Distributed unknown input observers for discrete-time LTI systems

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***Abstract*—In this paper we consider the problem of distributed estimation of the state of a discrete-time, linear and time-invariant (LTI) state-space model affected by disturbances. We assume that there is a connected network of sensors having access to some output measurements as well as to part of the control inputs applied to the system. Such sensors exchange information with the goal of achieving consensus and providing an asymptotically correct estimate of the original system state. Necessary and sufficient conditions for the existence of distributed unknown input observers with augmented states that achieve both goals are derived. The problem solution exploits the theory of decentralized output feedback control, thus making it possible to inherit the algorithms available for the solution of that problem.**

I. Introduction

The ever-increasing spread of complex networked systems has brought with it a series of new challenges in the control community. Indeed, when dealing with systems comprising a multitude of components and/or scattered over a large area, it is important to implement the desired control actions in a distributed fashion, by this meaning that each sensor/node in the network has to accomplish the required task by exploiting only the locally available information and the interactions with its neighbors. Among the various problems that the researchers in this field have tried to solve in a distributed manner, there is certainly the state estimation one. However, while in a centralized scenario we can assume to have access to all the components of the input applied to the system and to measure all the output components to reconstruct the system state, in a distributed architecture each sensor receives only a subset of the input and output components, and hence it may not be able to accomplish the task autonomously. Therefore, it is necessary to exploit cooperation among the agents, and resort, for instance, to a consensus strategy, in which each sensor shares its own state estimate with its neighbors in the communication network.

During the last decades several algorithms for state estimation have been adapted to the distributed case, such as the Kalman filter, in a stochastic setting [9], [13], and the Luenberger observer, in a deterministic framework. Focusing on the deterministic set-up, the works of Park and Martins [14], [15], [16] address the problem of distributed state estimation for a discrete-time autonomous system by relying on a network of Luenberger-like observers. Each local observer, endowed with an additional internal state, measures a portion of the output vector and computes a state estimate using its own measurements and the state estimates of other local observers shared through a communication graph. The introduction of an augmented state allows to reduce the design of the distributed observer to the synthesis of a decentralized dynamic output feedback controller, and hence to use existing literature results. The idea of using an extended observer has been applied also to continuous-time systems. In [19] Wang and Morse propose a linear time-invariant distributed observer for an m-channel continuous-time linear system, where $m-1$ estimators have dimension equal to the state dimension n and one estimator is endowed with an augmented state of dimension $n+m-1$. By using results from classical decentralized control theory, they show that, under suitable assumptions, it is possible to freely assign the spectrum of the overall distributed observer.

In [10] the state estimation problem is solved by means of a distributed Luenberger observer, based on the decomposition of each node state space into a detectable and an undetectable part. Then, cooperation is exploited to compensate the locally undetectable parts. Another approach to the design of distributed observers has been proposed in [12], building upon the fact that a given node may be able to reconstruct a portion of the state by means of an appropriate Luenberger observer, using only its own measurements. Therefore, the node only needs to exchange information with neighbors to estimate the portion of the state that is not locally detectable.

In the papers mentioned so far, the distributed state estimation problem is addressed assuming that no disturbance affects the system dynamics. However, in a real-life scenario, it is highly unlikely that there are no disturbances and that each node has access to all the input components. Therefore, in recent times, the control community has started to investigate the distributed state estimation problem in the presence of unknown inputs. In [22] a distributed state estimation scheme for linear continuous-time systems subject to unknown inputs that is capable of reconstructing the global system state has been proposed. The authors derive existence conditions consistent with the results on centralized unknown input observers available in the literature. The additional condition that allows the estimation scheme to be implemented in a distributed manner is the absence of vectors lying in the intersection of the undetectable subspaces of all nodes in the network. The same condition has been exploited in [2] to derive a distributed unknown input observer (DUIO), whose design procedure builds upon the one proposed in [22], but is simpler and more distributed. In [20] a slightly stronger sufficient condition for the existence of a distributed observer for a continuous-time linear system has been proposed: the existence of at least one detectable node. However, the strengthening of the condition is motivated by the fact that the main focus of [20] is a data-driven implementation of the distributed estimation scheme, which requires a solvability condition that can be checked directly on data. All the conditions provided in [2], [20] and

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[22] are only sufficient for the existence of a DUIO, and applicable only to continuous-time systems, since they are all based on a high-gain mechanism, that cannot be extended to the discrete-time case.

In this paper we aim to fill this gap, and hence to design a distributed unknown input observer for a discrete-time LTI system. Specifically, inspired by the papers of Martins and Park, that resort to the theory of decentralized output feedback control, we consider an estimation scheme in which each sensor is endowed with an augmented state. We derive necessary and sufficient conditions to guarantee that, by using the proposed architecture, the state estimates provided by each node in the network asymptotically reach consensus and the consensus value coincides with the true system state. The problem solution leverages some concepts and tools borrowed from classical decentralized control theory, and in particular from dynamic output feedback control. It is worth highlighting that although all the results are derived in a discrete-time setting, they easily extend to continuous-time systems.

The paper is organized as follows. In Section II the problem of interest is formally defined. In Section III necessary and sufficient conditions for the problem solvability are provided. The numerical example presented in Section IV validates the effectiveness of the proposed scheme. Finally, Section V concludes the paper. Some technical lemmas used in the proofs of the main results are reported in the Appendix.

Notation. Given two integers h, k, with $h \leq k$, we let $[h,k]$ denote the set $\{h, h+1, \dots, k\}$. $\mathbb{0}_{m\times n}$ is the zero matrix of size $m \times n$, $\mathbb{0}_n$ and $\mathbb{1}_n$ are the all-zero and all-one vectors of size n, respectively. Suffixes will be omitted when the dimensions are irrelevant or can be deduced from the context. The *Moore-Penrose pseudoinverse* of a matrix Q is denoted by $Q^\dagger$. The *spectrum* of a square matrix Q is denoted by $\sigma(Q)$. A square symmetric matrix Q is *positive (semi)definite* if $x^\top Qx > 0$ ($x^\top Qx \geq 0$) for every $x \neq \mathbb{0}$. The *Kronecker product* is denoted by $\otimes$. Given matrices $M_i, i \in [1,p]$, the block diagonal matrix whose ith diagonal block is the matrix M_i is denoted by $\mathrm{diag}(M_i)$. The (i,j)th entry of a matrix M is denoted by $[M]_{i,j}$.

For every subset $\mathcal{J}$ of an integer set $[1,N]$, $N \geq 1$, we denote by $\mathcal{J}^c$ the complement of $\mathcal{J}$ with respect to $[1,N]$, namely $\mathcal{J}^c = [1,N] \setminus \mathcal{J}$. If $\mathcal{J} = \{j_1, j_2, \dots, j_k\} \subseteq [1,N]$, we denote by $S_\mathcal{J}$ the selection matrix obtained by juxtaposing the (N-dimensional) canonical vectors $\mathbb{e}_{j_\ell}, j_\ell \in \mathcal{J}$, i.e., $S_\mathcal{J} \triangleq \begin{bmatrix} \mathbb{e}_{j_1} & \dots & \mathbb{e}_{j_k} \end{bmatrix} \in \mathbb{R}^{N\times k}$. Accordingly, for any $N \times N$ matrix M, $MS_\mathcal{J}$ is the submatrix of M obtained by selecting the columns corresponding to the indices in $\mathcal{J}$, while $S_\mathcal{J}^\top M$ is the submatrix of M obtained by selecting the rows corresponding to the indices in $\mathcal{J}$.

An *undirected, weighted graph* is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = [1,N]$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} \in \mathbb{R}^{N\times N}$ is the symmetric, nonnegative, weighted *adjacency matrix* which satisfies $[\mathcal{A}]_{i,j} = [\mathcal{A}]_{j,i} > 0$ if and only if $(i,j) \in \mathcal{E}$. An undirected graph is *connected* if for every $i, j \in \mathcal{V}$ there exist $k \in \mathbb{Z}_+$ and vertices $i_1, \dots, i_k \in \mathcal{V}$ such that $(i, i_1), (i_1, i_2), \dots, (i_k, j)$ are all edges in $\mathcal{E}$. The *Laplacian* associated with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is the matrix $\mathcal{L} \in \mathbb{R}^{N\times N}$ whose entries are defined as follows $[\mathcal{L}]_{i,j} = -[\mathcal{A}]_{i,j}$, if $i \neq j$, $[\mathcal{L}]_{i,i} = \sum_{k=1}^N [\mathcal{A}]_{i,k}$. Since $\mathcal{A} = \mathcal{A}^\top$, then $\mathcal{L} = \mathcal{L}^\top$.

II. Problem Formulation

The problem set-up we adopt is similar to the one adopted in [2], [20], [22]. The main difference is that here we consider a discrete-time dynamics. More specifically, we consider the following discrete-time LTI system

$$x(t+1) = Ax(t) + Bu(t) + Ed(t), \tag{1}$$

where $t \in \mathbb{Z}_+$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $d(t) \in \mathbb{R}^q$ is the unknown process disturbance, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $E \in \mathbb{R}^{n\times q}$. The system outputs are measured through a sensor network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ comprising N heterogeneous sensor nodes. At each time t, each node of the network provides an output signal, that represents an indirect measurement of the state and is given by

$$y_i(t) = C_i x(t), \quad \forall i \in \mathcal{V} = [1,N], \tag{2}$$

where $y_i(t) \in \mathbb{R}^{p_i}$, and $C_i \in \mathbb{R}^{p_i\times n}$. Moreover, we assume that each sensor node has access only to a subset of the input entries, and hence for every $i \in \mathcal{V}$ we can split the entries of the control input $u(t)$ into two parts: the measurable part $u_i(t)$, and the unmeasurable part $u_i^u(t)$. Consequently, for every $i \in \mathcal{V}$, we can always express $Bu(t)$ as:

$$Bu(t) = B_i^m u_i(t) + B_i^u u_i^u(t), \tag{3}$$

where $u_i(t) \in \mathbb{R}^{m_i}$, $B_i^m \in \mathbb{R}^{n\times m_i}$, $u_i^u(t) \in \mathbb{R}^{m-m_i}$, $B_i^u \in \mathbb{R}^{n\times(m-m_i)}$. Since $d(t)$ is also unknown for each node, the overall unknown input at node i and the associated system matrix can be represented as

$$w_i(t) \triangleq \begin{bmatrix} u_i^u(t)^\top & d(t)^\top \end{bmatrix}^\top \in \mathbb{R}^{r_i}, \tag{4a}$$

$$D_i \triangleq [B_i^u \ E] \in \mathbb{R}^{n\times r_i}, \tag{4b}$$

where $r_i \triangleq m - m_i + q$. Consequently, for every $i \in \mathcal{V}$, the system dynamics, from the perspective of the ith sensor node, can be described as:

$$x(t+1) = Ax(t) + B_i^m u_i(t) + D_i w_i(t). \tag{5}$$

In the following, we will denote by $\mathcal{T}_i$ the system described by the pair of equations (1)–(2) or, equivalently, by the pair (5)–(2). It is worth highlighting that even if it is convenient to adopt a different description for each sensor node, to underline the fact that each of them has access to different information, the system whose state needs to be estimated is unique and described as in (1).

Remark 1. *There is no loss of generality (see [22]) in assuming that the matrix D_i is of full column rank r_i, since in the following analysis the specific expression of $w_i(t)$ plays no role and hence $w_i(t)$ can be redefined.*

We consider a distributed state estimation scheme in which each sensor node $i \in \mathcal{V}$ is equipped with an (augmented unknown input) observer (in the following called DUIO_i) and generates an estimate of the state of system (1) at time t, $\hat{x}_i(t)$, using only the locally available information, namely the input and output signals $u_i(t)$ and $y_i(t)$, and the local communication with its neighbors. A distributed state estimation scheme is a distributed unknown input observer (DUIO) [2], [22] if all the estimates provided by the sensors in the network converge asymptotically to the true state of system (1), independently of the initial conditions, of the control input and of the disturbance acting on the system.

Definition 2. *Given the systems $\mathcal{T}_i$, $i \in \mathcal{V}$, interacting through a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, a set of observers* DUIO_i, $i \in \mathcal{V}$, *is a* distributed unknown input observer *if*

$$e_i(t) \triangleq x(t) - \hat{x}_i(t) \underset{t\to+\infty}{\longrightarrow} \mathbb{0}, \quad \forall\, i \in \mathcal{V},$$

independently of the initial conditions, the input u and the disturbance d.

We assume that the ith sensor node generates the estimate $\hat{x}_i(t)$ through the following augmented observer:

$$\underset{i\in\mathcal{V}}{\text{DUIO}_i} : \begin{cases} z_i(t+1) = N_i z_i(t) + M_i u_i(t) + L_i y_i(t) \\ \qquad\qquad - K_i \displaystyle\sum_{j=1}^{N} [\mathcal{A}]_{i,j}[\hat{x}_j(t) - \hat{x}_i(t)] - Q_i \xi_i(t), \\ \xi_i(t+1) = S_i \xi_i(t) + R_i \displaystyle\sum_{j=1}^{N} [\mathcal{A}]_{i,j}[\hat{x}_j(t) - \hat{x}_i(t)], \\ \hat{x}_i(t) = z_i(t) + H_i y_i(t), \end{cases} \tag{6}$$

where $[z_i(t)^\top \ \xi_i(t)^\top]^\top \in \mathbb{R}^{n+\beta_i}$ is the augmented state of the ith DUIO, $\hat{x}_i(t)$ is the estimate of $x(t)$ provided by node i. $N_i, K_i \in \mathbb{R}^{n\times n}$, $M_i \in \mathbb{R}^{n\times m_i}$, $L_i, H_i \in \mathbb{R}^{n\times p_i}$, $Q_i \in \mathbb{R}^{n\times\beta_i}$, $S_i \in \mathbb{R}^{\beta_i\times\beta_i}$ and $R_i \in \mathbb{R}^{\beta_i\times n}$ are matrices to be designed. $[\mathcal{A}]_{i,j}$ is the (i,j)th entry of the adjacency matrix $\mathcal{A}$ of the communication graph $\mathcal{G}$. Let us introduce the global vector

$$z_G(t) \triangleq \begin{bmatrix} z_1^\top(t) & \cdots & z_i^\top(t) & \cdots & z_N^\top(t) \end{bmatrix}^\top.$$

The vectors $\xi_G(t), u_G(t), y_G(t)$ and $\hat{x}_G(t)$ are defined in an analogous way. Also, we introduce the block diagonal matrices

$$N \triangleq \text{diag}(N_i), M \triangleq \text{diag}(M_i), L \triangleq \text{diag}(L_i), K \triangleq \text{diag}(K_i),$$

$$Q \triangleq \text{diag}(Q_i), S \triangleq \text{diag}(S_i), R \triangleq \text{diag}(R_i), H \triangleq \text{diag}(H_i).$$

This allows us to rewrite the equations of the overall DUIO as follows:

$$\text{DUIO} : \begin{cases} z_G(t+1) = N z_G(t) + M u_G(t) + L y_G(t) \\ \qquad\qquad + K(\mathcal{L} \otimes I_n)\hat{x}_G(t) - Q\xi_G(t), \\ \xi_G(t+1) = S\xi_G(t) - R(\mathcal{L} \otimes I_n)\hat{x}_G(t), \\ \hat{x}_G(t) = z_G(t) + H y_G(t). \end{cases} \tag{7}$$

Assumption 1 (Communication network)**.** *The undirected, weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is connected, which amounts to saying [11] that the Laplacian $\mathcal{L}$ is a symmetric positive semidefinite matrix with 0 as an eigenvalue of multiplicity 1.*

Remark 3. *The idea of resorting to an augmented observer to be able to later exploit the theory of decentralized output feedback [6], [7] has been inspired by the milestone work of Park and Martins [14], [15], [16]. However, as one can easily see by a direct comparison, the choice of the observer equations here is quite different from those adopted in the aforementioned references. Indeed, in [14], [15], [16] the core structure of the observer represents an extension of a Luenberger observer, while in this paper we adopt a more complex structure that is closer to those adopted in [2], [22], due to the fact that with a Luenberger observer it is not possible to decouple the estimation error update from the effects of unknown inputs. Also, we introduce a consensus term expressed through the Laplacian of the communication graph.*

We are now ready to provide the formal statement of the problem we are going to solve.

Problem 1. *Given the systems $\mathcal{T}_i$, $i \in \mathcal{V}$, interacting through the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, satisfying Assumption 1, determine under what conditions block diagonal matrices[1] N, M, L, K, Q, S, R, and H can be found, so that the state estimates provided by the nodes of the distributed state estimation scheme (7) achieve consensus and the common state estimate converges to the true state value. This amounts to saying that the global estimation error $e_G(t) \triangleq (\mathbb{1} \otimes x(t)) - \hat{x}_G(t)$ (defined by concatenation of the estimation errors of the individual nodes $e_i(t), i \in \mathcal{V}$) converges to $\mathbb{0}$ as $t \to +\infty$, regardless of the initial conditions, of the inputs and of the disturbances acting on the systems.*

III. Necessary and sufficient conditions for problem solvability

In this section we provide necessary and sufficient conditions for the solvability of Problem 1, stated in the previous section. If we introduce the block diagonal matrices

$$B^m \triangleq \text{diag}(B_i^m), \quad C \triangleq \text{diag}(C_i), \quad D \triangleq \text{diag}(D_i),$$

then, similarly to [18, Sections 2–3] and [22, Section 4], one can describe the dynamics of the estimation error $e_G(t) = (\mathbb{1} \otimes x(t)) - \hat{x}_G(t)$ in compact form as follows

$$\begin{aligned} & e_G(t+1) = \big[N + K(\mathcal{L} \otimes I_n)\big] e_G(t) \\ & + \big[(I_{Nn} - HC)B^m - M\big] u_G(t) \\ & + \big[(I_{Nn} - HC)(I_N \otimes A) - N(I_{Nn} - HC) - LC\big] x_G(t) \\ & + (I_{Nn} - HC) D w_G(t) + Q\xi_G(t), \end{aligned} \tag{8}$$

where $w_G(t)$ is defined analogously to $z_G(t)$. Clearly, in order to decouple the estimation error dynamics from the effects of the (control and unknown) inputs and of the initial condition of x, the following conditions must hold

$$M = (I_{Nn} - HC)B^m, \tag{9a}$$

$$(I_{Nn} - HC)D = \mathbb{0}, \tag{9b}$$

$$N = (I_{Nn} - HC)(I_N \otimes A) - (L - NH)C. \tag{9c}$$

Correspondingly, equation (8) becomes

$$e_G(t+1) = (N + K(\mathcal{L} \otimes I_n))e_G(t) + Q\xi_G(t). \tag{10}$$

The joint dynamics of $e_G(t)$ and $\xi_G(t)$ is then given by

$$\begin{bmatrix} e_G(t+1) \\ \xi_G(t+1) \end{bmatrix} = \Phi \begin{bmatrix} e_G(t) \\ \xi_G(t) \end{bmatrix}, \tag{11}$$

[1] In the sequel when saying that the matrices N, M, L, K, Q, S, R, and H are block diagonal, we will always mean that for each of them the ith (not necessarily square) diagonal block has dimensions compatible with the corresponding variables of the system $\mathcal{T}_i$ or the corresponding DUIO_i (6).

where

$$\Phi \triangleq \left[\begin{array}{c|c} N + K(\mathcal{L} \otimes I_n) & Q \\ \hline R(\mathcal{L} \otimes I_n) & S \end{array}\right], \tag{12}$$

and N satisfies (9c).
We first observe that (9a) is always feasible, so we can disregard it in the following. Necessary and sufficient conditions for the solvability of equation (9b) are given in Lemma 4 below, whose proof is a trivial extension of the single agent case [4], [5], being all matrices in (9b) block diagonal, and hence is omitted.

Lemma 4. *[20] The following facts are equivalent.*

(i) Equation (9b) admits a block diagonal solution $H = \mathrm{diag}(H_i), H_i \in \mathbb{R}^{n \times p_i}$.
(ii) For every $i \in \mathcal{V}$*, there exists* $H_i \in \mathbb{R}^{n \times p_i}$ *such that*

$$H_i C_i D_i = D_i. \tag{13}$$

(iii) For every $i \in \mathcal{V}$*,* $\mathrm{rank}(C_i D_i) = \mathrm{rank}(D_i) = r_i$.

So, in the sequel we will always make the following:

Assumption 2 (Unknown input decoupling condition)**.** *For every* $i \in \mathcal{V}$*,* $\mathrm{rank}(C_i D_i) = \mathrm{rank}(D_i) = r_i$.

We now consider condition (9c). Since all matrices in (9c) are block diagonal, for every choice of L, N and H the matrix $F \triangleq L - NH$ is block diagonal. And conversely, if we choose block diagonal matrices F, N and H, then there exists a block diagonal L such that $F = L - NH$. This allows us to rewrite (9c) as

$$N = (I_{Nn} - HC)(I_N \otimes A) - FC, \tag{14}$$

where F is a free block diagonal matrix parameter, whose diagonal blocks have the same sizes of the diagonal blocks of L. To conclude, we are reduced to finding block diagonal matrices H, F, K, Q, R and S such that condition (9b) is satisfied and the state estimation error, whose dynamics obeys (11), asymptotically converges to zero. In the following proposition we give necessary and sufficient conditions for the existence of matrices N, K, Q, R and S such that the state estimation error asymptotically goes to zero, independently of the initial conditions.

Proposition 5. *Let* N *be a block diagonal matrix described as in (14), with* H *satisfying (9b). The following facts are equivalent:*

(i) There exist block diagonal matrices K, Q, R *and* S *such that for every choice of the initial conditions, the vector* $\begin{bmatrix} e_G(t)^\top & \xi_G(t)^\top \end{bmatrix}^\top$*, whose dynamics is described as in (11), satisfies* $\lim_{t\to+\infty} e_G(t) = 0$.
(ii) There exist block diagonal matrices K, Q, R *and* S *such that the matrix* Φ *described as in (12) is Schur stable.*

Proof. The implication *(ii)* $\Rightarrow$ *(i)* is trivial, since the Schur stability of Φ ensures that $e_G(t)$ asymptotically converges to $\mathbb{0}$, independently of the initial conditions. So, the only implication that needs to be proved is *(i)* $\Rightarrow$ *(ii)*.
Suppose that there exist block diagonal matrices K, Q, R, and S such that the vector $e_G(t)$ of the autonomous system (11) tends to zero asymptotically for every choice of $e_G(0)$ and $\xi_G(0)$. This amounts to saying that the output $e_G(t)$ of the autonomous system

$$\begin{aligned} \begin{bmatrix} e_G(t+1) \\ \xi_G(t+1) \end{bmatrix} &= \Phi \begin{bmatrix} e_G(t) \\ \xi_G(t) \end{bmatrix}, \\ e_G(t) &= \begin{bmatrix} I_{Nn} & \mathbb{0} \end{bmatrix} \begin{bmatrix} e_G(t) \\ \xi_G(t) \end{bmatrix}, \end{aligned}$$

converges asymptotically to zero for every choice of the initial state. By Lemma 10 in Appendix A, if the pair $\left(\begin{bmatrix} I_{Nn} & \mathbb{0} \end{bmatrix}, \Phi\right)$ is detectable then the whole state vector $\begin{bmatrix} e_G(t)^\top & \xi_G(t)^\top \end{bmatrix}^\top$ converges to zero asymptotically and, since this is true for every choice of the initial condition, this ensures that Φ is Schur. On the other hand, if the pair $\left(\begin{bmatrix} I_{Nn} & \mathbb{0} \end{bmatrix}, \Phi\right)$ is not detectable, this means (see, again, Lemma 10) that the pair (Q, S) is not detectable and hence in particular not observable. Since the matrices Q and S are block diagonal, the detectability/observability of the pair (Q, S) is equivalent to the detectability/observability of each pair $(Q_i, S_i), i \in \mathcal{V}$. Let $T = \mathrm{diag}(T_i) \in \mathbb{R}^{\sum_i \beta_i \times \sum_i \beta_i}$ be a nonsingular block diagonal matrix that reduces each pair (Q_i, S_i) to standard observability form. Therefore, by applying the change of basis associated to the block diagonal matrix $\mathrm{diag}(I_{Nn}, T)$, the matrix Φ in the new basis (after a reordering of the last $\sum_i \beta_i$ rows and columns) becomes

$$\left[\begin{array}{c|c|c} N + K(\mathcal{L} \otimes I_n) & Q_{11} & \mathbb{0} \\ \hline R_{11}(\mathcal{L} \otimes I_n) & S_{11} & \mathbb{0} \\ \hline R_{21}(\mathcal{L} \otimes I_n) & S_{21} & S_{22} \end{array}\right],$$

where $Q_{11}, R_{11}, R_{21}, S_{11}, S_{21}$ and S_{22} are block diagonal matrices, and the pair (Q_{11}, S_{11}) is observable. This implies that the dynamics of $e_G(t)$ is given by

$$e_G(t+1) = \left[\begin{array}{c|c|c} N + K(\mathcal{L} \otimes I_n) & Q_{11} & \mathbb{0} \end{array}\right] \begin{bmatrix} e_G(t) \\ \xi_G^o(t) \\ \xi_G^u(t) \end{bmatrix},$$

where $\xi_G^o(t)$ and $\xi_G^u(t)$ are the observable and unobservable part of the vector $\xi_G(t)$, respectively, and hence it does not depend on the dynamics of $\xi_G^u(t)$. Therefore, if we reduce the augmented state of the DUIO in (7), by considering only the vector $\left[\begin{array}{c|c} z(t)^\top & \xi_G^o(t)^\top \end{array}\right]^\top$, we have found a choice of block diagonal matrices, namely K, Q_{11}, R_{11}, and S_{11}, such that the vector $e_G(t)$ still converges to zero asymptotically and the pair (Q_{11}, S_{11}) is observable (and hence detectable). By Lemma 10, this is equivalent to saying that the matrix Φ corresponding to this choice of matrices is Schur stable. □

As a result of Proposition 5, from now on, in order to ensure that each DUIO$_i$ asymptotically tracks the state of system (1), and hence the global state estimation error converges to zero, we need to find a block diagonal matrix H, whose diagonal blocks satisfy (13) for every $i \in \mathcal{V}$, and block diagonal matrices F, K, Q, R and S such that Φ is Schur stable, where $N = (I_{Nn} - HC)(I_N \otimes A) - FC$.
In order to solve this problem we resort to the theory of decentralized output feedback control, and to the concept of decentralized fixed modes [6], [7]. To this end, we set

$$\tilde{A} \triangleq N = (I_{Nn} - HC)(I_N \otimes A) - FC, \ \tilde{B} \triangleq I_{Nn}, \ \tilde{C} \triangleq \mathcal{L} \otimes I_n, \tag{15}$$

and we assume

$$\tilde{B} = \left[\begin{array}{c|c|c} \tilde{B}_1 & \dots & \tilde{B}_N \end{array}\right] = \left[\begin{array}{c|c|c} I_n & \dots & \mathbb{0} \\ \vdots & \ddots & \vdots \\ \mathbb{0} & \dots & I_n \end{array}\right],$$

$$\tilde{C} = \left[\begin{array}{c} \tilde{C}_1 \\ \hline \vdots \\ \hline \tilde{C}_N \end{array}\right] = \left[\begin{array}{ccc} \ell_{11} I_n & \dots & \ell_{1N} I_n \\ \hline \vdots & \ddots & \vdots \\ \hline \ell_{N1} I_n & \dots & \ell_{NN} I_n \end{array}\right].$$

The matrix Φ becomes

$$\Phi = \left[\begin{array}{c|c} \tilde{A} + \tilde{B}K\tilde{C} & \tilde{B}Q \\ \hline R\tilde{C} & S \end{array}\right]. \quad (16)$$

If we temporarily assume that $\tilde{A}$ is fixed (i.e., we assume that H and F have been chosen), we know from [6], [7] that there exist block diagonal matrices K, Q, R and S such Φ is Schur stable if and only if all the decentralized fixed modes of the triple $(\tilde{C}, \tilde{A}, \tilde{B})$, by this meaning all λ's such that

$$\lambda \in \bigcap_{K=\mathrm{diag}(K_i)} \sigma(\tilde{A} + \tilde{B}K\tilde{C}),$$

have moduli smaller than 1. By relying upon [6], [7], we derive the following set of equivalent conditions.

Theorem 6. *Suppose that the matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are defined as in (15) and that $\tilde{A}$ is fixed (namely H and F have been chosen). Then the following facts are equivalent.*

(i) There exist block diagonal matrices $K = \mathrm{diag}(K_i), K_i \in \mathbb{R}^{n\times n}$, $Q = \mathrm{diag}(Q_i), Q_i \in \mathbb{R}^{n\times\beta_i}$, $R = \mathrm{diag}(R_i), R_i \in \mathbb{R}^{\beta_i\times n}$, and $S = \mathrm{diag}(S_i), S_i \in \mathbb{R}^{\beta_i\times\beta_i}$, such that the matrix Φ in (16) is Schur stable.

(ii) The decentralized fixed modes of the triple $(\tilde{C}, \tilde{A}, \tilde{B})$ have moduli smaller than 1.

(iii) For every $\lambda \in \mathbb{C}$, with $|\lambda| \geq 1$, and every $\mathcal{J} \subseteq \mathcal{V}$

$$\mathrm{rank}\left(\left[\begin{array}{c|c} \lambda I_{Nn} - \tilde{A} & \tilde{B}\tilde{S}_{\mathcal{J}} \\ \hline \tilde{S}^\top_{\mathcal{J}^c}(\mathcal{L}\otimes I_n) & \mathbb{0} \end{array}\right]\right) \geq Nn, \quad (17)$$

where $\tilde{S}_{\mathcal{J}} \triangleq S_{\mathcal{J}} \otimes I_n$, and $\tilde{S}_{\mathcal{J}^c} \triangleq S_{\mathcal{J}^c} \otimes I_n$.

(iv)

$$\mathrm{rank}\left(\left[\begin{array}{c} \lambda I_{Nn} - \tilde{A} \\ \hline \mathcal{L}\otimes I_n \end{array}\right]\right) = Nn, \quad \forall \lambda \in \mathbb{C}, |\lambda| \geq 1.$$

(v) The pair $((\mathcal{L}\otimes I_n), \tilde{A})$ is detectable.

(vi) $\nexists (v, \lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that

$$\tilde{A}(\mathbb{1}\otimes v) = \lambda(\mathbb{1}\otimes v).$$

Proof. The equivalence of *(i)*, *(ii)* and *(iii)* is true for general matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$, and has been proved in [6], [7]. Similarly, *(iv)* $\Leftrightarrow$ *(v)* is a standard result for discrete-time LTI state space models. So, we will only prove the equivalence of *(iii)* and *(iv)*, and the equivalence of *(iv)* and *(vi)*.

(iii) $\Rightarrow$ *(iv)*. Condition *(iii)* holds for every subset $\mathcal{J} \subseteq \mathcal{V}$, so in particular it holds for $\mathcal{J} = \emptyset$. When so, $\tilde{B}\tilde{S}_{\mathcal{J}}$ is the empty matrix, while $\tilde{S}^\top_{\mathcal{J}^c} = I_{Nn}$. Therefore for $\mathcal{J} = \emptyset$ the rank condition (17) becomes:

$$\mathrm{rank}\left(\left[\begin{array}{c} \lambda I_{Nn} - \tilde{A} \\ \hline \mathcal{L}\otimes I_n \end{array}\right]\right) \geq Nn, \quad \forall \lambda \in \mathbb{C}, |\lambda| \geq 1,$$

that can be verified only with the equality sign.

(iv) $\Rightarrow$ *(iii)*. We observe that since $\tilde{B} = I_{Nn}$, then $\tilde{B}\tilde{S}_{\mathcal{J}} = \tilde{S}_{\mathcal{J}}$. We first note that for $\mathcal{J} = \mathcal{V}$, the rank condition (17) becomes:

$$\mathrm{rank}\left(\left[\begin{array}{c|c} \lambda I_{Nn} - \tilde{A} & I_{Nn} \end{array}\right]\right) \geq Nn, \quad \forall \lambda \in \mathbb{C}, |\lambda| \geq 1,$$

that is always verified with the equality sign. For every $\mathcal{J} \subset \mathcal{V}$, with $\mathcal{J} \neq \emptyset$ and hence $1 \leq |\mathcal{J}| \leq N-1$, the matrix $\tilde{S}_{\mathcal{J}}$ has always full column rank, equal to $|\mathcal{J}|n$. This ensures that

$$\mathrm{rank}\left(\left[\begin{array}{c|c} \lambda I_{Nn} - \tilde{A} & \tilde{B}\tilde{S}_{\mathcal{J}} \end{array}\right]\right) \geq |\mathcal{J}|n.$$

At the same time, the matrix

$$\tilde{S}^\top_{\mathcal{J}^c}(\mathcal{L}\otimes I_n) = (S^\top_{\mathcal{J}^c}\otimes I_n)(\mathcal{L}\otimes I_n) = (S^\top_{\mathcal{J}^c}\mathcal{L})\otimes I_n$$

has full row rank $|\mathcal{J}^c|n = (N - |\mathcal{J}|)n$ (see Lemma 14, in Appendix B). This ensures that for every $\mathcal{J} \subseteq \mathcal{V}$, with $\mathcal{J} \neq \emptyset$, the matrix in *(iii)* has rank (at least) Nn, independently of $\lambda \in \mathbb{C}$. Therefore if the matrix in *(iv)* has rank Nn, then condition *(iii)* holds also for $\mathcal{J} = \emptyset$, and hence for every $\mathcal{J} \subseteq \mathcal{V}$.

(iv) $\Leftrightarrow$ *(vi)*. Condition *(iv)* holds if and only if there is no pair $(w, \lambda), w \in \mathbb{R}^{Nn}, w \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that

$$\tilde{A}w = \lambda w, \quad \text{and} \quad (\mathcal{L}\otimes I_n)w = 0. \quad (18)$$

By Assumption 1, the matrix $\mathcal{L}$ has a single eigenvalue in 0 and the corresponding eigenspace is $\mathrm{span}\{\mathbb{1}_N\}$. By the properties of the Kronecker product, this implies that $\ker(\mathcal{L}\otimes I_n) = \{w \in \mathbb{R}^{Nn} : w = \mathbb{1}_N \otimes v, \exists\ v \in \mathbb{R}^n\}$. Consequently, (18) holds for some $(w, \lambda), w \in \mathbb{R}^{Nn}, w \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, if and only if there exists $(v, \lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that $\tilde{A}(\mathbb{1}_N \otimes v) = \lambda(\mathbb{1}_N \otimes v)$. $\square$

To summarize, Proposition 5 allows us to say that there exists a DUIO described as in (7) if and only if there exist:
- a block diagonal matrix H, whose diagonal blocks satisfy (13) for every $i \in \mathcal{V}$, and
- block diagonal matrices F, K, Q, R and S
such that Φ is Schur stable.
On the other hand, by Theorem 6, this is the case if and only if we can find:
- a block diagonal matrix H, whose diagonal blocks satisfy (13) for every $i \in \mathcal{V}$, and
- a block diagonal matrix F,
such that $\tilde{A} = N = (I_{Nn} - HC)(I_N \otimes A) - FC$ has no eigenvectors of the form $\mathbb{1}_N \otimes v$, for some $v \in \mathbb{R}^n$, corresponding to some unstable eigenvalue $\lambda \in \mathbb{C}, |\lambda| \geq 1$.

Since all matrices involved at this stage are block diagonal, we are reduced ourselves to the problem of determining under what conditions we can find matrices H_i satisfying (13) and matrices F_i, $i \in \mathcal{V}$, so that $\nexists (v, \lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that

$$[(I_n - H_iC_i)A - F_iC_i]v = \lambda v, \quad \forall\ i \in \mathcal{V}. \quad (19)$$

In the following proposition, we provide an equivalent condition for the existence of such matrices H_i and F_i, $i \in \mathcal{V}$.

Proposition 7. *The following facts are equivalent.*

(i) For every $i \in \mathcal{V}$, there exist matrices $H_i \in \mathbb{R}^{n\times p_i}$ satisfying (13) and $F_i \in \mathbb{R}^{n\times p_i}$ so that $\nexists (v, \lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that (19) holds.

(ii) $\nexists(v,\lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, *such that*

$$\begin{cases}(I_n - \bar{H}_i C_i)Av = \lambda v, \\ C_i v = \mathbb{0},\end{cases} \quad \forall\, i \in \mathcal{V}, \tag{20}$$

where $\bar{H}_i$ is the particular solution of (13) given by $\bar{H}_i = D_i(C_i D_i)^\dagger$.

Proof. Throughout the proof, we will steadily refer to the notation used in Appendix A, namely, for every $i \in \mathcal{V}$, we will denote by P_i the matrix $I_n - H_i C_i$ for a generic H_i satisfying (13) (and hence parametrized as in (26)), and by $\bar{P}_i$ the same matrix corresponding to $H_i = \bar{H}_i$.
(i) ⇒ (ii). We proceed by contradiction and we show that ¬*(ii)* ⇒ ¬*(i)*. Assume that there exists $(v,\lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that (20) holds for every $i \in \mathcal{V}$, namely

$$\begin{cases}\bar{P}_i Av = \lambda v, \\ C_i v = \mathbb{0},\end{cases} \quad \forall\, i \in \mathcal{V}.$$

By Lemmas 11 and 12 in Appendix A, for every $i \in \mathcal{V}$ and every P_i, we have that $\sigma_u(C_i, \bar{P}_i A) \subseteq \sigma_u(C_i, P_i A)$, namely the set of eigenvalues of the undetectable subsystem $\mathcal{UD}(C_i, \bar{P}_i A)$ of the pair $(C_i, \bar{P}_i A)$ is included in the set of eigenvalues of the undetectable subsystem $\mathcal{UD}(C_i, P_i A)$ of the pair $(C_i, P_i A)$, and the eigenvectors of $\bar{P}_i A$ belonging to $\mathcal{UD}(C_i, \bar{P}_i A)$ are also eigenvectors of $P_i A$ belonging to $\mathcal{UD}(C_i, P_i A)$. Therefore, for every $i \in \mathcal{V}$ and every P_i, we have that $P_i Av = \lambda v$, and $C_i v = \mathbb{0}$. Consequently, for every $i \in \mathcal{V}$, every $H_i \in \mathbb{R}^{n \times p_i}$ satisfying (13), and every $F_i \in \mathbb{R}^{n \times p_i}$, we get $[(I_n - H_i C_i)A - F_i C_i]v = [P_i A - F_i C_i]v = \lambda v$. This contradicts *(i)*.
(ii) ⇒ (i). Again, we proceed by contradiction and we show that ¬*(i)* ⇒ ¬*(ii)*. Assume that, for every $i \in \mathcal{V}$, for every $F_i \in \mathbb{R}^{n \times p_i}$, and for every $H_i \in \mathbb{R}^{n \times p_i}$ satisfying (13), $\exists(v,\lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that (19) holds. This, in particular, holds true if for every $i \in \mathcal{V}$ we choose $H_i = \bar{H}_i$ and $F_i = \bar{F}_i$, where $\bar{F}_i$ is any matrix that moves all the eigenvalues of $\bar{P}_i A$ inside the unit circle, except those belonging to $\sigma_u(C_i, \bar{P}_i A)$ (see Lemma 13 in Appendix A). This amounts to saying that for any $\lambda \in \mathbb{C}, |\lambda| \geq 1$,

$$\lambda \in \sigma(\bar{P}_i A - \bar{F}_i C_i) \;\Leftrightarrow\; \lambda \in \sigma_u(C_i, \bar{P}_i A).$$

Moreover, exploiting again Lemma 13, if $v \in \mathbb{R}^n, v \neq \mathbb{0}$, is an eigenvector of $\bar{P}_i A - \bar{F}_i C_i$ corresponding to some $\lambda \in \sigma_u(C_i, \bar{P}_i A)$, then $\bar{P}_i Av = \lambda v$ and $C_i v = \mathbb{0}$. Consequently, $[(I_n - \bar{H}_i C_i)A - \bar{F}_i C_i]v = [\bar{P}_i A - \bar{F}_i C_i]v = \lambda v$, implies that (20) holds. This contradicts *(ii)*. □

We are now ready to state the main result of the paper, in which we provide necessary and sufficient conditions for the solvability of Problem 1.

Theorem 8. *The following facts are equivalent.*

(i) There exists a distributed unknown input observer (DUIO) of the form (7) for system (1).

(ii) There exist block diagonal matrices $H = \operatorname{diag}(H_i), H_i \in \mathbb{R}^{n \times p_i}$, $F = \operatorname{diag}(F_i), F_i \in \mathbb{R}^{n \times p_i}$, $K = \operatorname{diag}(K_i), K_i \in \mathbb{R}^{n \times n}$, $Q = \operatorname{diag}(Q_i), Q_i \in \mathbb{R}^{n \times \beta_i}$, $R = \operatorname{diag}(R_i), R_i \in \mathbb{R}^{\beta_i \times n}$, and $S = \operatorname{diag}(S_i), S_i \in \mathbb{R}^{\beta_i \times \beta_i}$ such that

$$\Phi = \left[\begin{array}{c|c}(I_{Nn} - HC)(I_N \otimes A) - FC + K(\mathcal{L} \otimes I_n) & Q \\ \hline R(\mathcal{L} \otimes I_n) & S\end{array}\right],$$

is Schur stable, and H satisfies (9b).

(iii) For every $i \in \mathcal{V}$, there exist matrices $F_i \in \mathbb{R}^{n \times p_i}$ and $H_i \in \mathbb{R}^{n \times p_i}$ satisfying (13) so that $\nexists(v,\lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that (19) holds.

(iv) $\nexists(v,\lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that (20) holds, where $\bar{H}_i = D_i(C_i D_i)^\dagger$.

The proof of the previous theorem is a direct consequence of Propositions 5 and 7, and of Theorem 6, and hence is omitted.

IV. Example

Example 9. *Consider a network of $N = 3$ sensors described by the following Laplacian (of a connected communication graph)*

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

and a system of the form (1)-(2) with matrices:

$$A = \begin{bmatrix} 0.6 & 0.6 & -0.6 \\ 0 & 1.2 & -0.6 \\ 0 & 0 & -0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Assume that the matrices in (4b) weighting the overall unknown input for each sensor are:

$$D_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_2 = D_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be easily checked that condition (iii) of Lemma 4 is satisfied, and hence there exist matrices $H_i, i \in \mathcal{V}$, satisfying (13). Assume $H_i = \bar{H}_i = D_i(C_i D_i)^\dagger$, $\forall\, i \in \mathcal{V}$, so that

$$\bar{H}_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad \bar{H}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \bar{H}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It is easy to verify that $\nexists(v,\lambda), v \in \mathbb{R}^n, v \neq \mathbb{0}, \lambda \in \mathbb{C}, |\lambda| \geq 1$, such that (20) holds, and hence there exists a DUIO of the form (7) for the system. From $\bar{H}_i$ we derive matrices $M_i, i \in \mathcal{V}$. Then, for every $i \in \mathcal{V}$, we design the matrices F_i (and hence N_i and L_i) in order to place in zero all the movable eigenvalues of the matrices $P_i A - F_i C_i$ (referring to the notation adopted in Appendix A). We get

$$N_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.6 & -0.6 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.6 & 0 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.2 & 1.4 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 \\ -0.6 & 0 \\ 0 & 0 \end{bmatrix}$$

$$N_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.8 & -2.6 \\ 0 & 0.8 & 2.6 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 \\ -0.6 & 0 \\ 0.6 & 0 \end{bmatrix}.$$

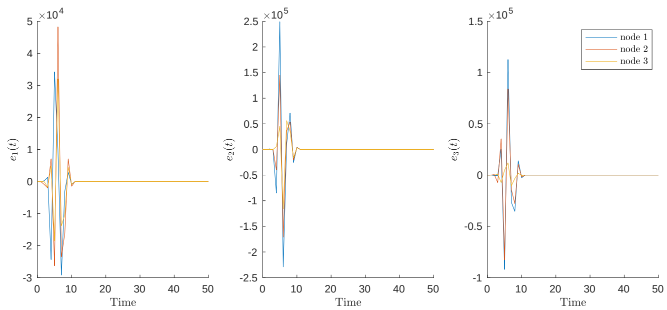

Fig. 1. Dynamics of the state estimation error component-wise

We are now remained to design the matrices K_i, Q_i, R_i and S_i for every $i \in \mathcal{V}$. To this end, we leverage the results in [3] and [16], and we make the system controllable and observable from the first sensor node (channel) by randomly selecting $K_i, i \in [2,3]$ (since this can be achieved for almost every choice of such matrices). In particular, we set

$$K_2 = \begin{bmatrix} 0.16 & -2.21 & -0.61 \\ 2.10 & 1.56 & 2.05 \\ 1.90 & 0.70 & -2.77 \end{bmatrix}, \; K_3 = \begin{bmatrix} -1.62 & 0.13 & 0.66 \\ 1.79 & 1.61 & 0.91 \\ -0.42 & -0.64 & -1.11 \end{bmatrix}.$$

Then, we design the matrices K_1, Q_1, R_1 and S_1 following the algorithm for the synthesis of a centralized dynamic output feedback controller proposed in [1]. By placing in zero all the eigenvalues of the matrix

$$\Phi_1 \triangleq \left[\begin{array}{c|c} \tilde{A} + \sum_{i=2}^{3} \tilde{B}_i K_i \tilde{C}_i & \tilde{B}_1 Q_1 \\ \hline R_1 \tilde{C}_1 & S_1 \end{array}\right],$$

we obtain

$$K_1 = \begin{bmatrix} -1.96 & -2.96 & -2.96 \\ -8.93 & -7.93 & -8.93 \\ 1.98 & 1.98 & 2.98 \end{bmatrix}, \; Q_1 = \begin{bmatrix} -20.33 & 0.30 \\ -51.17 & 3.52 \\ 2.59 & 3.83 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; S_1 = \begin{bmatrix} 7.11 & -0.47 \\ 1 & 0 \end{bmatrix}.$$

The dynamics of the state estimation error component-wise and for each node in the network is shown in Figure 1.

V. Conclusions

In this paper we investigated the problem of designing a distributed unknown input observer for a discrete-time LTI system. More specifically, we considered an augmented observer in which each sensor node is endowed with an additional internal state, and has access only to its partial output measurements as well as to part of the control inputs applied to the system. We showed that, under a connectivity assumption on the network graph, it is possible to exploit cooperation among the agents and obtain that the state estimates provided by each node reach consensus and asymptotically align with the true state. By resorting to the theory of decentralized output feedback, we provided necessary and sufficient conditions for the existence of such a DUIO. The current analysis remains valid in continuous-time and can be easily extended to the case of directed graphs.

Appendix

A. Technical results about detectability

Lemma 10. *Consider the following system of equations*

$$\begin{aligned} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} &= \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Gamma \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ y(t) &= \left[\begin{array}{c|c} I_{n_1} & \mathbb{0} \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1(t), \end{aligned}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, and $\Gamma \triangleq \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$. The following facts are equivalent.

(i) For every $x_1(0) \in \mathbb{R}^{n_1}$ and every $x_2(0) \in \mathbb{R}^{n_2}$, $y(t) \underset{t\to+\infty}{\longrightarrow} \mathbb{0}$ implies $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \underset{t\to+\infty}{\longrightarrow} \mathbb{0}$.

(ii) The pair $(\left[\begin{array}{c|c} I_{n_1} & \mathbb{0} \end{array}\right], \Gamma)$ is detectable.

(iii) The pair (A_{12}, A_{22}) is detectable.

Proof. (i) $\Leftrightarrow$ (ii). This equivalence is a standard result for linear state-space models (see, e.g., [17]).

(ii) $\Leftrightarrow$ (iii). By applying the PBH observability test, we have that the pair $(\left[\begin{array}{c|c} I_{n_1} & \mathbb{0} \end{array}\right], \Gamma)$ is detectable if and only if for every $z \in \mathbb{C}^{n_1+n_2}, |z| \geq 1$, it holds

$$\operatorname{rank}\left(\left[\begin{array}{c|c} zI_{n_1} - A_{11} & -A_{12} \\ \hline -A_{21} & zI_{n_2} - A_{22} \\ \hline I_{n_1} & \mathbb{0} \end{array}\right]\right) = n_1 + n_2,$$

which, in turn, holds if and only if

$$\operatorname{rank}\left(\left[\begin{array}{c} -A_{12} \\ \hline zI_{n_2} - A_{22} \end{array}\right]\right) = n_2.$$

The previous condition is equivalent to the detectability of the pair (A_{12}, A_{22}). □

We consider a matrix pair[2] $(C, \bar{A})$, where $\bar{A} \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$ that is not detectable. Then there exists [21] a nonsingular square matrix $T \in \mathbb{R}^{n\times n}$ that reduces $(C, \bar{A})$ to *standard detectability form*, by this meaning that

$$T^{-1}\bar{A}T = \begin{bmatrix} \bar{A}_{11} & \mathbb{0} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad CT = \begin{bmatrix} C_1 & \mathbb{0} \end{bmatrix}, \tag{21}$$

where $\bar{A}_{ii} \in \mathbb{R}^{n_i\times n_i}, i = 1, 2$, and $C_1 \in \mathbb{R}^{p\times n_1}$, the pair $(C_1, \bar{A}_{11})$ is detectable, while $\bar{A}_{22}$ has all the eigenvalues of modulus greater than or equal to 1. We refer to $\sigma(\bar{A}_{22})$ as the *set of eigenvalues of the undetectable subsystem* of the pair $(C, \bar{A})$ and denote it by $\sigma_u(C, \bar{A})$. Clearly, $\sigma_u(C, \bar{A}) \subseteq \sigma(\bar{A})$. It is well-known that, given $\lambda \in \mathbb{C}$, with $|\lambda| \geq 1$, then $\lambda \in \sigma_u(C, \bar{A})$ if and only if the PBH observability matrix loses rank in λ, i.e.,

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I_n - \bar{A} \\ C \end{bmatrix}\right) < n. \tag{22}$$

We define the *undetectable subspace* of the pair $(C, \bar{A})$ as the set [21] (see, also, [2], [22])

$$\mathcal{UD}(C, \bar{A}) \triangleq X^{no}(C, \bar{A}) \cap \ker(\psi_{u,\bar{A}}(\bar{A})), \tag{23}$$

[2]In this section we will resort to a simplified notation, but all the results stated/derived in the sequel will later be used for specific matrix pairs associated with each ith sensor node, $i \in \mathcal{V}$, and hence the pairs will come with suffixes.

where

$$X^{no}(C,\bar{A}) \triangleq \ker\left(\begin{bmatrix} C \\ C\bar{A} \\ \vdots \\ C\bar{A}^{n-1}\end{bmatrix}\right),$$

while $\psi_{u,\bar{A}}(z)$ denotes the (monic) divisor of the minimal annihilating polynomial of $\bar{A}$, $\psi_{\bar{A}}(z)$, comprising all its unstable zeros. If the pair $(C,\bar{A})$ is in standard detectability form, then

$$\mathcal{UD}(C,\bar{A}) = \mathrm{span}\{\mathbb{e}_i : i \in [n_1+1,n]\}.$$

If not, and T is the nonsingular square matrix that reduces $(C,\bar{A})$ to standard detectability form (21), then

$$\mathcal{UD}(C,\bar{A}) = \mathrm{span}\{T\mathbb{e}_i : i \in [n_1+1,n]\}.$$

Finally, a nonzero vector $v \in \mathbb{R}^n$ is an eigenvector of $\bar{A}$ belonging to $\mathcal{UD}(C,\bar{A})$ if and only if there exists $\lambda \in \sigma_u(C,\bar{A})$ such that

$$\begin{bmatrix}\lambda I_n - \bar{A} \\ C\end{bmatrix} v = \mathbb{0}. \tag{24}$$

Let us introduce a full column matrix $D \in \mathbb{R}^{n\times r}$ and consider the equation

$$HCD = D, \tag{25}$$

in the unknown matrix $H \in \mathbb{R}^{n\times p}$, which is solvable if and only if $\mathrm{rank}(CD) = \mathrm{rank}(D) = r$. The set of solutions H of (25) can be parametrized as follows [5]:

$$H = D(CD)^\dagger + Y(I_n - CD(CD)^\dagger), \tag{26}$$

where $Y \in \mathbb{R}^{n\times n}$ is arbitrary. In the following we will steadily refer to the following notation:

$$\bar{H} \triangleq D(CD)^\dagger,\ \ P \triangleq I_n - HC, \text{ for } H \text{ generic},\ \ \bar{P} \triangleq I_n - \bar{H}C.$$

We observe that as $\mathrm{rank}(\bar{H}) = r$, then [8] the corresponding $\bar{P}$ has rank $n-r$, while for a generic H satisfying (25) we can only claim that $\mathrm{rank}(P) \le n-r$ since $PD = \mathbb{0}$ and hence $\mathrm{im}(D) \subseteq \ker(P)$. We have the following results.

Lemma 11.
(i) The set $\sigma_u(C,\bar{P}A)$ of eigenvalues of the undetectable subsystem of $(C,\bar{P}A)$ is given by

$$\sigma_u(C,\bar{P}A) = \left\{z\in\mathbb{C}, |z|\ge 1 : \mathrm{rank}\left(\begin{bmatrix} zI-A & -D \\ C & \mathbb{0}\end{bmatrix}\right) < n+r\right\};$$

(ii) $\sigma_u(C,\bar{P}A) \subseteq \sigma_u(C,PA)$, for every other P.

Proof. The following proof is inspired by the proof of Theorems 1 and 2 in [5].
(i) For every $z \in \mathbb{C}$, it holds that

$$\begin{aligned}
&\mathrm{rank}\left(\begin{bmatrix} zI-A & -D \\ C & \mathbb{0}\end{bmatrix}\right) \\
\ge\ &\mathrm{rank}\left(\begin{bmatrix} P & \mathbb{0} \\ D^\dagger & \mathbb{0} \\ \mathbb{0} & I\end{bmatrix}\begin{bmatrix} zI-A & -D \\ C & \mathbb{0}\end{bmatrix}\begin{bmatrix} I & \mathbb{0} \\ zD^\dagger - D^\dagger A & I\end{bmatrix}\right) \\
=\ &\mathrm{rank}\left(\begin{bmatrix} zP-PA & \mathbb{0} \\ \mathbb{0} & I \\ C & \mathbb{0}\end{bmatrix}\right) \\
=\ &r + \mathrm{rank}\left(\begin{bmatrix} I & zH \\ \mathbb{0} & I\end{bmatrix}\begin{bmatrix} zP-PA \\ C\end{bmatrix}\right) \\
=\ &r + \mathrm{rank}\left(\begin{bmatrix} zI-PA \\ C\end{bmatrix}\right),
\end{aligned}$$

where the inequality in the first row becomes an equality if $P = \bar{P}$. The previous reasoning has two consequences:
1) for every $z \in \mathbb{C}$ we have that

$$\mathrm{rank}\left(\begin{bmatrix} zI-\bar{P}A \\ C\end{bmatrix}\right) + r = \mathrm{rank}\left(\begin{bmatrix} zI-A & -D \\ C & \mathbb{0}\end{bmatrix}\right),$$

thus proving that *(i)* holds. 2) For every $z \in \mathbb{C}$,

$$\begin{aligned}\mathrm{rank}\left(\begin{bmatrix} zI-\bar{P}A \\ C\end{bmatrix}\right) + r &= \mathrm{rank}\left(\begin{bmatrix} zI-A & -D \\ C & \mathbb{0}\end{bmatrix}\right) \\ &\ge \mathrm{rank}\left(\begin{bmatrix} zI-PA \\ C\end{bmatrix}\right) + r,\end{aligned} \tag{27}$$

and hence

$$\mathrm{rank}\left(\begin{bmatrix} zI-PA \\ C\end{bmatrix}\right) \le \mathrm{rank}\left(\begin{bmatrix} zI-\bar{P}A \\ C\end{bmatrix}\right). \tag{28}$$

Consequently, if $\lambda \in \mathbb{C}, |\lambda| \ge 1$,

$$\begin{aligned}
\lambda \in \sigma_u(C,\bar{P}A) \ &\Leftrightarrow\ \mathrm{rank}\left(\begin{bmatrix} \lambda I-\bar{P}A \\ C\end{bmatrix}\right) < n \\
&\Rightarrow\ \mathrm{rank}\left(\begin{bmatrix} \lambda I-PA \\ C\end{bmatrix}\right) < n \\
&\Leftrightarrow\ \lambda \in \sigma_u(C,PA).
\end{aligned}$$

□

Lemma 12. *If $v \in \mathbb{R}^n, v \neq \mathbb{0}$, is an eigenvector of $\bar{P}A$ belonging to $\mathcal{UD}(C,\bar{P}A)$, and hence corresponding to some $\lambda \in \mathbb{C}, |\lambda| \ge 1$, then it is also an eigenvector of PA belonging to $\mathcal{UD}(C,PA)$ and corresponding to the same λ, for every P.*

Proof. This proof is, in turn, inspired by the proof of Theorem 2 in [5]. The following equivalent conditions hold (see (24)):

$$\begin{aligned}
&\begin{bmatrix} \lambda I-\bar{P}A \\ C\end{bmatrix} v = \mathbb{0} \ \Leftrightarrow\ \begin{bmatrix} \lambda I-\bar{P}A & \mathbb{0} \\ \mathbb{0} & I \\ C & \mathbb{0}\end{bmatrix}\begin{bmatrix} v \\ \mathbb{0}\end{bmatrix} = \mathbb{0} \\
\Leftrightarrow\ &\begin{bmatrix} I & \mathbb{0} & -\lambda\bar{H} \\ \mathbb{0} & I & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & I\end{bmatrix}\begin{bmatrix} \lambda\bar{P}-\bar{P}A & \mathbb{0} \\ \mathbb{0} & I \\ C & \mathbb{0}\end{bmatrix}\begin{bmatrix} v \\ \mathbb{0}\end{bmatrix} = \mathbb{0} \\
\Leftrightarrow\ &\begin{bmatrix} \lambda I-A & -D \\ C & \mathbb{0}\end{bmatrix}\begin{bmatrix} I & \mathbb{0} \\ \lambda D^\dagger - D^\dagger A & I\end{bmatrix}\begin{bmatrix} v \\ \mathbb{0}\end{bmatrix} = \mathbb{0} \\
\Leftrightarrow\ &\begin{bmatrix} (I-DD^\dagger)(\lambda I-A) \\ C\end{bmatrix} v = \mathbb{0}.
\end{aligned}$$

This, in particular, implies that v is an eigenvector of $\bar{P}A$ belonging to $\mathcal{UD}(C,\bar{P}A)$ (and corresponding to λ) if and only if[3]

$$\begin{cases} Cv = \mathbb{0}, \\ (\lambda I - A)v \in \ker(I - DD^{\dagger}) = \operatorname{im}(D), \end{cases} \tag{29}$$

which is equivalent to

$$\begin{bmatrix} v \\ w \end{bmatrix} \in \ker\left(\begin{bmatrix} \lambda I - A & -D \\ C & \mathbb{0} \end{bmatrix}\right), \quad \exists w \in \mathbb{R}^r. \tag{30}$$

On the other hand, if (30) holds then

$$\mathbb{0} = \begin{bmatrix} P & \lambda H \\ \mathbb{0} & I_p \end{bmatrix} \begin{bmatrix} \lambda I - A & -D \\ C & \mathbb{0} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \lambda I - PA \\ C \end{bmatrix} v.$$

This implies that each eigenvector of $\bar{P}A$ that belongs to $\mathcal{UD}(C,\bar{P}A)$ is also an eigenvector of PA belonging to $\mathcal{UD}(C,PA)$ (and it corresponds to the same eigenvalue). $\square$

Lemma 13. *Let $(C,\bar{A})$, with $\bar{A} \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, be an undetectable pair and let $\sigma_u(C,\bar{A})$ be the set of eigenvalues of the undetectable subsystem of the pair $(C,\bar{A})$. Then there exists $L \in \mathbb{R}^{n\times p}$ such that*

(i) the set of the unstable eigenvalues of $\bar{A} - FC$ coincides with $\sigma_u(C,\bar{A})$;

(ii) if (v,λ), with $v \in \mathbb{R}^n, v \neq \mathbb{0}$, and $\lambda \in \mathbb{C}, |\lambda| \geq 1$, satisfies $(\bar{A} - FC)v = \lambda v$, then $\bar{A}v = \lambda v$ and $Cv = \mathbb{0}$.

Proof. Let $T \in \mathbb{R}^{n\times n}$ be a nonsingular matrix that reduces the pair $(C,\bar{A})$ to standard detectability form (21), with $(C_1,\bar{A}_{11})$ detectable, and $\sigma(\bar{A}_{22}) = \sigma_u(C,\bar{A})$. Let F_1 be a matrix such that $\bar{A}_{11} - F_1C_1$ is Schur. Then clearly the set of unstable eigenvalues in

$$\sigma\left(\begin{bmatrix} \bar{A}_{11} & \mathbb{0} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} - \begin{bmatrix} F_1 \\ \mathbb{0} \end{bmatrix} \begin{bmatrix} C_1 & \mathbb{0} \end{bmatrix}\right)$$

coincides with $\sigma_u(C,\bar{A})$. This implies that the matrix

$$L \triangleq T \begin{bmatrix} F_1 \\ \mathbb{0} \end{bmatrix}$$

guarantees that the set of unstable eigenvalues in $\sigma(\bar{A} - FC)$ coincides with $\sigma_u(C,\bar{A})$, and hence *(i)* holds.

On the other hand, we observe that condition

$$\begin{bmatrix} \bar{A}_{11} - F_1C_1 & \mathbb{0} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for some $\lambda \in \mathbb{C}, |\lambda| \geq 1$, and $[v_1^\top \ v_2^\top]^\top \neq \mathbb{0}$, implies $v_1 = \mathbb{0}$ (as $\lambda \notin \sigma(\bar{A}_{11} - F_1C_1)$). Therefore

$$\begin{bmatrix} C_1 & \mathbb{0} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbb{0}.$$

But this implies that

$$(\bar{A} - FC)v = \lambda v, \ |\lambda| \geq 1 \ \Rightarrow \ Cv = \mathbb{0},$$

and hence it is also true that $\bar{A}v = \lambda v$ and $Cv = \mathbb{0}$. This proves *(ii)*. $\square$

B. Properties of the Laplacian

Lemma 14. *Let $\mathcal{L} \in \mathbb{R}^{N\times N}$ be the Laplacian of an undirected, weighted and connected graph $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{A})$, and let $\mathcal{I}$, with $1 \leq |\mathcal{I}| \leq N-1$, be a subset of $\mathcal{V} = [1,N]$. Consider the block matrix $\tilde{S}_{\mathcal{I}}$ given by $\tilde{S}_{\mathcal{I}} = S_{\mathcal{I}} \otimes I_n$, where $S_{\mathcal{I}} \in \mathbb{R}^{N\times|\mathcal{I}|}$ is the selection matrix corresponding to $\mathcal{I}$. Then, $\tilde{S}_{\mathcal{I}}^\top(\mathcal{L} \otimes I_n)$ is of full row rank, i.e.,*

$$\operatorname{rank}(\tilde{S}_{\mathcal{I}}^\top(\mathcal{L} \otimes I_n)) = |\mathcal{I}|n.$$

Proof. It entails no loss of generality assuming that $\mathcal{I} = [1,k]$, where $k = |\mathcal{I}|$, since we can always reduce ourselves to this case by means of permutations. If $k = N-1$, then we can exploit Lemma 3 in [20] to claim that $S_{\mathcal{I}}^\top \mathcal{L} S_{\mathcal{I}}$ is positive definite and hence nonsingular. On the other hand, if $1 \leq k < N-1$, then set $\mathcal{H} \triangleq [1,N-1]$, let $S_{\mathcal{H}}$ be the corresponding selection matrix and let $\hat{S}_{\mathcal{I}}$ be the $(N-1)\times k$ selection matrix corresponding to $\mathcal{I} = [1,k]$ regarded as a subset of $[1,N-1]$. Then

$$S_{\mathcal{I}}^\top \mathcal{L} S_{\mathcal{I}} = \hat{S}_{\mathcal{I}}^\top [S_{\mathcal{H}}^\top \mathcal{L} S_{\mathcal{H}}] \hat{S}_{\mathcal{I}}.$$

The matrix $S_{\mathcal{H}}^\top \mathcal{L} S_{\mathcal{H}}$ is positive definite because $|\mathcal{H}| = N-1$. On the other hand, every principal submatrix of a positive definite matrix is positive definite, in turn, and this proves that $\hat{S}_{\mathcal{I}}^\top [S_{\mathcal{H}}^\top \mathcal{L} S_{\mathcal{H}}] \hat{S}_{\mathcal{I}}$ is positive definite.
This implies that $\tilde{S}_{\mathcal{I}}^\top(\mathcal{L} \otimes I_n)\tilde{S}_{\mathcal{I}} = (S_{\mathcal{I}}^\top \mathcal{L} S_{\mathcal{I}}) \otimes I_n$ is positive definite and hence $\tilde{S}_{\mathcal{I}}^\top(\mathcal{L} \otimes I_n)$ is of full row rank. $\square$

[3] The equality $\ker(I - DD^{\dagger}) = \operatorname{im}(D)$ can be proved as follows. From $(I - DD^{\dagger})D = \mathbb{0}$, we deduce that $\operatorname{im}(D) \subseteq \ker(I - DD^{\dagger})$, and this implies that $\operatorname{rank}(I - DD^{\dagger}) \leq n - r$, where we exploited the fact that D is of full column rank. On the other hand, there exists an orthonormal matrix $T \in \mathbb{R}^{n\times n}$, i.e., such that $TT^\top = T^\top T = I$, for which

$$T^\top D = \begin{bmatrix} \Delta \\ \mathbb{0} \end{bmatrix},$$

where $\Delta \in \mathbb{R}^{r\times r}$ is a nonsingular matrix. Therefore, we have

$$T^\top(I - DD^{\dagger})T = \begin{bmatrix} I_r - \Delta(DD^{\dagger})^{-1}\Delta^\top & \mathbb{0} \\ \mathbb{0} & I_{n-r} \end{bmatrix},$$

which implies that $\operatorname{rank}(I - DD^{\dagger}) \geq n-r$. Thus, we conclude that $\operatorname{rank}(I - DD^{\dagger}) = n - r$, and hence $\ker(I - DD^{\dagger}) = \operatorname{im}(D)$.

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