a reference output, and we consider the problem of designing a distributed data-driven control law for a family of discrete-time heterogeneous LTI agents, connected through a directed graph, in order to synchronize the agents' outputs to the reference one. The agents of the network are split into two categories: leaders, with direct access to the exosystem output, and followers, that only receive information from their neighbors. All agents aim to achieve output synchronization by means of a state feedback that makes use of their own states as well as of an estimate of the exogenous system state, provided by an internal state observer. Such observer has a different structure for leaders and followers. Necessary and sufficient conditions for the existence of a solution are first derived in the model-based set-up and then in a data-driven context. An example illustrates both the implementation procedure and the performance of the proposed approach.**

I. Introduction

Output regulation and output synchronization are fundamental control problems, that have attracted the interest of the researchers since the early seventies. The solutions to these problems, which are now part of the background of all control engineers, rely on the Internal Model Principle and have their foundations in some milestone papers from giants of the control community [5], [7]–[9].

In the early years of this century, there has been a shift from centralized control to distributed control mainly driven by the growing complexity of modern systems, and the need for greater scalability, fault tolerance, and real-time responsiveness. Distributed control enables decentralized decision-making, improved resilience, and better handling of large-scale, networked environments such as power grids, autonomous vehicles, and industrial automation. This paradigm shift has triggered a surge of research aiming to provide distributed solutions to classic control problems, including output regulation and synchronization. Reference [20] by Su and Huang pioneered the research on this topic, first exploring how cooperation among agents may allow all of them to synchronize their outputs to a desired reference trajectory generated by an exosystem. This first contribution stimulated a long list of papers on this topic (see [2], [4], [12], [13], [15], [17], to cite a few). The interested reader is referred to [3] for an extensive review of the literature.

In recent years, the control systems community has witnessed another major paradigm shift: from traditional model-based approaches to data-driven methods. This transition is driven by the growing availability of large-scale data, the advancements in machine learning, the increasingly higher performance of computational methods and lower costs of storage. This shift is reflected in a new trend of research, aiming to explore how to achieve output synchronization for multi-agent LTI systems when the agent models are not available, but extensive data about their dynamics have been collected in a preliminary offline phase.

To the best of our knowledge, [14] is the first paper where data-driven output synchronization for (heterogeneous) multi-agent systems (MASs) is investigated. In [14] agents are described by state-space models that are not affected by the exosystem dynamics whose output they want to track. However, a subset of the agents, called "leaders", has access to the *exosystem state* and they share it with their neighbors to ensure that each agent achieves an asymptotic estimate of the exosystem state. The matrices of the state equations are assumed to be unknown, but the matrices involved in the output equations, as well as the exosystem model, are known. The design of the state feedback control protocol is based only on data that are collected offline for each agent. In [21] the data-driven cooperative output regulation problem is investigated, by imposing a state feedback control strategy that relies on a dead-beat controller. A network of heterogeneous agents is considered, in which the state dynamics of *all* agents are affected by the *exosystem state*. All the matrices involved in the agent dynamics are supposed to be unknown. In [18] the data-driven output containment control problem for heterogeneous multi-agent systems is investigated. The agents dynamics is not affected by the exosystem and all the systems matrices are unknown. It worth remarking that in all these references the exosystem state is accessible to a subset of the agents, nonetheless all agents still implement a state observer to estimate the exosystem state that updates based on the state estimates of their neighbors. Reference [16] also deals with data driven output regulation, but the authors achieve the goal by introducing multiple copies of the exogeneous systems. Also, the problem solution is obtained via RL techniques. Finally, in [24] a data-driven algorithm is proposed to simultaneously solve an optimal control problem and identify the matrices which describe the system dynamics.

In this paper the output synchronization problem for a discrete-time heterogeneous LTI MAS is investigated. The agents of the network split into two categories: leaders, whose dynamics is affected by the *exosystem output* they want to track, and followers, that are not affected by it. In order to synchronize its output with the exosystem one, each

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agent implements a state-feedback control strategy, making use of its own state and of its estimate of the exosystem state. However, leaders produce the exosystem state estimate by using a Luenberger observer, while followers achieve this goal by exchanging information about the *exosystem output* with their neighbors. Necessary and sufficient conditions for the existence of a solution, first by resorting to a model-based approach and then by relying only on data, are derived, together with a (partial) parametrization of the problem solutions. It is shown that, under suitable assumptions on the informativity of the collected data, the data-driven solution is equivalent to the model-based one. An illustrative example concludes the paper showing the soundness of the proposed method.

The main contributions of this paper compared to the previous literature on this topic are the following ones:
- First of all, we assume what we regard as a more rational problem set-up, by making the state observers the agents implement consistent with their models. Indeed, if the exosystem directly affects the agent dynamics it makes sense to assume that such agents (the leaders) are aware of this and take advantage of this information to obtain an estimate of the exosystem state in the most efficient way. This means that the state estimate relies on first-hand information received from the exosystem, and not on the state estimates of their neighbors. Meanwhile, agents whose dynamics is not affected by the exosystem (the followers) cannot reasonably have access to direct measurements from the exosystem, and hence the best they can do is to exchange information with their neighbors.
- We assume that leaders have access to the *exosystem output*. Since they can provide exact information on such output to their neighbors, it makes sense to exchange the exosystem output estimate in order to correct the estimate of the exosystem state (not the exosystem state estimate, nor the agent output). This makes the overall system dynamics simpler, effective, and easy to design.
- While the previous two comments hold true both for the model-based and the data-driven solutions, focusing on the data-driven solution we can claim that our set-up and solutions are more articulated and complete than those investigated in [14], [18]. On the other had, compared to [21] and [16] the conditions for solvability we provide are much simpler.
- We provide (see Propositions 4 and 7) novel conditions to verify when data are informative for stabilization by state feedback [14], [22] and a parametrization of the solutions.

Notation. The sets of real numbers and nonnegative integers are denoted by $\mathbb{R}$ and $\mathbb{Z}_+$, respectively. Given two integers h and k, with $h \leq k$, we let $[h,k]$ denote the set $\{h, h+1, \dots, k\}$. $\mathbb{1}_n$ is the all-one vector of size n. Suffixes will be omitted when the dimensions are irrelevant or can be deduced from the context. Given any matrix Q, its *Moore-Penrose pseudoinverse* [1] is denoted by $Q^\dagger$. If Q is of full row rank, then $Q^\dagger = Q^\top(QQ^\top)^{-1}$, and it is a particular *right inverse* of Q, by this meaning any (full column rank) matrix $Q^\#$ such that *(s.t.)* $QQ^\# = I$.

We use $\mathrm{im}(Q)$ to represent the *column space* of Q. The *spectrum* of a square matrix Q is denoted by $\sigma(Q)$ and is the set of all its eigenvalues. The *Kronecker product* is denoted by $\otimes$. Given matrices $M_i, i \in [1,p]$, the block diagonal matrix whose ith diagonal block is the matrix M_i is denoted either by $\mathrm{diag}(M_i), i \in [1,p]$, or by $\mathrm{diag}(M_1, M_2, \dots, M_p)$, while given vectors $v_i, i \in [1,p]$, the column stacking of these vectors is denoted either by $\mathrm{col}(v_i), i \in [1,p]$, or by $\mathrm{col}(v_1, v_2, \dots, v_p)$. The (i,j)th entry of a matrix M is denoted by $[M]_{i,j}$.

A *weighted, directed graph* (digraph) is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\} = [1,N]$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} \in \mathbb{R}^{N\times N}$ is the nonnegative, weighted *adjacency matrix* which satisfies $[\mathcal{A}]_{i,j} > 0$ if and only if *(iff)* $(j,i) \in \mathcal{E}$. We assume that $[\mathcal{A}]_{i,i} = 0$ for every $i \in \mathcal{V}$. The *in-degree* of the node i is $d_i = \sum_{j=1}^N [\mathcal{A}]_{i,j}$. The *in-degree matrix* $\mathcal{D}$ is the diagonal matrix defined as $\mathcal{D} = \mathrm{diag}(d_i), i \in [1,N]$. The *Laplacian* associated with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is defined as $\mathcal{L} \doteq \mathcal{D} - \mathcal{A}$. A digraph is *connected* if there is a (directed) *path* from every node to every other node. Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, a *directed spanning tree* is a subgraph of $\mathcal{G}$ that includes all the vertices $\mathcal{V}$ and has a single root node from which all the other nodes can be reached, without forming any cycle.

II. Error-Feedback Output Synchronization: Problem Statement

Consider an exosystem described by the equations

$$x_0(t+1) = Sx_0(t), \tag{1a}$$
$$y_0(t) = Rx_0(t), \tag{1b}$$

where $t \in \mathbb{Z}_+$, $x_0(t) \in \mathbb{R}^{n_0}$ and $y_0(t) \in \mathbb{R}^p$ are the state and the output of the exogenous system, respectively. We make the following standard assumptions on the exosystem.

Assumption 1. *[12], [14], [15] All the eigenvalues of S are simple and lie on the unit circle.*

Assumption 2. *[6], [14] The pair (R,S) is observable.*

Consider a multi-agent system consisting of N heterogeneous agents. Without loss of generality, we assume that the first N_l agents (in the following referred to as *leaders*) are affected by the exosystem output $y_0(t)$, while the remaining $N_f \doteq N - N_l$ agents (the *followers*) have not. We let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote the directed graph describing the interactions among the N agents, and by $\mathcal{G}_0 = (\{0\} \cup \mathcal{V}, \mathcal{E}_0, \mathcal{A}_0)$ the extended digraph including also the node 0 corresponding to the exosystem. The set $\mathcal{E}_0$ is obtained by adding to the edges in $\mathcal{E}$ the edges from the exosystem node to the N_l nodes representing the leaders. We assume that the weights of all such edges are unitary. Under the previous assumptions, the adjacency matrix $\mathcal{A}_0$ is uniquely identified from $\mathcal{A}$, and the Laplacian matrix associated with it can be expressed in partitioned form as follows:

$$\mathcal{L}_0 = \begin{bmatrix} 0 & \mathbb{0}_{N_l}^\top & \mathbb{0}_{N_f}^\top \\ \mathbb{1}_{N_l} & \mathcal{L}_{ll} & \mathcal{L}_{lf} \\ \mathbb{0}_{N_f} & \mathcal{L}_{fl} & \mathcal{L}_{ff} \end{bmatrix} \tag{2}$$

where $\mathcal{L}_{ll} \in \mathbb{R}^{N_l \times N_l}$ and $\mathcal{L}_{ff} \in \mathbb{R}^{N_f \times N_f}$.
We introduce the following assumption on the digraph $\mathcal{G}_0$.

Assumption 3. *[3], [12], [14], [15] The digraph $\mathcal{G}_0$ contains a directed spanning tree with root node 0.*

The reason why we split the agents in leaders and followers is because, unlike previous works on this topic, we assume that if the exosystem output affects the dynamics of an agent, the agent is aware of receiving direct information from the source it aims at synchronizing with and is able to transfer the received information to its neighbors. This makes such an agent a leader in the communication process. On the other hand, agents whose dynamics are not affected by the exogenous system can only rely on the information received from their neighbors, and hence act as followers.

Under the previous assumptions, the leader dynamics are

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t) + E_i y_0(t), \tag{3a}$$
$$y_i(t) = C_i x_i(t) + D_i u_i(t) + F_i y_0(t), \tag{3b}$$

for $i \in [1, N_l]$, while the followers dynamics are

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t), \tag{4a}$$
$$y_i(t) = C_i x_i(t) + D_i u_i(t), \tag{4b}$$

$i \in [N_l + 1, N]$, where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $y_i(t) \in \mathbb{R}^p$ are the state, input, and output of the ith agent, respectively. The matrices A_i, B_i, C_i, D_i, E_i, and F_i are real matrices of suitable dimensions. As it is standard practice in output regulation/synchronization literature, we define the *output tracking error* of the ith agent as $e_i(t) \doteq y_i(t) - y_0(t)$. Based on the leaders and followers descriptions (3) and (4), it follows that for $i \in [1, N_l]$:

$$e_i(t) = C_i x_i(t) + D_i u_i(t) + F_i R x_0(t) - R x_0(t), \tag{5}$$

while for $i \in [N_l + 1, N]$

$$e_i(t) = C_i x_i(t) + D_i u_i(t) - R x_0(t). \tag{6}$$

In this set-up, every ith agent needs to estimate the state of the exosystem, in order to design a state feedback control law that depends on both its state x_i and on its estimate z_i of the exosystem state x_0, described as follows:

$$u_i(t) = K_i(x_i(t) - \Pi_i z_i(t)) + \Gamma_i z_i(t), \tag{7}$$

where the matrices $K_i \in \mathbb{R}^{m_i \times n_i}$, $\Pi_i \in \mathbb{R}^{n_i \times n_0}$ and $\Gamma_i \in \mathbb{R}^{m_i \times n_0}$, $i \in [1, N]$, are design parameters. Leaders have direct access to $y_0(t)$ and hence can rely upon a Luenberger observer [19] to generate the estimate $z_i(t)$ of $x_0(t)$:

$$z_i(t+1) = S z_i(t) - L(y_0(t) - R z_i(t)), \tag{8}$$

$i \in [1, N_l]$, where $L \in \mathbb{R}^{n_0 \times p}$, is the observer gain to be designed. On the other hand, followers need to exchange information with their neighbors to obtain a reliable asymptotic estimate of $x_0(t)$. By extending the philosophy underlying the Luenberger observer, followers update their estimate of the exosystem state by collecting from their neighbors the best estimate they can provide of the exosystem output. This means the real exosystem output in the case where the jth neighbor is a leader, and

$$\hat{y}_{0,j}(t) \doteq R z_j(t)$$

in the case where the jth neighbor is a follower. Consequently, the state estimate $z_i(t)$ for the ith follower, $i \in [N_l + 1, N]$, updates according to:

$$\begin{aligned} z_i(t+1) =& S z_i(t) + \frac{1}{1+d_i} H \Bigg[\sum_{j=1}^{N_l} [\mathcal{A}]_{i,j} (y_0(t) - \hat{y}_{0,i}(t)) \\ &+ \sum_{j=N_l+1}^{N} [\mathcal{A}]_{i,j} (\hat{y}_{0,j}(t) - \hat{y}_{0,i}(t)) \Bigg] \\ =& S z_i(t) + \frac{1}{1+d_i} H \Bigg[\sum_{j=1}^{N_l} [\mathcal{A}]_{i,j} (y_0(t) - R z_i(t)) \\ &+ \sum_{j=N_l+1}^{N} [\mathcal{A}]_{i,j} (R z_j(t) - R z_i(t)) \Bigg], \end{aligned} \tag{9}$$

where $[\mathcal{A}]_{i,j}$ is the (i,j)th entry of the adjacency matrix $\mathcal{A}$, while $H \in \mathbb{R}^{n_0 \times p}$ is a matrix parameter to be designed.

Remark 1. *To the best of our knowledge, the idea of relying on $R z_i(t)$ as an estimate of $y_0(t)$, to update the state estimate dynamics, is original. It provides a simpler algorithm to design the matrices of the observer-based state-feedback controllers compared with the approach that directly employs the agent outputs $y_i(t)$ (see [3], [25]).*

In this scenario, the error-feedback output synchronization problem is stated as follows.

Problem 1. *Consider the exosystem (1) and the MAS whose leaders are described as in (3), $i \in [1, N_l]$, and whose followers are described as in (4), $i \in [N_l+1, N]$, and assume that Assumptions 1, 2, and 3 hold.*
Design, if possible, matrices $K_i \in \mathbb{R}^{m_i \times n_i}$, $\Pi_i \in \mathbb{R}^{n_i \times n_0}$, $\Gamma_i \in \mathbb{R}^{m_i \times n_0}$, $i \in [1, N]$, $L \in \mathbb{R}^{n_0 \times p}$ and $H \in \mathbb{R}^{n_0 \times p}$ so that the overall system, consisting of all the leaders and followers, as well as the state observer equations (8) and (9), under the state feedback control law (7), $i \in [1, N]$, satisfies the following two conditions:

1) *if $x_0(t)$ is identically zero, the system is asymptotically stable;*
2) *for all initial conditions $x_0(0)$, $x_i(0)$, and $z_i(0)$,*

$$\lim_{t \to \infty} e_i(t) = \mathbb{0}_p, \quad \forall i \in [1, N]. \tag{10}$$

III. Problem Solution: Model-Based Approach

For the subsequent analysis it is worth introducing (see [15]) the ith agent *state estimation error*

$$\delta_i(t) \doteq z_i(t) - x_0(t), \tag{11}$$

and *state tracking error*

$$\varepsilon_i(t) \doteq x_i(t) - \Pi_i z_i(t). \tag{12}$$

A. Leader Nodes Dynamics

Let us define the global vector corresponding to the leader state dynamics as $x_l(t) \doteq \text{col}(x_i(t))$, $i \in [1, N_l]$, and define $u_l(t)$, $y_l(t)$, $z_l(t)$, $e_l(t)$, $\delta_l(t)$, and $\varepsilon_l(t)$ in a similar way. Accordingly, we introduce the matrix $A_l \doteq \text{diag}(A_i)$, $i = [1, N_l]$, and define B_l, C_l, D_l, E_l, F_l, K_l, Π_l and Γ_l in a similar way. Finally, we define $S_l \doteq (I_{N_l} \otimes S)$ and $R_l \doteq (I_{N_l} \otimes R)$. We can rewrite the leader dynamics (3) in compact form as

$$\begin{aligned} x_l(t+1) =& A_l x_l(t) + B_l u_l(t) + E_l(\mathbb{1}_{N_l} \otimes y_0(t)), && \text{(13a)}\\ y_l(t) =& C_l x_l(t) + D_l u_l(t) + F_l(\mathbb{1}_{N_l} \otimes y_0(t)). && \text{(13b)} \end{aligned}$$

Also, we can rewrite the state observers equations (8) and the state feedback control laws (7) in compact form as

$$\begin{aligned} z_l(t+1) =& (S_l + L_l R_l) z_l(t) - L_l(\mathbb{1}_{N_l} \otimes y_0(t)), && \text{(14a)}\\ u_l(t) =& K_l(x_l(t) - \Pi_l z_l(t)) + \Gamma_l z_l(t). && \text{(14b)} \end{aligned}$$

Substituting (14b) inside (13), we obtain

$$\begin{aligned} x_l(t+1) =& (A_l + B_l K_l) x_l(t) + B_l(\Gamma_l - K_l \Pi_l) z_l(t) \\ &+ E_l(\mathbb{1}_{N_l} \otimes y_0(t)), && \text{(15a)}\\ y_l(t) =& (C_l + D_l K_l) x_l(t) + D_l(\Gamma_l - K_l \Pi_l) z_l(t) \\ &+ F_l(\mathbb{1}_{N_l} \otimes y_0(t)). && \text{(15b)} \end{aligned}$$

The dynamics of the state estimation error, of the state tracking error and of the output tracking error become

$$\begin{aligned} \delta_l(t+1) =& (S_l + L_l R_l)\delta_l(t), && \text{(16a)}\\ \varepsilon_l(t+1) =& (A_l + B_l K_l)\varepsilon_l(t) && \text{(16b)}\\ &+ [A_l\Pi_l + B_l\Gamma_l - \Pi_l(S_l + L_l R_l)]\delta_l(t) \\ &+ (A_l\Pi_l + B_l\Gamma_l + E_l R_l - \Pi_l S_l)(\mathbb{1}_{N_l} \otimes x_0(t)), \\ e_l(t) =& (C_l + K_l D_l)\varepsilon_l(t) + (C\Pi_l + D_l\Gamma_l)\delta_l(t) && \text{(16c)}\\ &+ (C\Pi_l + D_l\Gamma_l + F_l R_l - R_l)(\mathbb{1}_{N_l} \otimes x_0(t)). \end{aligned}$$

Note that the leader dynamics is completely decoupled from the follower dynamics. By resorting to a change of basis we can replace the state vector $[x_l(t)^\top\ z_l(t)^\top\ (\mathbb{1}_{N_l} \otimes x_0(t))^\top]^\top$ with $[\delta_l(t)^\top\ \varepsilon_l(t)^\top\ (\mathbb{1}_{N_l} \otimes x_0(t))^\top]^\top$. Consequently, under Assumptions 1, 2, and 3, we can ensure that the internal dynamics of the leaders are asymptotically stable when $x_0(t)$ is identically zero (see point 1) of Problem 1) and the error $e_l(t) \to \mathbb{0}$ as $t \to \infty$ (see point 2) of Problem 1) iff

- c1) There exists a block diagonal matrix K_l s.t. $(A_l + B_l K_l)$ is Schur stable;
- c2) There exists a block diagonal matrix L_l s.t. $(S_l + L_l R_l)$ is Schur stable;
- c3) There exist (block diagonal) matrices Π_l and Γ_l s.t.

$$\begin{aligned} A_l\Pi_l + B_l\Gamma_l + E_l R_l =& \Pi_l S_l, && \text{(17a)}\\ C_l\Pi_l + D_l\Gamma_l + F_l R_l =& R_l. && \text{(17b)} \end{aligned}$$

Under condition c3), the dynamics of the state estimation errors, of the state tracking errors and of the output tracking errors become:

$$\begin{aligned} \delta_l(t+1) =& (S_l + L_l R_l)\delta_l(t), && \text{(18a)}\\ \varepsilon_l(t+1) =& (A_l + B_l K_l)\varepsilon_l(t) - (E_l + \Pi_l L_l) R_l \delta_l(t) && \text{(18b)}\\ e_l(t) =& (C_l + K_l D_l)\varepsilon_l(t) + (C\Pi_l + D_l\Gamma_l)\delta_l(t). && \text{(18c)} \end{aligned}$$

B. Follower Nodes Dynamics

Let us define the global vector corresponding to the follower state dynamics as $x_f(t) \doteq \text{col}(x_i(t))$, $i \in [N_l + 1, N]$, and define $u_f(t)$, $y_f(t)$, $z_f(t)$, $e_f(t)$, $\delta_f(t)$, and $\varepsilon_f(t)$ in a similar way. Accordingly, we introduce the matrices $A_f \doteq \text{diag}(A_i)$, $i \in [N_l+1, N]$ and, in a similar way, we define B_f, C_f, D_f, K_f, Π_f and Γ_f. Finally, we define $\mathcal{D}_f \doteq \text{diag}(d_i)$, $i \in [N_l + 1, N]$, where d_i is the in-degree of the ith node of the network, $S_f \doteq (I_{N_f} \otimes S)$ and $R_f \doteq (I_{N_f} \otimes R)$. We can rewrite the follower dynamics (4) in compact form as

$$\begin{aligned} x_f(t+1) =& A_f x_f(t) + B_f u_f(t), && \text{(19a)}\\ y_f(t) =& C_f x_f(t) + D_f u_f(t). && \text{(19b)} \end{aligned}$$

The state observer equations (9) and the state feedback laws (7) become

$$\begin{aligned} z_f(t+1) =& [S_f - (I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{ff} \otimes HR] z_f(t) && \text{(20a)}\\ &- [(I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{fl} \otimes H](\mathbb{1}_{N_l} \otimes y_0(t)), \\ u_f(t) =& K_f(x_f(t) - \Pi_f z_f(t)) + \Gamma_f z_f(t). && \text{(20b)} \end{aligned}$$

The dynamics of the state estimation errors, of the state tracking errors and of the output tracking errors for the followers are:

$$\begin{aligned} \delta_f(t+1) =& [S_f - (I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{ff} \otimes HR]\delta_f(t), && \text{(21a)}\\ \varepsilon_f(t+1) =& (A_f + B_f K_f)\varepsilon_f(t) && \text{(21b)}\\ &+ [A_f\Pi_f + B_f\Gamma_f - \Pi_f S_f \\ &+ \Pi_f(I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{ff} \otimes HR]\,\delta_f(t) \\ &+ (A_f\Pi_f + B_f\Gamma_f - \Pi_f S_f)(\mathbb{1}_{N_f} \otimes x_0(t)), \\ e_f(t) =& (C_f + K_f D_f)\varepsilon_f(t) && \text{(21c)}\\ &+ (C_f\Pi_f + D_f\Gamma_f)\delta_f(t) \\ &+ (C_f\Pi_f + D_f\Gamma_f - R_f)(\mathbb{1}_{N_f} \otimes x_0(t)). \end{aligned}$$

Note that the follower dynamics is decoupled, in turn, from the leader dynamics. By resorting to a change of basis we replace the state vector $[x_f(t)^\top\ z_f(t)^\top\ (\mathbb{1}_{N_f} \otimes x_0(t))^\top]^\top$ with $[\delta_f(t)^\top\ \varepsilon_f(t)^\top\ (\mathbb{1}_{N_f} \otimes x_0(t))^\top]^\top$. So, under the proposed assumptions, we can ensure that the internal dynamics of the followers are asymptotically stable (see point 1) of Problem 1) and $e_f(t) \to \mathbb{0}$ as $t \to \infty$ (see point 2) of Problem 1) iff

- c4) There exists a block diagonal matrix K_f s.t. $(A_f + B_f K_f)$ is Schur stable;
- c5) There exists a matrix H s.t. $\left[S_f - (I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{ff} \otimes HR\right]$ is Schur stable;
- c6) There exist (block diagonal) matrices Π_f and Γ_f s.t.

$$\begin{aligned} A_f\Pi_f + B_f\Gamma_f =& \Pi_f S_f, && \text{(22a)}\\ C_f\Pi_f + D_f\Gamma_f =& R_f. && \text{(22b)} \end{aligned}$$

Under condition c6), the dynamics of (21) become:

$$\delta_f(t+1) = \left[S_f - (I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{ff} \otimes HR\right]\delta_f(t), \quad (23a)$$
$$\varepsilon_f(t+1) = (A_f + B_f K_f)\varepsilon_f(t) + \Pi_f\left[(I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{ff} \otimes HR\right]\delta_f(t) \quad (23b)$$
$$e_f(t) = (C_f + K_f D_f)\varepsilon_f(t) + R_f\delta_f(t). \quad (23c)$$

C. Model-Based Solution

As a consequence of the previous analysis, the problem solvability reduces to the possibility of satisfying conditions c1)-c6). We have the following result.

Theorem 1. *Consider the exosystem (1) and the MAS with leaders described as in (3), $i \in [1, N_l]$, and followers described as in (4), $i \in [N_l + 1, N]$. Under Assumptions 1, 2, and 3, Problem 1 is solvable iff*

i) for each $i \in [1, N]$, the pair (A_i, B_i) is stabilizable,

and there exist matrices Π_i, Γ_i, of suitable dimensions, s.t.

ii) for each $i \in [1, N_l]$,

$$A_i\Pi_i + B_i\Gamma_i + E_iR = \Pi_iS, \quad (24a)$$
$$C_i\Pi_i + D_i\Gamma_i + F_iR = R, \quad (24b)$$

iii) for each $i \in [N_l + 1, N]$,

$$A_i\Pi_i + B_i\Gamma_i = \Pi_iS, \quad (25a)$$
$$C_i\Pi_i + D_i\Gamma_i = R. \quad (25b)$$

Proof: We first observe that, by Assumption 2, the pair (R, S) is observable. This ensures that there exists L such that $S + LR$ is Schur stable, and hence c2) holds for $L_l = I_{N_l} \otimes L$. On the other hand, Assumptions 1 and 3 ensure that there exists a matrix H such that $S - \lambda HR$ is Schur stable for every $\lambda \in \sigma((I_{N_f} + \mathcal{D}_f)^{-1}\mathcal{L}_{ff})$ (see Lemmas 11 and 12 in the Appendix). Corresponding to this matrix H condition c5) holds.

We observe, now, that there exist matrices $K_i, i \in [1, N]$, such that conditions c1) and c4) hold if and only if each pair (A_i, B_i) is stabilizable, which is exactly condition i). Finally, conditions c3) and c6) are equivalent to ii) and iii). ☐

IV. Data-Driven Approach

In this section we assume that Assumptions 1, 2, and 3 hold, and we introduce a new assumption.

Assumption 4. *All the matrices that describe the leaders and followers are unknown, while the matrices S and R which describe the exogenous system dynamics are known.*

We assume to have collected offline output measurements from the exogenous system, as well as input, state and output measurements from the leader and follower systems on a finite time window of (sufficiently large) length $T+1$: $y_0^d \doteq \{y_0^d(t)\}_{t=0}^{T-1}$, $u_i^d \doteq \{u_i^d(t)\}_{t=0}^{T-1}$, $x_i^d \doteq \{x_i^d(t)\}_{t=0}^{T}$ and $y_i^d \doteq \{y_i^d(t)\}_{t=0}^{T-1}$, $i = [1, N]$. Accordingly, we set

$$Y_0^p \doteq \left[y_0^d(0), \ldots, y_0^d(T-1)\right] \in \mathbb{R}^{p \times T}$$
$$U_i^p \doteq \left[u_i^d(0), \ldots, u_i^d(T-1)\right] \in \mathbb{R}^{m_i \times T}$$
$$X_i^p \doteq \left[x_i^d(0), \ldots, x_i^d(T-1)\right] \in \mathbb{R}^{n_i \times T}$$
$$X_i^f \doteq \left[x_i^d(1), \ldots, x_i^d(T)\right] \in \mathbb{R}^{n_i \times T}$$
$$Y_i^p \doteq \left[y_i^d(0), \ldots, y_i^d(T-1)\right] \in \mathbb{R}^{p \times T}$$

In order to solve Problem 1 in a data-driven set-up, we need to understand how conditions i), ii) and iii) in Theorem 1 translate in terms of data matrices. Note that while condition ii) pertains only leaders and condition iii) only followers, condition i) pertains both. However, due to the different state-space models generating the collected data, the conditions we will derive for leaders and followers will be different.

A. Leader Nodes Dynamics

We start by introducing the set of all leader systems $(A_i, B_i, E_i, C_i, D_i, F_i)$ that are compatible with the collected data $(Y_0^p, U_i^p, X_i^p, X_i^f, Y_i^p)$, for $i \in [1, N_l]$:

$$\Sigma_i^l \doteq \left\{(A_i, B_i, E_i, C_i, D_i, F_i) : \begin{bmatrix} X_i^f \\ Y_i^p \end{bmatrix} = \begin{bmatrix} A_i & B_i & E_i \\ C_i & D_i & F_i \end{bmatrix}\begin{bmatrix} X_i^p \\ U_i^p \\ Y_0^p \end{bmatrix}\right\}.$$

We also introduce for every $i \in [1, N_l]$ the set

$$\Sigma_i^{0,l} \doteq \left\{(A_i^0, B_i^0, E_i^0, C_i^0, D_i^0, F_i^0) : \begin{bmatrix} A_i^0 & B_i^0 & E_i^0 \\ C_i^0 & D_i^0 & F_i^0 \end{bmatrix}\begin{bmatrix} X_i^p \\ U_i^p \\ Y_0^p \end{bmatrix} = \begin{bmatrix} \mathbb{0} \\ \mathbb{0} \end{bmatrix}\right\}.$$

The first aspect we want to address is condition i) for leaders. In a model-based set-up, the stabilizability of each pair (A_i, B_i) is necessary and sufficient for the existence of a matrix K_i such that $A_i + B_iK_i$ is Schur stable. When we rely on data, we are not able to identify the specific sextuple $(A_i, B_i, E_i, C_i, D_i, F_i)$ that has generated the data, and hence in particular the specific pair (A_i, B_i). This means that we need to be able to design from data a matrix K_i that makes $A_i + B_iK_i$ Schur stable for every pair (A_i, B_i) that is compatible with the collected data. For this reason, we resort to the following definition and characterization.

Definition 2. *[14, Definition 3] The data $(Y_0^p, U_i^p, X_i^p, X_i^f, Y_i^p)$ are said to be* informative for stabilization by state feedback *if there exists a feedback gain K_i s.t. $A_i + B_iK_i$ is Schur stable for all $(A_i, B_i, E_i, C_i, D_i, F_i) \in \Sigma_i^l$.*

Proposition 3. *[14, Proposition 2] The data $(Y_0^p, U_i^p, X_i^p, X_i^f, Y_i^p)$ are informative for stabilization by state feedback iff*

i) X_i^p is of full row rank;

ii) There exists a right inverse $(X_i^p)^\#$ of X_i^p, s.t.

iia) $X_i^f(X_i^p)^\#$ is Schur stable;

iib) $Y_0^p(X_i^p)^\# = \mathbb{0}$.

When so, the stabilizing feedback gain is $K_i = U_i^p(X_i^p)^\#$.

It is worth noticing that, as a consequence of Proposition 3, if the data $(Y_0^p, U_i^p, X_i^p, X_i^f, Y_i^p)$ are informative for stabilization by state feedback, then X_i^p is of full row rank, and the matrix

$$\Psi_i \doteq \begin{bmatrix} X_i^p \\ Y_0^p \end{bmatrix} \in \mathbb{R}^{(n_i+p)\times T} \tag{27}$$

satisfies $\operatorname{rank}(\Psi_i) = \operatorname{rank}(X_i^p) + \operatorname{rank}(Y_0^p) = n_i + \operatorname{rank}(Y_0^p)$. This allows us to introduce a parametrization of all possible right inverses $(X_i^p)^\#$ of X_i^p that satisfy $iib)$ of Proposition 3, as well as a method to check if in this set there exists at least one matrix for which also $iia)$ holds.

Proposition 4. *Let Ψ_i be defined as in (27), with X_i^p of full row rank, and $\operatorname{rank}(\Psi_i) = n_i + \operatorname{rank}(Y_0^p)$, and let $\Psi_i^\dagger$ be its Moore-Penrose inverse. The following facts are equivalent:*

1) There exists a matrix $(X_i^p)^\#$ such that $\Psi_i(X_i^p)^\# = \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}$ and $X_i^f(X_i^p)^\#$ is Schur.

2) The pair $\left(X_i^f \Psi_i^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}, X_i^f \left(I_T - \Psi_i^\dagger \Psi_i\right)\right)$ is stabilizable.

3) $\operatorname{rank} \left[X_i^f \Psi_i^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} - \lambda I_{n_i} \;\middle|\; X_i^f \left(I_T - \Psi_i^\dagger \Psi_i\right) \right] = n_i$, $\forall\ \lambda \in \mathbb{C},\ |\lambda| \geq 1$.

Proof: Let L_0 and R_0 be a full column rank matrix and a full row rank matrix, respectively, such that $Y_0^p = L_0 R_0$, so that

$$\Psi_i = \begin{bmatrix} I_{n_i} & \mathbb{0} \\ \mathbb{0} & L_0 \end{bmatrix} \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}.$$

Note that in the previous factorization the matrix on the left is of full column rank, while the matrix on the right is of full row rank and such common rank coincides with $\operatorname{rank}(\Psi_i)$. Consequently, by the properties of the Moore-Penrose inverse [10], we can claim that

$$\Psi_i^\dagger = \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}^\dagger \begin{bmatrix} I_{n_i} & \mathbb{0} \\ \mathbb{0} & L_0 \end{bmatrix}^\dagger.$$

Now, set

$$\Psi_i^\# \doteq \Psi_i^\dagger + \left(I_T - \Psi_i^\dagger \Psi_i\right) Q_i, \tag{28}$$

where Q_i is a free matrix parameter.
We want to prove that a matrix $(X_i^p)^\#$ satisfies

$$\Psi_i (X_i^p)^\# = \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} \tag{29}$$

if and only if it satisfies

$$(X_i^p)^\# = \Psi_i^\# \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}, \tag{30}$$

for some $\Psi_i^\#$ is expressed as in (28), i.e., for some Q_i.
Clearly, if (30) holds for some Q_i, then

$$\begin{aligned} \Psi_i (X_i^p)^\# &= \Psi_i \left[\Psi_i^\dagger + \left(I_T - \Psi_i^\dagger \Psi_i\right) Q_i\right] \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} \\ &= \Psi_i \Psi_i^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} = \begin{bmatrix} I_{n_i} & \mathbb{0} \\ \mathbb{0} & L_0 L_0^\dagger \end{bmatrix} \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} = \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} \end{aligned}$$

which means that (29) holds.
Conversely, suppose that (29) holds. Since L_0 is of full column rank, this means that

$$\begin{bmatrix} X_i^p \\ R_0 \end{bmatrix} (X_i^p)^\# = \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}.$$

Since $\begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}$ is of full row rank, by Lemma 13, we can claim that $(X_i^p)^\#$ can be expressed as

$$(X_i^p)^\# = \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} + \left(I_T - \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}^\dagger \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}\right) Q_i \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} \tag{31}$$

for some matrix Q_i. It is easy to see that

$$\begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} = \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}^\dagger \begin{bmatrix} I_{n_i} & \mathbb{0} \\ \mathbb{0} & L_0 \end{bmatrix}^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}$$

as well as

$$\begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}^\dagger \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix} = \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}^\dagger \begin{bmatrix} I_{n_i} & \mathbb{0} \\ \mathbb{0} & L_0 \end{bmatrix}^\dagger \begin{bmatrix} I_{n_i} & \mathbb{0} \\ \mathbb{0} & L_0 \end{bmatrix} \begin{bmatrix} X_i^p \\ R_0 \end{bmatrix}.$$

This shows that (31) coincides with (30), where $\Psi_i^\#$ is described as in (28) for some Q_i.
1) $\Leftrightarrow$ 2) From the previous analysis, it follows that condition 1) holds if and only if there exists Q_i such that

$$\begin{aligned} X_i^f (X_i^p)^\# &= X_i^f \left(\Psi_i^\dagger + \left(I_T - \Psi_i^\dagger \Psi_i\right) Q_i\right) \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} \\ &= \left(X_i^f \Psi_i^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}\right) + \left(X_i^f \left(I_T - \Psi_i^\dagger \Psi_i\right)\right) Q_i \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix} \end{aligned}$$

is Schur. This is equivalent to saying that the pair $\left(X_i^f \Psi_i^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}, X_i^f \left(I_T - \Psi_i^\dagger \Psi_i\right)\right)$ is stabilizable (condition 2)).
2) $\Leftrightarrow$ 3) Follows from the PBH reachability test [19]. □

We now address condition $ii)$ in Theorem 1 from a data-driven perspective.

Proposition 5. *For every $i \in [1, N_l]$, given the data $(Y_0^p, U_i^p, X_i^p, X_i^f, Y_i^p)$, the following conditions are equivalent:*

1) There exist matrices Π_i and Γ_i such that (24) hold for all the sextuples $(A_i, B_i, E_i, C_i, D_i, F_i) \in \Sigma_i^l$.

2) There exists a matrix M_i such that

$$X_i^f M_i = X_i^p M_i S, \tag{32a}$$
$$Y_i^p M_i = R, \tag{32b}$$
$$Y_0^p M_i = R. \tag{32c}$$

Proof: The proof is inspired by the proof in [14, Lemma 3] and has some similarities with the proof in [21, Lemma 2].
1) $\Rightarrow$ 2) If we denote by $(A_i, B_i, E_i, C_i, D_i, F_i)$ the "real" sextuple that generated the data, then clearly $(A_i, B_i, E_i, C_i, D_i, F_i) \in \Sigma_i^l$ and the sextuples

$(\hat{A}_i, \hat{B}_i, \hat{E}_i, \hat{C}_i, \hat{D}_i, \hat{F}_i)$ compatible with the data are those and only those that satisfy

$$\begin{bmatrix} A_i - \hat{A}_i & B_i - \hat{B}_i & E_i - \hat{E}_i \\ C_i - \hat{C}_i & D_i - \hat{D}_i & F_i - \hat{F}_i \end{bmatrix} \begin{bmatrix} X_i^p \\ U_i^p \\ Y_0^p \end{bmatrix} = \begin{bmatrix} \mathbb{0} \\ \mathbb{0} \end{bmatrix},$$

which implies that $(A_i - \hat{A}_i, B_i - \hat{B}_i, E_i - \hat{E}_i, C_i - \hat{C}_i, D_i - \hat{D}_i, F_i - \hat{F}_i) \in \Sigma_i^{0,l}$. Consequently, if the equations (24) hold for all the sextuples compatible with the data, this means that for all $(\hat{A}_i, \hat{B}_i, \hat{E}_i, \hat{C}_i, \hat{D}_i, \hat{F}_i) \in \Sigma_i^l$, it must hold that

$$\begin{bmatrix} A_i - \hat{A}_i & B_i - \hat{B}_i & E_i - \hat{E}_i \\ C_i - \hat{C}_i & D_i - \hat{D}_i & F_i - \hat{F}_i \end{bmatrix} \begin{bmatrix} \Pi_i \\ \Gamma_i \\ R \end{bmatrix} = \begin{bmatrix} \mathbb{0} \\ \mathbb{0} \end{bmatrix}. \tag{33}$$

This implies that

$$\ker_{\mathrm{L}}\left(\begin{bmatrix} X_i^p \\ U_i^p \\ Y_0^p \end{bmatrix}\right) \subseteq \ker_{\mathrm{L}}\left(\begin{bmatrix} \Pi_i \\ \Gamma_i \\ R \end{bmatrix}\right)$$

where $\ker_{\mathrm{L}}(Q) \doteq \{v : v^\top Q = \mathbb{0}^\top\}$. This ensures that there exists a matrix M_i such that

$$\begin{bmatrix} X_i^p \\ U_i^p \\ Y_0^p \end{bmatrix} M_i = \begin{bmatrix} \Pi_i \\ \Gamma_i \\ R \end{bmatrix}.$$

This implies that $X_i^p M_i = \Pi_i, U_i^p M_i = \Gamma_i, Y_0^p M_i = R$, and hence (32c) holds. Moreover, equation (24a) becomes

$$\begin{aligned} & A_i \Pi_i + B_i \Gamma_i + E_i R = \Pi_i S, \\ \Rightarrow\ & A_i X_i^p M_i + B_i U_i^p M_i + E_i Y_0^p M_i = X_i^p M_i S, \\ \Rightarrow\ & X_i^f M_i = X_i^p M_i S, \end{aligned}$$

while equation (24b) becomes

$$\begin{aligned} & C_i \Pi_i + D_i \Gamma_i + F_i R = R, \\ \Rightarrow\ & C_i X_i^p M_i + D_i U_i^p M_i + F_i Y_0^p M_i = Y_0^p M_i, \\ \Rightarrow\ & Y_i^p M_i = Y_0^p M_i, \end{aligned}$$

that are exactly (32a) and (32b).
2) $\Rightarrow$ 1) Suppose that there exists a matrix M_i such that (32) hold. Set $\Pi_i \doteq X_i^p M_i$, and $\Gamma_i \doteq U_i^p M_i$. For every sextuple $(A_i, B_i, E_i, C_i, D_i, F_i) \in \Sigma_i^l$, condition (32a), making use of (32c), leads to

$$\begin{aligned} & (A_i X_i^p + B_i U_i^p + E_i Y_0^p) M_i = X_i^p M_i S \\ \Rightarrow\ & A_i X_i^p M_i + B_i U_i^p M_i + E_i Y_0^p M_i = X_i^p M_i S \\ \Rightarrow\ & A_i \Pi_i + B_i \Gamma_i + E_i R = \Pi_i S \end{aligned}$$

that is equal to (24a).
Similarly, starting from (32b) and using (32c), for every sextuple $(A_i, B_i, E_i, C_i, D_i, F_i) \in \Sigma_i^l$ we get

$$\begin{aligned} & (C_i X_i^p + D_i U_i^p + F_i Y_0^p) M_i = R \\ \Rightarrow\ & C_i X_i^p M_i + D_i U_i^p M_i + F_i Y_0^p M_i = R \\ \Rightarrow\ & C_i \Pi_i + D_i \Gamma_i + F_i R = R. \end{aligned}$$

that is equal to (24b). □

B. Follower Nodes Dynamics

For $i \in [N_l+1, N]$, based on equations (4), we define the set of all quadruples (A_i, B_i, C_i, D_i), representing followers, that are compatible with the data $(U_i^p, X_i^p, X_i^f, Y_i^p)$ as

$$\Sigma_i^f \doteq \left\{ (A_i, B_i, C_i, D_i) : \begin{bmatrix} X_i^f \\ Y_i^p \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} X_i^p \\ U_i^p \end{bmatrix} \right\}.$$

Also, as in the previous subsection, we introduce the set

$$\Sigma_i^{0,f} \doteq \left\{ (A_i^0, B_i^0, C_i^0, D_i^0) \ : \begin{bmatrix} A_i^0 & B_i^0 \\ C_i^0 & D_i^0 \end{bmatrix} \begin{bmatrix} X_i^p \\ U_i^p \end{bmatrix} = \begin{bmatrix} \mathbb{0} \\ \mathbb{0} \end{bmatrix} \right\}.$$

We now address the data-driven characterization of condition $i)$ in Theorem 1 for the followers.

Definition 6. *[22, Definition 12] The data $(U_i^p, X_i^p, X_i^f, Y_i^p)$ are are said to be* informative for stabilization by state feedback *if there exists a feedback gain K_i such that $A_i + B_i K_i$ is Schur stable for all $(A_i, B_i, C_i, D_i) \in \Sigma_i^f$.*

The characterization of informativity for stabilization by state feedback provided in Proposition 7, below, is similar to, but much simpler than, the one provided in Proposition 4 and is omitted due to space constraints.

Proposition 7. *The following facts are equivalent:*

i) The data $(U_i^p, X_i^p, X_i^f, Y_i^p)$ are informative for stabilization by state feedback.
ii) X_i^p is of full row rank, and there exists a right inverse $(X_i^p)^\#$ of X_i^p such that $X_i^f (X_i^p)^\#$ is Schur.
iii) X_i^p is of full row rank and the pair $\left(X_i^f (X_i^p)^\dagger, X_i^f (I_T - (X_i^p)^\dagger X_i^p)\right)$ is stabilizable.
iv) X_i^p is of full row rank, and $\mathrm{rank}\left(\left[X_i^f (X_i^p)^\dagger - \lambda I_{n_i} \;\middle|\; X_i^f (I_T - (X_i^p)^\dagger X_i^p)\right]\right) = n_i$, $\forall\ \lambda \in \mathbb{C},\ |\lambda| \geq 1$.

Note that corresponding to every $(X_i^p)^\#$ satisfying $ii)$, we obtain the stabilizing feedback gain $K_i = U_i^p (X_i^p)^\#$ (see [22]). The following result is the analogous, for followers, of Proposition 5 and hence its proof is omitted.

Proposition 8. *For every $i \in [N_l + 1, N]$, given the data $(U_i^p, X_i^p, X_i^f, Y_i^p)$, the following conditions are equivalent:*

1) There exist matrices Π_i and Γ_i such that (25) hold for all quadruples $(A_i, B_i, C_i, D_i) \in \Sigma_i^f$.
2) There exists a matrix M_i such that

$$X_i^f M_i = X_i^p M_i S, \tag{34a}$$
$$Y_i^p M_i = R. \tag{34b}$$

C. Data-driven solution

By putting together the model-solution given in Theorem 1, and the characterizations given for leaders in Propositions 4 and 5, and for followers in Propositions 7 and 8, we obtain the complete data-driven solution of Problem 1.

Theorem 9. *Consider the exosystem (1) and the MAS with leaders described as in (3), $i \in [1, N_l]$, and followers described as in (4), $i \in [N_l+1, N]$. Assume that Assumptions*

[1](#), [2](#), [3](#) and [4](#) hold. Problem [1](#) is solvable based on the families of collected data $(Y_0^p, U_i^p, X_i^p, X_i^f), i \in [1, N]$, *iff*

i) For each $i \in [1, N_l]$,

ia) $\operatorname{rank}(\Psi_i) = n_i + \operatorname{rank}(Y_0^p)$.

ib) The pair $\left(X_i^f \Psi_i^\dagger \begin{bmatrix} I_{n_i} \\ \mathbb{0} \end{bmatrix}, X_i^f \left(I_T - \Psi_i^\dagger \Psi_i \right) \right)$ *is stabilizable.*

ic) There exists a matrix M_i *s.t. equations (32) hold.*

ii) For each $i \in [N_l + 1, N]$,

iia) X_i^p *is of full row rank.*

iib) The pair $\left(X_i^f (X_i^p)^\dagger, X_i^f (I_T - (X_i^p)^\dagger X_i^p) \right)$ *is stabilizable.*

iic) There exists a matrix M_i *s.t. equations (34) hold.*

Example 10. *Consider an exosystem and a group of 5 agents (2 leaders and 3 followers) connected through the binary (i.e.* $[\mathcal{A}]_{ij}$ *is either* 0 *or* 1*) digraph* $\mathcal{G}_0$ *depicted in Figure 1 and with dynamics described by the following matrices:*

$$
\begin{aligned}
&S = \begin{bmatrix} \sin(0.2) & \cos(0.2) \\ -\cos(0.2) & \sin(0.2) \end{bmatrix}, \ R = \begin{bmatrix} -1 & 1 \end{bmatrix}, \\
&A_1 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \ E_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \ B_4 = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, \\
&C_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \ D_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \ C_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}, \ C_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \\
&A_2 = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ E_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 1 & 1 \\ 10 & 3 \end{bmatrix}, \ B_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \\
&A_4 = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}, \ A_5 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}, \ B_5 = \begin{bmatrix} 1 & 3 & 1 \\ 5 & -3 & 6 \\ 0 & 5 & -1 \end{bmatrix}, \\
&C_4 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \ C_5 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}, \ D_5 = \begin{bmatrix} 3 & 6 & -1 \end{bmatrix}, \\
&F_1 = \begin{bmatrix} 5 \end{bmatrix}, \ D_2 = \begin{bmatrix} 3 \end{bmatrix}, \ F_2 = \begin{bmatrix} 3 \end{bmatrix}, \ D_3 = \begin{bmatrix} 6 \end{bmatrix}, \ D_4 = \begin{bmatrix} 3 \end{bmatrix}.
\end{aligned}
$$

We set $T = 6$*. The inputs* $u_i(t), t \in [0, 5]$*, and the initial states* $x_i(0)$ *and* $x_0(0)$*,* $i \in [1, 5]$*, have been randomly generated from a standard Gaussian distribution. We have collected the corresponding data matrices* $(Y_0^p, U_i^p, X_i^p, X_i^f, Y_i^p)$*. By relying on the previous analysis, we have obtained the following matrices:*

$$
\begin{aligned}
&L = \begin{bmatrix} -0.5719 \\ -0.4692 \end{bmatrix}, \ H = \begin{bmatrix} 0.1987 \\ -0.9801 \end{bmatrix}, \\
&K_1 = \begin{bmatrix} 0.7908 & 0.1046 & 0.5590 \\ -0.1677 & 0.2658 & 0.0935 \\ -1.3346 & 0.0327 & -0.8135 \end{bmatrix}, \\
&K_2 = \begin{bmatrix} -0.0001 & -2.8999 \end{bmatrix}, \ K_3 = \begin{bmatrix} -1.0303 & -0.5076 \end{bmatrix}, \\
&K_4 = \begin{bmatrix} -2.4279 & 0.7161 & -0.0281 \end{bmatrix}, \\
&K_5 = \begin{bmatrix} 3.5372 & 1.1530 & -1.7219 \\ -0.6923 & -0.2701 & -0.5844 \\ -3.4587 & -1.3458 & 0.4763 \end{bmatrix}, \ \Pi_1 = \begin{bmatrix} 9.4737 & -0.7312 \\ -0.8750 & 3.8708 \\ 0.0693 & -3.3919 \end{bmatrix} \\
&\Pi_2 = \begin{bmatrix} 0.3327 & 3.4521 \\ -1.0572 & 1.1880 \end{bmatrix}, \ \Pi_3 = \begin{bmatrix} 0.0399 & 0.2869 \\ 0.4135 & -1.0947 \end{bmatrix}, \\
&\Pi_4 = \begin{bmatrix} -0.7908 & 0.3916 \\ 0.6203 & 1.4994 \\ -2.8368 & -1.0040 \end{bmatrix}, \ \Pi_5 = \begin{bmatrix} 0.2158 & 0.0351 \\ -0.4961 & 0.1923 \\ 0.1232 & -0.1110 \end{bmatrix}, \\
&\Gamma_1 = \begin{bmatrix} 5.5430 & -0.1230 \\ 4.3996 & -3.3488 \\ -10.1109 & 1.6856 \end{bmatrix}, \ \Gamma_5 = \begin{bmatrix} 0.0329 & 0.0156 \\ -0.0861 & 0.1020 \\ -0.0710 & -0.0326 \end{bmatrix}, \\
&\Gamma_2 = \begin{bmatrix} 0.7972 & -3.3640 \end{bmatrix}, \ \Gamma_3 = \begin{bmatrix} -0.2422 & 0.3013 \end{bmatrix}, \\
&\Gamma_4 = \begin{bmatrix} 2.3536 & 0.2073 \end{bmatrix}.
\end{aligned}
$$

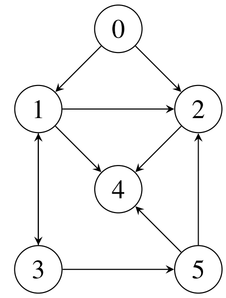

Fig. 1: Graph $\mathcal{G}_0$.

We have then tested the proposed data driven solution, randomly generating from a standard Gaussian distribution the initial condition $x_i(0)$*,* $i = [1, 5]$*, and* $x_0(0)$*, and the results shown in Figure 2 highlight the excellent performance of the output synchronization algorithm.*

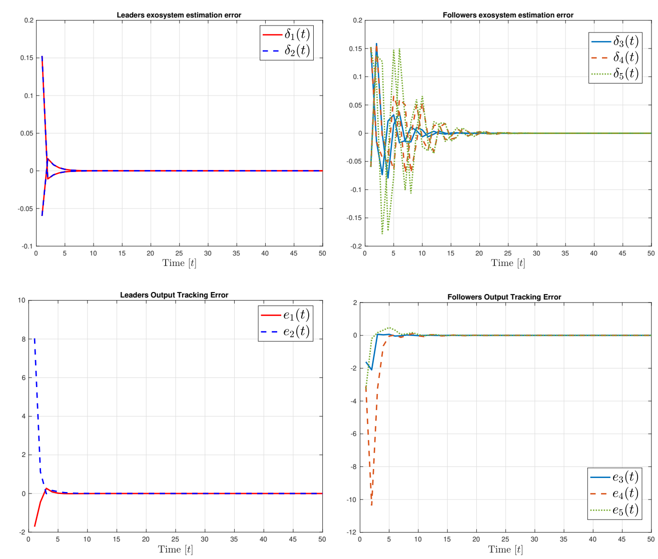

Fig. 2: Plot of the leaders (left) and followers (right) estimation error $\delta_i(t)$ (top) and output tracking error $e_i(t)$ (bottom).

Appendix

Lemma 11. *Under Assumption 3, each eigenvalue* λ *of* $\sigma((I_{N_f} + \mathcal{D}_f)^{-1} \mathcal{L}_{ff})$ *(see (2)) satisfies the following conditions: a)* $\lambda \neq 0$*; b)* $\lambda \in \{z \in \mathbb{C} : |z - 1| < 1\}$.

Proof: a) Consider the condensed graph $\mathcal{G}_c$, obtained from $\mathcal{G}_0$ by merging all leader nodes and the node 0 representing the exosystem. There is an edge from this new node to any of the follower nodes if and only if there was an edge from one of the leaders to that follower in $\mathcal{G}_0$. Conversely, there is an edge from any follower to the new node if and only if there was an edge from that follower to one of the leaders in $\mathcal{G}_0$. It is easy to see that the Laplacian associated with this new digraph is related to $\mathcal{L}_0$ in (2) as follows:

$$
\mathcal{L}_c = \begin{bmatrix} N_l + \mathbb{1}_{N_l}^\top \mathcal{L}_{ll} \mathbb{1}_{N_l} & \mathbb{1}_{N_l}^\top \mathcal{L}_{lf} \\ \mathcal{L}_{fl} \mathbb{1}_{N_l} & \mathcal{L}_{ff}. \end{bmatrix}
$$

Assumption 3, introduced for $\mathcal{G}_0$, still holds for $\mathcal{G}_c$. But then we can resort to Lemma 1 in [20] to claim that $\mathcal{L}_{ff}$ is nonsingular square. As $(I+\mathcal{D}_f)^{-1}$ is nonsingular, too, then a) holds.

b) By Gershgorin's Circle theorem [11], we can say that

$$\sigma((I_{N_f}+\mathcal{D}_f)^{-1}\mathcal{L}_{ff}) \subset \bigcup_{j\in[N_l+1,N]} \left\{ z\in\mathbb{C} \ : \ \left| z-\frac{d_j}{1+d_j}\right| \le \sum_{k=N_l+1}^{N} \frac{[\mathcal{A}]_{jk}}{1+d_j} \right\},$$

but, by Assumption 3, $\sum_{k=N_l+1}^{N}[\mathcal{A}]_{jk} \le d_j$, and hence for all $j\in[N_l+1,N]$ and every $\lambda\in\sigma((I_{N_f}+\mathcal{D}_f)^{-1}\mathcal{L}_{ff})$:

$$\left|\lambda-\frac{d_j}{1+d_j}\right| \le \frac{d_j}{1+d_j}$$

that, together with a), implies b). □

Lemma 12. *Set $\sigma((I_{N_f}+\mathcal{D}_f)^{-1}\mathcal{L}_{ff}) = \{\lambda_1,\lambda_2,\dots,\lambda_{N_f}\}$. Under Assumptions 1, 2 and 3, there exists a matrix H such that $S-\lambda_i HR$ is Schur stable for all $i\in[1,N_f]$.*

Proof: First, we want to prove that if (R,S) is observable, then there exists a vector $v\neq 0$ such that $(v^\top R,S)$ is observable. Let T be a nonsingular matrix s.t. $T^{-1}ST=\operatorname{diag}(\mu_1,\dots,\mu_{n_0})$, with $|\mu_j|=1$, $\mu_k\neq\mu_j$ if $k\neq j$. It follows that (R,S) is observable if and only if $(RT,T^{-1}ST)$ is observable, and by the PBH observability criterion this is the case if and only if RT does not have null columns. On the other hand, $(v^\top R,S)$ is observable if and only if $(v^\top RT,T^{-1}ST)$ is observable, which happens if and only if $v^\top RT$ does not have null entries.
Let w_i denote the ith column of RT, then $v^\top w_i=0$ if and only if $v\in(\operatorname{im}(w_i))^\perp$. Since $\bigcup_{i=1}^{n_0}(\operatorname{im}(w_i))^\perp \subsetneq \mathbb{R}^p$, it follows that exists $v\in\mathbb{R}^p\setminus\left(\bigcup_{i=1}^{n_0}(\operatorname{im}(w_i))^\perp\right)$, namely exists $v\neq\mathbb{0}$ such that $v^\top RT$ does not have null entries, and thus $(v^\top R,S)$ is observable.
Now it remains to show that if $(v^\top R,S)$ is observable then there exists $q\in\mathbb{R}^{n_0}$ such that $S-\lambda_i qv^\top R$ is Schur for all $i\in[1,N_f]$, but this is the dual result of [23, Theorem 3.2]. Note that we can apply such result because condition (17) in [23] holds, due to Lemma 11. So, the result holds for $H=qv^\top$. □

Lemma 13. *Let $A\in\mathbb{R}^{n\times r}$ and $B\in\mathbb{R}^{k\times r}$, and assume that*

$$\Psi \doteq \begin{bmatrix} A\\ B\end{bmatrix} \in\mathbb{R}^{(n+k)\times r}$$

is of full row rank. A matrix $C\in\mathbb{R}^{r\times n}$ satisfies

$$\Psi C = \begin{bmatrix} I_n\\ \mathbb{0}\end{bmatrix} \tag{35}$$

if and only if

$$C = \left(\Psi^\dagger + (I_r-\Psi^\dagger\Psi)Q\right)\begin{bmatrix} I_n\\ \mathbb{0}\end{bmatrix} \tag{36}$$

for some matrix $Q\in\mathbb{R}^{r\times(n+k)}$.

Proof: Since Ψ is of full row rank, its right inverses are those and those only that can be expressed as

$$\Psi^\# = \Psi^\dagger + \left(I_r-\Psi^\dagger\Psi\right)Q, \tag{37}$$

as Q varies in $\mathbb{R}^{r\times(n+k)}$. Therefore if C is expressed as in (36), for some Q, then obviously (35) holds. Conversely, if C satisfies (35), the fact that B is of full row rank ensures that

$$\begin{bmatrix} A\\ B\end{bmatrix}\begin{bmatrix} C & B^\dagger\end{bmatrix} = \begin{bmatrix} I_n & *\\ \mathbb{0} & I_k\end{bmatrix},$$

for some matrix $*$ and hence there exists $B^\#$ such that

$$\Psi\begin{bmatrix} C & B^\#\end{bmatrix} = \begin{bmatrix} A\\ B\end{bmatrix}\begin{bmatrix} C & B^\#\end{bmatrix} = \begin{bmatrix} I_n & \mathbb{0}\\ \mathbb{0} & I_k\end{bmatrix}.$$

This immediately implies that $\begin{bmatrix} C & B^\#\end{bmatrix}$ is a right inverse of Ψ and hence can be expressed as in (37) for some Q. Therefore (36) holds. □

References

[1] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications.* Springer, New York, USA, 2003.

[2] H. Cai, F. L. Lewis, G. Hu, and J. Huang. The adaptive distributed observer approach to the cooperative output regulation of linear multi-agent systems. *Automatica*, 75:299–305, 2017.

[3] C. Chen, F. L. Lewis, K. Xie, Y. Lyu, and S. Xie. Distributed output data-driven optimal robust synchronization of heterogeneous multi-agent systems. *Automatica*, 153:111030, 2023.

[4] K. Chen, J. Wang, Z. Zhao, G. Lai, and Y. Lyu. Output consensus of heterogeneous multiagent systems: A distributed observer-based approach. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 52(1):370–376, 2022.

[5] E. Davison. The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Transactions on Automatic Control*, 21(1):25–34, 1976.

[6] G. de Carolis, S. Galeani, and M. Sassano. Data-driven, robust output regulation in finite time for lti systems. *International Journal of Robust and Nonlinear Control*, 28(18):5997–6015, 2018.

[7] B. A. Francis. The linear multivariable regulator problem. *SIAM Journal on Control and Optimization*, 15(3):486–505, 1977.

[8] B. A. Francis and W. M. Wonham. The internal model principle for linear multivariable regulators. *Appl. Math. Optim.*, 2(2):170–194, 1975.

[9] B.A. Francis and W.M. Wonham. The internal model principle of control theory. *Automatica*, 12(5):457–465, 1976.

[10] T. N. E. Greville. Note on the generalized inverse of a matrix product. *SIAM Review*, 8(4):518–521, 1966.

[11] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge Univ. Press, Cambridge (GB), 1985.

[12] J. Huang. The cooperative output regulation problem of discrete-time linear multi-agent systems by the adaptive distributed observer. *IEEE Transactions on Automatic Control*, 62(4):1979–1984, 2017.

[13] S. Huo, Y. Zhang, F. L. Lewis, and C. Sun. Observer-based resilient consensus control for heterogeneous multiagent systems against cyber-attacks. *IEEE Transactions on Control of Network Systems*, 10(2):647–658, 2023.

[14] J. Jiao, H.J. van Waarde, H. L. Trentelman, M. K. Camlibel, and S. Hirche. Data-driven output synchronization of heterogeneous leader-follower multi-agent systems. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 466–471, 2021.

[15] B. Kiumarsi and F. L. Lewis. Output synchronization of heterogeneous discrete-time systems: A model-free optimal approach. *Automatica*, 84:86–94, 2017.

[16] L. Lin and J. Huang. Data-driven cooperative output regulation via distributed internal model. *arXiv preprint arXiv:2502.14336*, 2025.

[17] H. Modares, S. P. Nageshrao, G. A. Delgado Lopes, R. Babuška, and F. L. Lewis. Optimal model-free output synchronization of heterogeneous systems using off-policy reinforcement learning. *Automatica*, 71:334–341, 2016.

[18] M. Sader, W. Li, Y. Yin, Z. Li, D. Huang, Z. Liu, X. He, and C. Shang. Data-driven output containment control of heterogeneous multiagent systems: A hierarchical scheme. In *2024 63rd IEEE Conference on Decision and Control (CDC)*, pages 6889–6895, 2024.

[19] E.D. Sontag. *Mathematical Control Theory. Deterministic Finite Dimensional Systems.* Springer-Verlag, New York,2nd edition, 1998.

[20] Y. Su and J. Huang. Cooperative output regulation of linear multi-agent systems. *IEEE Transactions on Automatic Control*, 57(4):1062–1066, 2012.

[21] E. Tian, G. Zhai, D. Liang, and J. Liu. Cooperative output regulation of unknown linear multiagent systems: When deadbeat control meets data-driven framework. *IEEE Transactions on Industrial Informatics*, 20(5):7556–7564, 2024.

[22] H. J. van Waarde, J. Eising, M. K. Camlibel, and H. L. Trentelman. The informativity approach: To data-driven analysis and control. *IEEE Control Systems Magazine*, 43(6):32–66, 2023.

[23] K. You and L. Xie. Network topology and communication data rate for consensusability of discrete-time multi-agent systems. *IEEE Transactions on Automatic Control*, 56(10):2262–2275, 2011.

[24] Y. Zhou, G. Wen, J. Zhou, H. Liu, and J. Lü. Data-driven output consensus tracking control for heterogeneous multi-agent systems with a dynamic leader. *IEEE Transactions on Control of Network Systems*, pages 1–10, 2025.

[25] L. Zhu and Z. Chen. Data informativity for robust output regulation. *IEEE Transactions on Automatic Control*, 69(10):7075–7080, 2024.