On the consensus of homogeneous multi-agent systems with positivity constraints

Maria Elena Valcher *Fellow, IEEE,* and Irene Zorzan

Abstract

This paper investigates the consensus problem for multi-agent systems, under the assumptions that the agents are homogeneous and described by a single-input positive state-space model, the mutual interactions are cooperative, and the static state-feedback law that each agent adopts to achieve consensus preserves the positivity of the overall system. Necessary conditions for the problem solvability, that allow to address only the special case when the state matrix is irreducible, are provided. Under the irreducibility assumption, equivalent sets of sufficient conditions are derived. Special conditions either on the system description or on the Laplacian of the communication graph allow to obtain necessary and sufficient conditions for the problem solvability. Finally, by exploiting some results about robust stability either of positive systems or of polynomials, further sufficient conditions for the problem solvability are derived. Numerical examples illustrate the proposed results.

I. Introduction

Multi-agent systems and consensus problems have been very lively research topics in the last decade. Early contributions on these subjects date back to the seventies [9], followed by a few additional contributions in the eighties and nineties [40], [43], but it was only ten years ago that milestone papers like [15], [25], [30], [36], stimulated a wide interest in these problems within the Systems and Control community. The reason for such an impressive success must be credited to the number of diverse and challenging practical problems that can be formalized as consensus problems among agents: flocking and swarming in animal groups, dynamics of opinion forming, coordination in sensor networks, clock synchronization, distributed tasks among mobile robots/vehicles. In all these contexts, the common ingredient is the existence

Dip. di Ingegneria dell'Informazione, Univ. di Padova, via Gradenigo 6/B, 35131 Padova, Italy, phone: +39-049-827-7795 - fax: +39-049-827-7614, `meme@dei.unipd.it, irene.zorzan@studenti.unipd.it`.

of different individuals/units (the agents), each of them performing a task and communicating with its neighbours to achieve a common goal. Consensus algorithms propose distributed control strategies that each agent implements, based on the received information, to reach a common target, to converge to some common value or set of values (see, e.g. [22], [36], [39]).

In a number of these contexts, the information that the agents acquire and update, based on the communication with the other agents, and on which they search for consensus, is the value of variables that are intrinsically nonnegative. This is the case, for instance, when dealing with wireless sensor networks in greenhouses [7], since the parameters that the sensors measure and exchange are light intensity, humidity, and CO_2 concentration, and the sensors must converge to some common values for these parameters, based on which shading or artificial lights will be controlled, watering/heating systems will be activated, CO_2 will be injected, and so on.

Another interesting problem, formalized as a consensus problem with positivity constraint, is the emissions control for a fleet of Plugin Hybrid Vehicles [23]. Each vehicle has a parallel power-train configuration that allows for any arbitrary combination of the power generated by the combustion engine and the electric motor. Moreover, the vehicles can communicate. Under these assumptions, an algorithm was proposed in [23] to regulate in a cooperative way the CO_2 emissions, so that no vehicle has a higher emission level than the others.

Finally, the distributed multi-vehicle coordination problem through local information exchange investigated in [34], [35] is an example of consensus problem among agents (the vehicles) whose dynamics is described by a linear positive state-space model.

In addition to the previous contributions, specifically addressing the positive consensus problem, there have been a number of contributions dealing with properties and performances (stability, stabilization, L_1-gain, optimal and distributed control) of positive multi-agent systems and more generally of interconnected positive systems [10], [11], [12], [33], closely related to the positive consensus problem.

Stimulated by this stream of research and by the aforementioned application problems, this paper aims at investigating the consensus problem for homogeneous multi-agent systems, whose agents are described by a continuous-time, single-input, positive state-space model. The agents are supposed to be cooperative and the communication graph describing the agents' mutual interactions is a weighted, undirected and connected graph. Also, agents adopt a distributed state-feedback control strategy, based on the information available on the states of their neighbouring agents. As the agents' states are intrinsically nonnegative, a natural requirement to introduce, in addition to consensus, is the positivity of the overall controlled multi-agent system. This amounts to saying that the state feedback law adopted to achieve consensus must constrain the state trajectories to remain in the positive orthant. The consensus problem under positivity constraint was first addressed in [44], under quite different assumptions on the communication structure, the feedback control law and the final goal. Indeed, first of all no communication graph was introduced and an n-dimensional supervisory output-feedback controller, instead of a distributed state-feedback control law, was adopted. This implies, in particular, that each agent was assumed to interact with all the other agents, an assumption that does not seem realistic as the number of the agents grows. Finally, consensus was imposed (for all the nonnegative initial conditions) to a common constant value, a requirement that constrains the spectrum of the state matrix of the agents' state-space description and has here been removed.

The paper is organized as follows. Section II provides some background material about positive matrices, graphs and Laplacians. In Section III the positive consensus problem is formalized. A set of necessary conditions is provided in Section IV. In particular, first we determine a necessary condition for the problem solvability that refers to the Frobenius normal form of the matrix A involved in the state-space representation of each agent, and to the corresponding block partitioned input-to-state matrix B. As a result of this necessary condition, the rest of the paper can focus on the special case of an irreducible state matrix A. Additional necessary conditions derived in Section IV prove to be fundamental for the subsequent derivations. The case when the spectral abscissa of A is zero is completely solved in Section V. Section VI focuses on a set of tighter requirements on the matrices $A - \lambda_i BK$ involved in the problem solution, thus leading to two families of sufficient conditions for the problem solvability. Section VII addresses three special cases for which additional constraints, either on the matrices A and B involved in the agents' description or on the Laplacian associated with their communication graph, allow to determine necessary and sufficient conditions for the solvability of the positive consensus problem. Finally, Section VIII provides sufficient conditions for the problem solvability that are based on the robust stability of positive systems [41] or on the robust stability of polynomials [5].

The manuscript encompasses, in revised form, most of the results included in the conference papers [45], [46], where we first proposed the positive consensus problem. But it also brings novel contribution, since Section VII.C and Section VIII are completely new. New remarks and examples have been introduced, to give a clear picture of this challenging problem whose complete solution is still under investigation.

II. Background material

If N is a positive integer, we denote by $[1, N]$ the finite set $\{1, 2, \ldots, N\}$. $\mathbf{1}_N$ is the N-dimensional vector with all entries equal to 1. $\mathbf{e}_i$ denotes the ith *canonical vector* (whose size is always clear from the context). A vector $\mathbf{v} = v_i \mathbf{e}_i$, $v_i > 0$, is called ith *monomial vector* or, generically, monomial vector. The *Kronecker (or tensor) product* of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as the matrix $C = A \otimes B \in \mathbb{R}^{pm\times qn}$:

$$C = [A \otimes B] := \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{bmatrix}.$$

A real square matrix A is *Hurwitz* if all its eigenvalues lie in the open left complex halfplane. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix (in particular, a vector) A_+ with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A_+ \geq 0$); if $A_+ \geq 0$ and at least one entry is positive, A_+ is a *positive matrix* ($A_+ > 0$), while if all its entries are positive it is a *strictly positive matrix* ($A_+ \gg 0$). A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. An $n \times n$ Metzler matrix A ($n > 1$) is *reducible* if there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*. In general, given a Metzler matrix A, a permutation matrix Π can be found such that (s.t.)

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1s} \\ 0 & A_{22} & \ldots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{ss} \end{bmatrix}, \tag{1}$$

where each diagonal block A_{ii}, of size $n_i \times n_i$, is either scalar ($n_i = 1$) or irreducible. (1) is usually known as *Frobenius normal form* of A [19], [28].

Given $A \in \mathbb{R}^{n\times n}$, we denote by $\lambda_{\max}(A) \in \mathbb{R}$ the *spectral abscissa* of A, defined as $\lambda_{\max}(A) := \max\{\Re(\lambda), \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A. For a Metzler matrix A, the following properties hold: $a)$ the spectral abscissa is always an eigenvalue (namely the eigenvalue with maximal real part is always real) and it is called *Frobenius eigenvalue*; $b)$ there exists a positive eigenvector (*Frobenius eigenvector*) $\mathbf{v}_F$ corresponding to $\lambda_{\max}(A)$. If, in addition, the Metzler matrix A is also irreducible, then $a)$ $\lambda_{\max}(A)$ is necessarily simple and $b)$ the Frobenius eigenvector $\mathbf{v}_F$ is strictly positive. Moreover, the following monotonicity property holds [41]: let $A, \bar{A} \in \mathbb{R}^{n\times n}$ be Metzler matrices such that $A \leq \bar{A}$, then $\lambda_{\max}(A) \leq \lambda_{\max}(\bar{A})$; if in addition $\bar{A}$ is irreducible, then $A < \bar{A}$ implies $\lambda_{\max}(A) < \lambda_{\max}(\bar{A})$.

An *undirected, weighted graph* is a triple [29], [31] $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}_+^{N\times N}$ is the (positive and symmetric) *adjacency matrix* of the weighted graph $\mathcal{G}$. In this paper we assume that $\mathcal{G}$ has no self-loops, namely $[\mathcal{A}]_{ii} = 0$ for every index $i \in [1, N]$. The graph is *connected* if for every $i, j \in \mathcal{V}, i \neq j$, there exists $k > 0$ such that $[\mathcal{A}^k]_{ij} > 0$. If $[\mathcal{A}]_{ij} > 0$ for every $i, j \in \mathcal{V}, i \neq j$, the graph $\mathcal{G}$ is called *complete*. We define the *Laplacian matrix* $\mathcal{L} \in \mathbb{R}^{N\times N}$ as $\mathcal{L} := \mathcal{C} - \mathcal{A}$, where $\mathcal{C} \in \mathbb{R}_+^{N\times N}$ is a diagonal matrix whose ith diagonal entry is the *weighted degree* of vertex i, i.e. $[\mathcal{C}]_{ii} := \sum_{l=1}^N [\mathcal{A}]_{il}$. Accordingly, the Laplacian matrix $\mathcal{L} = \mathcal{L}^\top$ takes the following form: versione pagina intera

$$\mathcal{L} = \begin{bmatrix} \ell_{11} & \ell_{12} & \dots & \ell_{1N} \\ \ell_{12} & \ell_{22} & \dots & \ell_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1N} & \ell_{2N} & \dots & \ell_{NN} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^N [\mathcal{A}]_{1j} & -[\mathcal{A}]_{12} & \dots & -[\mathcal{A}]_{1N} \\ -[\mathcal{A}]_{12} & \sum_{j=1}^N [\mathcal{A}]_{2j} & \dots & -[\mathcal{A}]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathcal{A}]_{1N} & -[\mathcal{A}]_{2N} & \dots & \sum_{j=1}^N [\mathcal{A}]_{Nj} \end{bmatrix}.$$

If $\mathcal{G}$ is connected then $\ell_{ii} > 0$ for every $i \in [1, N]$. Notice that all rows of $\mathcal{L}$ sum up to 0, and hence $\mathbf{1}_N$ is always a right eigenvector of $\mathcal{L}$ corresponding to the zero eigenvalue. The following lemma states a useful and well-known result about Laplacian matrices of undirected graphs.

Lemma 1. *[16], [36], [47] If the undirected, weighted graph $\mathcal{G}$ is connected, then $\mathcal{L}$ is a symmetric positive semidefinite matrix, and 0 is a simple eigenvalue of $\mathcal{L}$. As a consequence, the eigenvalues of $\mathcal{L}$, say $\lambda_i = \lambda_i(\mathcal{L})$, $i \in [1, N]$, are nonnegative and real, and they can always be sorted in non-decreasing order, namely as*

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N. \tag{2}$$

In the following, we will steadily assume that $\mathcal{G}$ is an undirected, weighted and connected graph. Consequently, both $\mathcal{A}$ and $\mathcal{L}$ are irreducible matrices [14].

Set, now, $\ell^* := \max_{i=1,\dots,N} \ell_{ii} > 0$. It is well-known [18] that if the eigenvalues of $\mathcal{L}$ are sorted as in (2), then $\ell^* \le \lambda_N$. In addition, since $\mathcal{L}$ is irreducible, then (see Theorem 3 in [18]) $\ell^* < \lambda_N$. Consequently it is always true that $0 < \ell^* < \lambda_N$.

Lemma 2. *[31] Let $\mathcal{G}$ be an undirected, weighted graph with N vertices and Laplacian $\mathcal{L}$ with eigenvalues sorted as in (2). If $\ell^* < \lambda_2$, then $\mathcal{G}$ is complete.*

III. Problem statement

Consider N agents, each of them described by the same n-dimensional continuous-time positive single-input system:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \qquad t \in \mathbb{R}_+,$$

where $\mathbf{x}_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$ are the state vector and the input of the ith agent, respectively, $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a non-Hurwitz Metzler matrix, and $B = [b_i] \in \mathbb{R}^n_+$ is a positive vector. The communication among the N agents is described by an undirected, weighted and connected *communication graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with $\mathcal{V} = \{1, \dots, N\}$ and $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}^{N\times N}_+$ irreducible, with $[\mathcal{A}]_{ii} = 0$. Notice that the assumption that the adjacency matrix is positive corresponds to assuming that the interactions between pairs of agents are cooperative. We assume that the nonnegative eigenvalues of the Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N\times N}$ (with $\ell_{ii} > 0$ for every $i \in [1, N]$) are sorted as in (2). We also introduce the assumption that the graph $\mathcal{G}$ is not complete, namely it is not true that each agent communicates with all the other agents. The completeness of $\mathcal{G}$ is, indeed, a not realistic assumption for N sufficiently large, and removing that case allows to slightly simplify the analysis. As a result, we can claim that $0 = \lambda_1 < \lambda_2 \le \ell^* < \lambda_N$.

Consider the state-feedback control law[1]:

$$u_i(t) = K \sum_{j=1}^{N} [\mathcal{A}]_{ij} [\mathbf{x}_j(t) - \mathbf{x}_i(t)],$$

[1]This state-feedback control law is known in the literature as De Groot's type law, since the first formal study of consensus is credited to DeGroot [9]. This kind of state-feedback law has been assumed in the literature as the standard consensus algorithm since the early contributions on the subject (see [30], [37] and references therein). Even nowadays, possibly with modifications that account for the existence of time-varying communications among the agents or for the fact that relationships among agents may be cooperative or competitive, this is the most common consensus protocol [42], [47], [49].

where $K \in \mathbb{R}^{1\times n}$ is a feedback matrix to be designed. If we denote by $\mathbf{x}(t) \in \mathbb{R}^{Nn}$ and $\mathbf{u}(t) \in \mathbb{R}^N$ the state vector and the input vector of the multi-agent system, respectively, i.e.

$$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1^\top(t) & \dots & \mathbf{x}_N^\top(t) \end{bmatrix}^\top$$
$$\mathbf{u}(t) := \begin{bmatrix} u_1(t) & \dots & u_N(t) \end{bmatrix}^\top,$$

the overall dynamics is described by:

$$\dot{\mathbf{x}}(t) = (I_N \otimes A)\mathbf{x}(t) + (I_N \otimes B)\mathbf{u}(t)$$
$$\mathbf{u}(t) = -(\mathcal{L} \otimes K)\mathbf{x}(t),$$

or equivalently by:

$$\dot{\mathbf{x}}(t) = [(I_N \otimes A) - (I_N \otimes B)(\mathcal{L} \otimes K)]\mathbf{x}(t). \tag{3}$$

The consensus problem with positivity constraints, or *positive consensus problem*, can be stated as follows: *find a feedback matrix* $K \in \mathbb{R}^{1\times n}$ *such that:*
(I) the overall system (3) is positive, i.e. $\mathbb{A} := (I_N \otimes A) - (I_N \otimes B)(\mathcal{L} \otimes K)$ *is a Metzler matrix;*
(II) the overall system (3) reaches consensus, i.e., $\lim_{t\to+\infty} \mathbf{x}_i(t) - \mathbf{x}_j(t) = 0,\ \forall i,j \in [1,N]$, *for almost all positive initial conditions.*[2]

The positive consensus problem can be restated in algebraic terms, as proved in the following proposition.

Proposition 1. *Define the matrix* $K^* = [k_i^*] \in \mathbb{R}_+^{1\times n}$ *as*[3]*:*

$$k_i^* := \begin{cases} \min_{\substack{j=1,\dots,n\\ j\neq i}} \dfrac{a_{ji}}{b_j}\dfrac{1}{\ell^*}, & \text{if } \exists\ j \neq i \text{ such that } b_j \neq 0; \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\ell^* = \max_{i=1,\dots,N} \ell_{ii} > 0$. *Then the positive consensus problem is solvable if and only if there exists a matrix* $K \in \mathbb{R}_+^{1\times n}$ *such that* $0 \leq K \leq K^*$ *and all matrices* $A - \lambda_i BK, i \in [2,N]$, *are Hurwitz.*

[2]Requiring that consensus is achieved for almost all initial conditions is a standard set-up, see e.g. [32], [49], to the point that often it is not even mentioned. In the special case of the positive consensus problem, we restrict our attention to positive initial conditions, and initial conditions for which consensus may not be achieved necessarily belong to the boundary of the positive orthant. The case when all nonnegative initial conditions lead to consensus would require to introduce the irreducibility assumption on both the matrix A and the matrix $\mathbb{A}$, see [44], constraints that seem unnecessary.

[3]Note that the only situation when K^* is not a finite row vector is when $B = b_i\mathbf{e}_i$ for some $i \in [1,n]$, and if so the only infinite entry is $k_i^* = +\infty$.

Proof. The positive consensus problem is solvable if and only if conditions (I) and (II) hold. As far as requirement (I) is concerned, we notice that $\mathbb{A}$ takes the following form:

$$\mathbb{A} = \begin{bmatrix} A-\ell_{11}BK & -\ell_{12}BK & \dots & -\ell_{1N}BK \\ -\ell_{12}BK & A-\ell_{22}BK & \dots & -\ell_{2N}BK \\ \vdots & \vdots & \ddots & \vdots \\ -\ell_{1N}BK & -\ell_{2N}BK & \dots & A-\ell_{NN}BK \end{bmatrix}$$

and hence $\mathbb{A}$ is Metzler if and only if (a) all blocks $A-\ell_{ii}BK$, $i \in [1, N]$, are Metzler and (b) all blocks $-\ell_{ij}BK$, $i, j \in [1, N], i \neq j$, are non-negative. Since $\ell_{ij} \leq 0$ for every $i, j \in [1, N], i \neq j$, and they cannot be all zero (if so $\mathcal{A}$ would be the zero matrix), condition (b) holds if and only if $BK \geq 0$, but since B is a positive column vector, this amounts to saying that $K \geq 0$.

On the other hand, by the definition of K^*, condition (a) holds if and only if $K \leq K^*$. Therefore, K makes $\mathbb{A}$ Metzler, namely condition (I) holds, if and only if $0 \leq K \leq K^*$.

Finally, for requirement (II) we can rely on a classical result about consensus [15], [47]: a necessary and sufficient condition for the agents to achieve consensus is that all matrices $A - \lambda_i BK, i \in [2, N]$, are Hurwitz. This completes the proof. □

Notice that Hurwitz stability of all matrices $A - \lambda_i BK, i \in [2, N]$, implies that the pair (A, B) needs to be stabilizable, and hence in the following we will always make this assumption. Note, also, that this is a special case of simultaneous stabilization problem, since we need to simultaneoulsy stabilize all the pairs $(A, \lambda_i B)$, $i \in [2, N]$, by resorting to state feedback matrices that belong to the hypercube of vertices 0 and K^*.

IV. Preliminary analysis: some necessary conditions

As a first step, we want to understand under what conditions on the structure of the matrices A and B the positive consensus problem is solvable. To this end, we preliminarily assume that the Metzler matrix A is in Frobenius normal form (1) and the positive vector B is partitioned consistently with A, namely

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{ss} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix}, \tag{4}$$

$$A-\ell^*BK^*=\left[\begin{array}{cccc|ccc} A_{11}-\ell^*B_1K_1^* & A_{12}-\ell^*B_1K_2^* & \dots & A_{1r}-\ell^*B_1K_r^* & A_{1r+1}-\ell^*B_1K_{r+1}^* & \dots & A_{1s}-\ell^*B_1K_s^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\ell^*B_rK_1^* & -\ell^*B_rK_2^* & \dots & A_{rr}-\ell^*B_rK_r^* & A_{rr+1}-\ell^*B_rK_{r+1}^* & \dots & A_{rs}-\ell^*B_rK_s^* \\ \hline & & & & A_{r+1r+1} & \dots & A_{r+1s} \\ & & & & \vdots & \ddots & \vdots \\ & & & & 0 & \dots & A_{ss} \end{array}\right] \tag{5}$$

where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are either scalar ($n_i = 1$) or irreducible matrices, and $B_i \in \mathbb{R}^{n_i}_+$. This is a not restrictive assumption, since we can always reduce ourselves to this situation by resorting to a suitable permutation matrix Π, and hence moving from the pair (A, B) to the pair $(\Pi^\top A\Pi, \Pi^\top B)$. It turns out that in order for the positive consensus problem to be solvable only one of the irreducible diagonal blocks A_{ii} can be non-Hurwitz. Specifically, we have the following result.

Proposition 2. *Assume without loss of generality (w.l.o.g.) that the Metzler matrix A and the positive vector B are described as in (4), where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are either scalar or irreducible matrices, and $B_i \in \mathbb{R}^{n_i}_+$. Set $r := \max\{i \in [1, s] \colon B_i \neq 0\}$. If the positive consensus problem is solvable, then A_{ii} is (Metzler and) Hurwitz for every $i \neq r$.*

Proof. Any matrix $K \in \mathbb{R}^{1\times n}_+$, with $0 \le K \le K^*$, can be partitioned in a way consistent with A and B, namely as $K = \begin{bmatrix} K_1 & K_2 & \dots & K_s \end{bmatrix}$, with $K_j \in \mathbb{R}^{1\times n_j}_+$. By the definition of K^*, $A - \ell^* BK^*$ is necessarily Metzler and takes the block-triangular form given in (5). If $r > 1$ the only way for this matrix to be Metzler is that $-\ell^* B_r K_j^* = 0$ for every $j \in [1, r-1]$, and since $\ell^* > 0$ and $B_r > 0$, this means that $K_j^* = 0$ for every $j \in [1, r-1]$ (if $r = 1$ the result is trivially true). So, if $K \in \mathbb{R}^{1\times n}_+$, $0 \le K \le K^*$, is any solution to the positive consensus problem, then all its blocks K_j, $j \in [1, r-1]$, must be zero. Consequently, each matrix $A - \lambda_i BK, i \in [2, N]$, takes the same block triangular form as A, with each diagonal block $A_{jj} - \lambda_i B_j K_j, j \neq r$, coinciding with the corresponding diagonal block A_{jj} in A. So, the Hurwitz property of $A - \lambda_i BK, i \in [2, N]$, implies that all the diagonal blocks A_{jj}, $j \neq r$, are (Metzler and) Hurwitz. □

The following corollary immediately follows from the previous Proposition 1.

Corollary 1. *Assume w.l.o.g. that the Metzler matrix A and the positive vector B are described as in (4), where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1,s]$, are either scalar or irreducible matrices, and $B_i \in \mathbb{R}_+^{n_i}$. Define the matrix $K^* \in \mathbb{R}_+^{1\times n}$ as in Proposition 1, and partition it accordingly to the partition of A and B. Set $r := \max\{i \in [1,s] \colon B_i \neq 0\}$ and let $K_r^* \in \mathbb{R}_+^{1\times n_r}$ be the rth block of K^*. The positive consensus problem is solvable if and only if*

i) *A_{ii} is (Metzler and) Hurwitz for every $i \neq r$;*

ii) *there exists a matrix $K_r \in \mathbb{R}_+^{1\times n_r}$, $0 \leq K_r \leq K_r^*$, that makes the matrices $A_{rr} - \lambda_i B_r K_r, i \in [2,N]$, Hurwitz.*

If the previous conditions hold, then the row matrix $K \in \mathbb{R}_+^{1\times n}$, having K_r as rth block and all remaining blocks equal to zero, is a solution.

Remark 1. *The previous corollary entails far rich consequences, since it tells us that once the non-Hurwitz Metzler matrix A is brought to Frobenius normal form (1), then the solvability of the positive consensus problem requires to first check that all the diagonal blocks of $\Pi^\top A \Pi$ are (Metzler and) Hurwitz, except for the (scalar or irreducible) diagonal block A_{rr}, and then to investigate the positive consensus problem for the pair (A_{rr}, B_r), by assuming as upper bound on the vector $K_r \in \mathbb{R}_+^{1\times n_r}$, the largest positive vector $K_r^* \in \mathbb{R}_+^{1\times n_r}$ such that*

$$\begin{cases} A_{rr} - \ell^* B_r K_r \quad \text{is Metzler} \\ A_{jr} - \ell^* B_j K_r \geq 0, \ \forall \ j \in [1, r-1]. \end{cases}$$

Example 1. *Consider the following single-input positive state-space model for the generic agent*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 0 & 1 & 4 & 1 & 1 \\ 0 & -2 & 1 & 4 & 1 & 1 \\ 0 & 1 & -3 & 2 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} u_i(t).$$

The pair (A, B) is stabilizable. A is in Frobenius normal form (1), with $s = 4$, $n_1 = n_4 = 1, n_2 = n_3 = 2$, and the only non-Hurwitz diagonal block is the one associated with the last *nonzero block in B. Specifically, $r = 3$ and*

$$A_{33} = \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore condition i) in Corollary 1 is satisfied. Assume that there are $N = 3$ agents and that the Laplacian matrix of the communication graph is the following one:

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

In this case $\ell^* = 2$, the eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0 < \lambda_2 = 1 < \lambda_3 = 3$ and $K^* = \left[\begin{array}{c|cc|cc|c} 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{array}\right]$. We note, however, that

$$A_{33} - \ell^* B_3 K_3^* = \begin{bmatrix} -3 & 1/2 \\ 1 & -3/2 \end{bmatrix},$$

so it is true that $A_{33} - \ell^* B_3 K_3^*$ is Metzler, but there exist matrices $K_3 > K_3^*$ such that $A_{33} - \ell^* B_3 K_3$ is Metzler, too. It is easy to see that the matrices $A_{rr} - \lambda_i B_r K_r^*, i \in [2, 3]$, are Hurwitz, and hence condition ii) in Corollary 1 holds. Therefore, the positive consensus problem is solvable. ♣

Corollary 1 immediately leads to the complete solution of the case when the diagonal block A_{rr} in (4) is scalar.

Corollary 2. *Assume that A, B and r are as in Corollary 1, and $n_r = 1$, namely A_{rr} and B_r are scalar. The positive consensus problem is solvable if and only if the following conditions hold: (a) A_{rr} is the only non-Hurwitz diagonal block of A, and (b) the scalar matrix $A_{rr} - \lambda_2 B_r k_r^*$ is negative (and hence Hurwitz). If so, a possible solution is given by the matrix $\bar{K} = \begin{bmatrix} 0_{n_1}^\top & \dots & 0_{n_{r-1}}^\top & k_r^* & 0_{n_{r+1}}^\top & \dots & 0_{n_s}^\top \end{bmatrix}$.*

Proof. For $n_r = 1$ Corollary 1 states what follows: the positive consensus problem is solvable if and only if A_{jj} is Hurwitz for every $j \neq r$, (namely condition (a) holds) and there exists a real number k_r, with $0 \leq k_r \leq k_r^*$, s.t.

$$A_{rr} - \lambda_i B_r k_r < 0, \qquad \forall\, i \in [2, N]. \tag{6}$$

But inequality (6) holds for some k_r, with $0 \leq k_r \leq k_r^*$, and every $i \in [2, N]$ if and only if it holds for $k_r = k_r^*$ and $i = 2$, which amounts to saying that condition (b) holds. This also shows that $\bar{K}$ solves the positive consensus problem. □

As already mentioned in Remark 1 (see also Example 1), the matrix K_r^* is not determined only by the constraint of keeping $A_{rr} - \ell^* B_r K_r^*$ Metzler and with off-diagonal entries as small as possible, but also by the additional constraints $A_{jr} - \ell^* B_j K_r^* \geq 0, j \in [1, r-1]$. Consequently, there might exist $K_r \in \mathbb{R}_+^{1\times n_r}$, $K_r > K_r^*$, such that $A_{rr} - \ell^* B_r K_r$ is still Metzler. Before proceeding it is thus convenient to define $\bar{K}_r^* = [\bar{k}_i^*] \in \mathbb{R}_+^{1\times n_r}$, $\bar{K}_r^* \geq K_r^*$, as

$$\bar{k}_i^* := \begin{cases} \min_{\substack{j=1,\dots,n_r \\ j\neq i}} \dfrac{[A_{rr}]_{ji}}{[B_r]_j} \dfrac{1}{\ell^*}, & \text{if } \exists\, j \neq i \text{ s.t. } b_j \neq 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Analogously, there might exist $\ell > \ell^*$ s.t.$A_{rr} - \ell B_r K_r^*$ is still Metzler, and hence we define

$$\begin{aligned} \bar{\ell}^* &:= \max\{\lambda \in \mathbb{R}_+ \colon A_{rr} - \lambda B_r K_r^* \text{ is Metzler}\} \\ &= \min_{\substack{i,j=1,\dots,n_r \\ j\neq i, [B_r]_i k_j^* \neq 0}} \frac{[A_{rr}]_{ij}}{[B_r]_i k_j^*} \geq \ell^*. \end{aligned}$$

Notice that if $\bar{\ell}^* = \ell^*$ then there exists $j \in [1, n_r]$ such that $\bar{k}_j^* = k_j^*$, while if $\bar{\ell}^* > \ell^*$ then $\bar{K}_r^* \gg K_r^*$. Also, it is always true that $\ell^* \bar{K}_r^* \geq \bar{\ell}^* K_r^*$.

Example 2. *Consider the multi-agent system consisting of $N = 3$ agents and described in Example 1. We have already seen that $\ell^* = 2, r = 3$ and $K_3^* = \begin{bmatrix} 1 & \frac{1}{4} \end{bmatrix}$. It is easy to see that $\bar{\ell}^* = 3 > 2 = \ell^*$, and hence $\bar{K}_3^* \gg K_3^*$, specifically $\bar{K}_3^* = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \end{bmatrix} \gg \begin{bmatrix} 1 & \frac{1}{4} \end{bmatrix} = K_3^*$.* ♣

In the following, to simplify the notation, we will drop the subscript r and hence refer to $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}_+^n$ and $K \in \mathbb{R}_+^{1\times n}$, with $0 \leq K \leq K^*$, under the steady assumption that $n \geq 2$, A is irreducible and the matrix K^* is assigned and satisfies the constraint $K^* \leq \bar{K}^*$, where $\bar{K}^*$ is the largest of the positive row vectors K such that $A - \ell^* BK$ is Metzler. Clearly, we rule out the trivial case when $K^* = 0$ and hence the problem has no solution.

We now provide three necessary conditions for the positive consensus problem solvability under the irreducibility assumption on A. The first one will be a key tool for the following analysis. The second one will provide a necessary condition on the spectrum of A. The last one is a technical result we will need in the following proofs.

Proposition 3. *Assume that A is an $n \times n, n \geq 2$, Metzler non-Hurwitz, irreducible matrix, $B \in \mathbb{R}^n_+$ is a positive vector and $K^* \in \mathbb{R}^{1\times n}_+$ is assigned. If the positive consensus problem is solvable, then*

i) *$A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix*[4]*;*

ii) *$\lambda_{\max}(A)$ is a simple eigenvalue, and it is the only nonnegative real eigenvalue of $\sigma(A)$;*

iii) *if A is non-singular, then $A^{-1}B$ is a positive vector and $K^*A^{-1}B > \frac{1}{\lambda_2} > 0$.*

Proof. i) Under the assumption that the communication graph $\mathcal{G}$ is undirected, connected but not complete, $\ell^* \geq \lambda_2$, and hence $A-\lambda_2 BK^* \geq A-\ell^* BK^*$ is necessarily Metzler. Assume, by contradiction, that $A-\lambda_2 BK^*$ is non-Hurwitz, namely $\lambda_{\max}(A-\lambda_2 BK^*) \geq 0$. Then, for every K such that $0 \leq K \leq K^*$, $A-\lambda_2 BK \geq A-\lambda_2 BK^*$ is a Metzler matrix, and by the monotonicity property of the spectral abscissa, it follows that $\lambda_{\max}(A-\lambda_2 BK) \geq \lambda_{\max}(A-\lambda_2 BK^*) \geq 0$. So, the positive consensus problem would not be solvable.

ii) The fact that $\lambda_{\max}(A)$ is a simple nonnegative eigenvalue follows from the irreducibility of A. The proof of the fact that there are no other nonnegative real eigenvalues relies on some results about the positive observer problem reported in [2]. If the positive consensus problem is solvable, we have already shown in i) that $(A-\lambda_2 BK^*)^\top = A^\top - (\lambda_2 K^{*\top})B^\top$ is Metzler Hurwitz. By Lemma 4.6 and Theorem 4.7 in [2], this implies that the positive system

$$\begin{aligned} \dot{\mathbf{z}}(t) &= A^\top \mathbf{z}(t), \\ \mathbf{y}(t) &= B^\top \mathbf{z}(t), \end{aligned}$$

admits a positive observer and hence the number of nonnegative real eigenvalues of $A^\top$, counting the multiplicity, is at most 1. As $\sigma(A) = \sigma(A^\top)$ and A is non-Hurwitz, then condition ii) holds.

We now show that, when A is non-singular, condition i) implies iii). As $A-\lambda_2 BK^*$ is Metzler and Hurwitz, its inverse exists and it is a negative matrix [33]. On the other hand, if A is nonsingular, recalling that $B > 0$, we have

$$0 > (A-\lambda_2 BK^*)^{-1}B = [I_n - \lambda_2 A^{-1}BK^*]^{-1}(A^{-1}B) = A^{-1}B(1-\lambda_2 K^* A^{-1}B)^{-1}.$$

Since $(1-\lambda_2 K^* A^{-1}B)$ is a scalar, all the nonzero entries of the vector $A^{-1}B$ must have the same sign. Suppose by contradiction that $A^{-1}B$ is a negative vector, namely $A^{-1}B = -\mathbf{v}$,

[4]In the special case when B is an ith monomial vector and hence the ith entry of K^* is $+\infty$, in order to define $A-\lambda_2 BK^*$ we assume that k_i^* is arbitrarily large but otherwise finite. This will be a steady assumption also in the following.

$\exists\ \mathbf{v} \in \mathbb{R}^n_+$. Let $\mathbf{w} \in \mathbb{R}^n_+$, $\mathbf{w} > 0$, be the left Frobenius eigenvector of A, so that $\mathbf{w}^\top A = \lambda_{\max}(A)\mathbf{w}^\top$. From $B = -A\mathbf{v}$, upon multiplying by $\mathbf{w}^\top$ on both sides, we get $0 \le \mathbf{w}^\top B = -\mathbf{w}^\top A\mathbf{v} = -\lambda_{\max}(A)\mathbf{w}^\top\mathbf{v} \le 0$, which implies $\mathbf{w}^\top B = 0$. But if this were the case, then $\mathbf{w}^\top(A - \lambda_2 BK^*) = \lambda_{\max}(A)\mathbf{w}^\top$, namely $\mathbf{w}^\top$ would be a left eigenvector of $A - \lambda_2 BK^*$ corresponding to the positive eigenvalue $\lambda_{\max}(A)$, thus contradicting the Hurwitz assumption on $A - \lambda_2 BK^*$. Hence, $A^{-1}B$ must be a nonnegative vector. Since $A^{-1}B(1 - \lambda_2 K^* A^{-1}B)^{-1} < 0$, this also means that $1 - \lambda_2 K^* A^{-1}B < 0$, namely $1 < \lambda_2 K^* A^{-1}B$, and since $\lambda_2 > 0$ the inequalities $K^* A^{-1}B > \frac{1}{\lambda_2} > 0$ follow. $\square$

V. THE CASE $\lambda_{\max}(A) = 0$

The case when A is irreducible and its spectral abscissa is 0 deserves an independent analysis that easily leads to the conclusion that under these conditions the positive consensus problem is always solvable.

Proposition 4. [5] *Assume that A is an $n \times n$, $n \ge 2$, irreducible Metzler matrix with $\lambda_{\max}(A) = 0$, and B is a positive vector. Then the positive consensus problem is always solvable and every $\bar{K}$ such that $0 < \bar{K} \ll K^*$ is a possible solution. If $\bar{\ell}^* \ge \lambda_N$ then every $\bar{K}$ such that $0 < \bar{K} \le K^*$ is a possible solution.*

Proof. Let $\mathbf{v}_F \gg 0$ be the Frobenius eigenvector of the irreducible Metzler matrix A. We first note that for every matrix K, the strictly positive vector $(\mathbf{1}_N \otimes \mathbf{v}_F)$ is an eigenvector of $\mathbb{A} = (I_N \otimes A) - (I_N \otimes B)(\mathcal{L} \otimes K)$ corresponding to the zero eigenvalue. For every $\bar{K}$, with $0 < \bar{K} \le K^*$, the matrix $\mathbb{A}$ is Metzler. On the other hand, if $0 < \bar{K} \ll K^*$ the matrices $A - \ell_{ii}B\bar{K}, i \in [1, N]$, have exactly the same nonzero pattern as the matrix A, and hence are irreducible. This implies (the proof is a minor modification of the proof of Lemma 2 in [44]) that $\mathbb{A}$ is irreducible. Therefore, for every $0 < \bar{K} \ll K^*$, $\mathbb{A}$ is an irreducible, Metzler matrix, having the strictly positive vector $(\mathbf{1}_N \otimes \mathbf{v}_F)$ as eigenvector corresponding to the zero eigenvalue. This ensures [28] that $\lambda_{\max}(\mathbb{A}) = 0$ and all the other eigenvalues have negative real part. Being $\sigma(\mathbb{A}) = \sigma(A) \cup \sigma(A - \lambda_2 BK) \cup \cdots \cup \sigma(A - \lambda_N BK)$ [47], [48], it follows that all matrices

[5] We are indebted with the Associate Editor, Fabian Wirth, for the final version of Proposition 4 that improves upon our original result.

$A-\lambda_i BK$ are Hurwitz, and hence consensus is achieved. On the other hand, if we assume that $\bar{\ell}^* \geq \lambda_N$, then for every $\bar{K}$, with $0 < \bar{K} \leq K^*$, we have

$$A > A - \lambda_i B\bar{K} \geq A - \lambda_i B\left(\frac{\bar{\ell}^*}{\lambda_N} K^*\right) = A - \bar{\ell}^* B\left(\frac{\lambda_i}{\lambda_N} K^*\right) \geq A - \bar{\ell}^* BK^*.$$

As the matrices $A-\lambda_i B\bar{K}$, $i \in [2, N]$, are lower bounded by a Metzler matrix, they are Metzler, too. On the other hand, by the irreducibility assumption on A and the monotonicity of the spectral abscissa, for every $i \in [2, N]$ we have

$$0 = \lambda_{\max}(A) > \lambda_{\max}\left(A - \lambda_i B\bar{K}\right), \tag{7}$$

i.e. $\bar{K}$ solves the positive consensus problem. □

Remark 2. *1) The reasoning adopted within the first part of the previous proof does not extend to the general case of arbitrary $\lambda_{\max}(A) \geq 0$, since the fact that $\mathbb{A}$ is an irreducible matrix having $\lambda_{\max}(A)$ as Frobenius eigenvalue does not ensure that the matrices $A - \lambda_i BK, i \in [2, N]$, are Hurwitz. 2) Condition $0 < K \ll K^*$ ensures the irreducibility of $\mathbb{A}$. When some of the entries of K coincide with their upperbound, it is possible that one or more of the diagonal blocks $A - \ell_{ii} BK, i \in [1, N]$, is not irreducible and hence $\mathbb{A}$ is not necessarily irreducible. Consensus may still be possible (in particular, as enlightened in the second part of the statement, if $\bar{\ell}^* \geq \lambda_N$) but it cannot be deduced through this path.*

Example 3. *Consider the positive single-input agent*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_i(t).$$

A is an irreducible, Metzler, non-Hurwitz matrix with $\lambda_{\max}(A) = 0$. The pair (A, B) is stabilizable. Assume that there are $N = 8$ agents and that the Laplacian matrix is:

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 2$, $\lambda_5 = \lambda_6 = \lambda_7 = 4$ and $\lambda_8 = 6$, while $\ell^* = 3$. We assume $K^* = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \end{bmatrix} < \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \bar{K}^*$, and hence $\bar{\ell}^* = \ell^*$. It is easy to see that for every $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ with $0 < K \leq K^*$ the matrices

$$A - \lambda_i BK = \begin{bmatrix} -1 - \lambda_i k_1 & 1 - \lambda_i k_2 \\ \\ 1 - 2\lambda_i k_1 & -1 - 2\lambda_i k_2 \end{bmatrix}, \ i \in [2, 8],$$

have negative traces and positive determinants and hence are Hurwitz. So, the positive consensus problem is solvable. ♣

As the case $\lambda_{\max}(A) = 0$ has already been solved, in the following we will steadily assume that $\lambda_{\max}(A) > 0$. By Proposition 3, a necessary condition for the positive consensus problem to be solvable is that A has no other nonnegative real eigenvalue, and therefore, in particular, $0 \notin \sigma(A)$. So, we will also assume that A is non-singular.

VI. Sufficient conditions for the problem solvability: the set $\mathcal{K}^{MH}$

In this section we provide a set of sufficient conditions for the solvability of the positive consensus problem that introduce additional constraints on the matrices $A - \lambda_i BK, i \in [2, N]$, with respect to that of being Hurwitz: we investigate the case when either one or all the solutions K, with $0 \leq K \leq K^*$, of the positive consensus problem make the resulting matrices $A - \lambda_i BK, i \in [2, N]$, not only Hurwitz but also Metzler.

Definition 1. *Given an $n \times n$, $n \geq 2$, irreducible Metzler A, a positive vector $B \in \mathbb{R}^n_+$, a positive matrix $K^* \in \mathbb{R}^{1\times n}_+$ and positive eigenvalues $0 < \lambda_2 \leq \cdots \leq \lambda_N$, we introduce the sets:*

$$\mathcal{K}^H_2 := \{K : 0 \leq K \leq K^*, \ A - \lambda_2 BK \ \text{Hurwitz}\}$$

$$\mathcal{K}^H := \{K : 0 \leq K \leq K^*, \ A - \lambda_i BK \ \text{Hurwitz}, i \in [2, N]\}$$

$$\mathcal{K}^{MH} := \{K \in \mathcal{K}^H : A - \lambda_i BK \ \text{Metzler}, \ i \in [2, N]\}.$$

It is clear that the set of solutions of the positive consensus problem, $\mathcal{K}^H$, is included in $\mathcal{K}^H_2$, and in turn the set of solutions of the positive consensus problem that make the matrices $A - \lambda_i BK, i \in [2, N]$, not only Hurwitz but also Metzler is a subset of $\mathcal{K}^H$. So, the following relationship holds: $\mathcal{K}_2^H \supseteq \mathcal{K}^H \supseteq \mathcal{K}^{MH}$. We want to determine necessary and sufficient conditions ensuring either that $\mathcal{K}^{MH} \neq \emptyset$ or that $\mathcal{K}^{MH} = \mathcal{K}^H$. To this end, we need to preliminarily investigate the structure of $\mathcal{K}_2^H$. We note that, as $\lambda_2 \leq \ell^*$, for every $K \in \mathcal{K}_2^H$, the matrix $A-\lambda_2 BK$ satisfies $A - \lambda_2 BK \geq A - \ell^* BK \geq A - \ell^* BK^*$, and hence, being lower-bounded by a Metzler matrix, it is Metzler, too (this means that $\mathcal{K}_2^H = \mathcal{K}_2^{MH} := \{K \in \mathcal{K}_2^H : A - \lambda_2 BK \text{ Metzler}\}$). Also, by Proposition 3, a necessary condition for the positive consensus problem to be solvable is that $K^* \in \mathcal{K}_2^H$, an assumption we will steadily make in the following. Lemma 3 below provides a characterization of the set $\mathcal{K}_2^H$.

Lemma 3. *Assume that $A \in \mathbb{R}^{n\times n}$ is Metzler, irreducible, and non-singular with $\lambda_{\max}(A) > 0$, $B \in \mathbb{R}_+^n, B > 0$, and $A - \lambda_2 BK^*$ is Metzler and Hurwitz. Then*

$$\mathcal{K}_2^H = \left\{ K : 0 \leq K \leq K^* \text{ and } KA^{-1}B > \frac{1}{\lambda_2} \right\}.$$

Proof. To prove the previous identity, we make the following observations:

1) Since $\mathcal{K}_2^H = \mathcal{K}_2^{MH}$, it is easy to prove, along the same lines of the proof we provided in [46] for $\mathcal{K}^{MH}$, that $\mathcal{K}_2^H$ is a convex set;
2) For every K with $0 \leq K \leq K^*$ the matrix $A-\lambda_2 BK$ is Metzler, so we need to understand for which K it is Hurwitz and for which K it is not. Clearly, $0 \notin \mathcal{K}_2^H$ and hence $\mathcal{K}_2^H \subsetneq \{K : 0 \leq K \leq K^*\}$;
3) By assumption $K^* \in \mathcal{K}_2^H$ and hence there exists $\varepsilon > 0$ such that for every $K \in B(K^*, \varepsilon)$, the ball of center K^* and radius ε, $A-\lambda_2 BK$ is Hurwitz. This ensures, in particular, that the set $\mathcal{K}_2^H$ intersects (possibly includes) the n faces $\mathcal{F}$ of the hypercube $\{K : 0 \leq K \leq K^*\}$ having one vertex in K^*.

In order to complete the description of $\mathcal{K}_2^H$, we only need to determine which matrices K in the interior of the hybercube $\{K : 0 \leq K \leq K^*\}$ belong to the boundary of $\mathcal{K}_2^H$. Clearly, such matrices K leave $A - \lambda_2 BK$ Metzler and irreducible, and hence they necessarily correspond to the case when $A-\lambda_2 BK$ loses the Hurwitz property by becoming singular (with all the remaining eigenvalues in the open left complex half-plane)[6]. This amounts to saying that $\det(A-\lambda_2 BK) = 0$, and since A is non-singular this means that $\det A \cdot \det(I_n - \lambda_2 A^{-1}BK) = 0$, and hence $1 - \lambda_2 KA^{-1}B = 0$, which means that $KA^{-1}B = 1/\lambda_2$. So, to conclude the interior of the convex set $\mathcal{K}_2^H$ consists of all the matrices in the interior of the hypercube $\{K : 0 \leq K \leq K^*\}$

[6]Note that this also means that this "lower boundary" of $\mathcal{K}_2^H$ belongs to the closure of $\mathcal{K}_2^H$, but not to $\mathcal{K}_2^H$ itself.

that are strictly greater than some matrix $\bar{K}$ belonging to the hyperplane $\bar{K}A^{-1}B = 1/\lambda_2$. By Proposition 3, part iii), the vector $A^{-1}B$ is positive and hence a matrix K belonging to the interior of the hypercube $\{K : 0 \le K \le K^*\}$ satisfies $K \gg \bar{K}$ for some vector $\bar{K}$ with $\bar{K}A^{-1}B = 1/\lambda_2$ if and only if $KA^{-1}B > 1/\lambda_2$. □

We now investigate under what conditions $\mathcal{K}^{MH} \neq \emptyset$. To this end we introduce the set

$$\mathcal{K}_N^M := \{K : 0 \le K \le K^*,\ A - \lambda_N BK \text{ Metzler}\}.$$

It is easy to see, by resorting to the definition of $\bar{K}^*$, that

$$\mathcal{K}_N^M = \{K : 0 \le K \le K^*\} \cap \{K : 0 \le K \le \frac{\ell^*}{\lambda_N}\bar{K}^*\},$$

and hence if we define $\hat{K} = [\hat{k}_j]$ as follows:

$$\hat{k}_j := \min\{k_j^*, \frac{\ell^*}{\lambda_N}\bar{k}_j^*\}, \qquad \forall\ j \in [1, n],$$

then $\mathcal{K}_N^M = \{K : 0 \le K \le \hat{K}\}$. We can now provide an answer to the previous problem.

Proposition 5. *Assume that $A \in \mathbb{R}^{n\times n}$ is Metzler, irreducible, and non-singular, with $\lambda_{\max}(A) > 0$, $B \in \mathbb{R}^n_+, B > 0$, and $A - \lambda_2 BK^*$ is Metzler and Hurwitz. Then $\mathcal{K}^{MH} = \mathcal{K}_2^H \cap \mathcal{K}_N^M$. Consequently, the following facts are equivalent:*

i) $\mathcal{K}^{MH} \neq \emptyset$*;*

ii) $\mathcal{K}_2^H \cap \mathcal{K}_N^M \neq \emptyset$*;*

iii) $A - \lambda_2 B\hat{K}$ *is (Metzler and) Hurwitz.*

Proof. If $K \in \mathcal{K}^{MH}$, then K satisfies $0 \le K \le K^*$, and makes all matrices $A - \lambda_i BK, i \in [2, N]$, Metzler and Hurwitz. This ensures that $A - \lambda_2 BK$ is Hurwitz (and hence $K \in \mathcal{K}_2^H$) and $A - \lambda_N BK$ is Metzler (and hence $K \in \mathcal{K}_N^M$). Conversely, let K be a matrix satisfying $0 \le K \le K^*$, and such that $A - \lambda_2 BK$ is Hurwitz and $A - \lambda_N BK$ is Metzler. As $A - \lambda_2 BK \ge A - \lambda_i BK \ge A - \lambda_N BK$ for every $i \in [2, N]$, the fact that the lower bound is Metzler ensures that all matrices are Metzler. The fact that the upper-bound of this set of Metzler matrices is Hurwitz ensures that all the matrices are Hurwitz. Consequently, $K \in \mathcal{K}^{MH}$. So, we have shown that $\mathcal{K}^{MH} = \mathcal{K}_2^H \cap \mathcal{K}_N^M$. The equivalence of i) and ii) is obvious from the previous part of the proof. To prove the equivalence of ii) and iii) observe that $\mathcal{K}_2^H \cap \mathcal{K}_N^M$ is the set of all matrices K that belong to the hypercube $\{K : 0 \le K \le \hat{K}\}$, and that satisfy $KA^{-1}B > 1/\lambda_2$. So, recalling that $A^{-1}B$ is a positive vector, either $\hat{K}$ satisfies $\hat{K}A^{-1}B > 1/\lambda_2$, namely it belongs to $\mathcal{K}_2^H$ and hence makes $A - \lambda_2 B\hat{K}$ (Metzler and) Hurwitz, or $\mathcal{K}_2^H \cap \mathcal{K}_N^M = \emptyset$. □

Note that, by putting together Proposition 5, Lemma 3 and the description of $\mathcal{K}_N^M$, we obtain

$$\mathcal{K}^{MH} = \left\{ K : 0 \leq K \leq \hat{K} \text{ and } KA^{-1}B > \frac{1}{\lambda_2} \right\}.$$

Example 4. *Consider the positive single-input agent*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_i(t)$$

A is an irreducible, Metzler, non-Hurwitz matrix with $\lambda_{\max}(A) > 0$. The pair (A, B) is stabilizable. If we assume that the Laplacian is the same as in Example 3, then we easily find that $\bar{K}^*$ remains the same. We assume that also K^* is the same. We observe that $\frac{\ell^*}{\lambda_N}\bar{K}^* = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \end{bmatrix} < \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \end{bmatrix} = K^*$, and hence $\hat{K} = \frac{\ell^*}{\lambda_N}\bar{K}^*$. It is easy to check that $A - \lambda_2 B\hat{K}$ is (Metzler and) Hurwitz, and indeed $\mathcal{K}^{MH} = \{K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} : 0 \leq k_1 \leq 1/12, 0 \leq k_2 \leq 1/6 \text{ and } 5k_1 + k_2 > 1/2\}$. ♣

We now address the case when $\mathcal{K}^H = \mathcal{K}^{MH}$. To this end we need this preliminary lemma.

Lemma 4. *Assume that $A \in \mathbb{R}^{n \times n}$ is Metzler, irreducible, and non-singular with $\lambda_{\max}(A) > 0$, $B \in \mathbb{R}^n_+, B > 0$. The following facts are equivalent:*

i) *$\bar{\ell}^* \geq \lambda_N$;*

ii) *$\bar{K}^* \frac{\ell^*}{\lambda_N} \geq K^*$;*

iii) *$A - \lambda_i BK^*$ is Metzler for every $i \in [2, N]$.*

Proof. i) ⇒ ii) By the way $\bar{\ell}^*$ has been defined, we have that $\bar{\ell}^* \leq \frac{a_{ij}}{b_i k_j^*}, \forall\, i, j, i \neq j$, and hence

$$k_j^* \leq \frac{a_{ij}}{b_i \bar{\ell}^*}, \qquad \forall\, i, j, i \neq j.$$

If i) holds, then $k_j^* \leq \frac{a_{ij}}{b_i \lambda_N} = \frac{\ell^*}{\lambda_N}\frac{a_{ij}}{b_i \ell^*}, \forall\, i, j, i \neq j$, and hence

$$k_j^* \leq \frac{\ell^*}{\lambda_N} \min_{\substack{i=1,\ldots,n \\ j \neq i, b_i \neq 0}} \frac{a_{ij}}{b_i \ell^*} = \frac{\ell^*}{\lambda_N}\bar{k}_j^*, \qquad \forall\, j \in [1, n],$$

thus proving that ii) holds.

ii) ⇒ iii) If $\bar{K}^* \frac{\ell^*}{\lambda_N} \geq K^*$, then for every $i \in [2, N]$

$$A - \lambda_i BK^* \geq A - \lambda_i B\left(\frac{\ell^*}{\lambda_N}\bar{K}^*\right) = A - \ell^* B\left(\frac{\lambda_i}{\lambda_N}\bar{K}^*\right) \geq A - \ell^* B\bar{K}^*.$$

Since all the matrices $A - \lambda_i BK^*$ are lower bounded by the Metzler matrix $A - \ell^* B\bar{K}^*$, they are Metzler, too, and hence condition iii) holds.
iii) $\Rightarrow$ i) By definition, $\bar{\ell}^* := \max\{\lambda \in \mathbb{R}_+ : A - \lambda BK^* \text{ is Metzler}\}$, so it is immediately seen that if iii) holds then $\lambda_i \leq \bar{\ell}^*$ for every $i \in [2, N]$, and hence i) holds. □

We can now make use of the previous lemma to derive the following result.

Proposition 6. *Assume that $A \in \mathbb{R}^{n\times n}$ is Metzler, irreducible, and non-singular with $\lambda_{\max}(A) > 0$, $B \in \mathbb{R}^n_+, B > 0$, and $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix. The following facts are equivalent*

i) $\bar{\ell}^* \geq \lambda_N$;
ii) $\bar{K}^* \frac{\ell^*}{\lambda_N} \geq K^*$;
iii) $A - \lambda_i BK^*$ *is Metzler and Hurwitz for every* $i \in [2, N]$;
iv) $\mathcal{K}^H = \mathcal{K}^{MH}$.

If any of the previous equivalent conditions hold, then $\mathcal{K}^H = \mathcal{K}^{MH}$ coincides with $\mathcal{K}^H_2$.

Proof. The equivalence of i), ii) and iii), under the assumption that $A - \lambda_2 BK^*$ is Metzler and Hurwitz, is an immediate consequence of Lemma 4. To prove that ii) and iv) are equivalent, observe that, in general, $\mathcal{K}^{MH} \subseteq \mathcal{K}^H \subseteq \mathcal{K}^H_2$. On the other hand, by Proposition 5,

$$\mathcal{K}^{MH} = \mathcal{K}^H_2 \cap \mathcal{K}^M_N = \mathcal{K}^H_2 \cap \{K : 0 \leq K \leq \hat{K}\}.$$

So, if ii) holds, then $\hat{K} = K^*$ and this implies that $\mathcal{K}^H_2 \cap \mathcal{K}^M_N = \mathcal{K}^H_2$. Consequently, $\mathcal{K}^{MH} = \mathcal{K}^H = \mathcal{K}^H_2$, namely iv) holds. This also proves the final statement of the proposition.

Conversely, assume that condition ii) does not hold. Consequently, $\mathcal{K}^{MH} = \mathcal{K}^H_2 \cap \mathcal{K}^M_N = \mathcal{K}^H_2 \cap \{K : 0 \leq K \leq \hat{K}\} \subsetneq \mathcal{K}^H_2$. In particular, by the structure of the sets $\mathcal{K}^H_2$ and $\{K : 0 \leq K \leq \hat{K}\}$, there exists $\bar{K}$ that satisfies two requirements (see Figure 1 for the case $n = 2$): (1) $\bar{K}$ belongs to the interior of $\mathcal{K}^H_2$; and (2) $\bar{K}$ belongs to the boundary of $\mathcal{K}^{MH}$, and specifically to some face $\mathcal{F}$ of $\{K : 0 \leq K \leq \hat{K}\}$ having $\hat{K}$ as one of its vertices.

Clearly, $A - \lambda_i B\bar{K}$ is Metzler and Hurwitz for every $i \in [2, N]$. On the other hand, an $\varepsilon > 0$ can be found such that $B(\bar{K}, \varepsilon) \subset \mathcal{K}^H_2$ and for every $\tilde{K} \in B(\bar{K}, \varepsilon)$ the matrices $A - \lambda_i B\tilde{K}, i \in [2, N]$, are Hurwitz. This implies that $\exists\ \tilde{K} \in \mathcal{K}^H \setminus \mathcal{K}^{MH}$, thus contradicting iv). □

Remark 3. *It is worth mentioning that [1] provides a complete parametrization, expressed as the set of solutions of a Linear Programming problem, of all the matrices K that make*

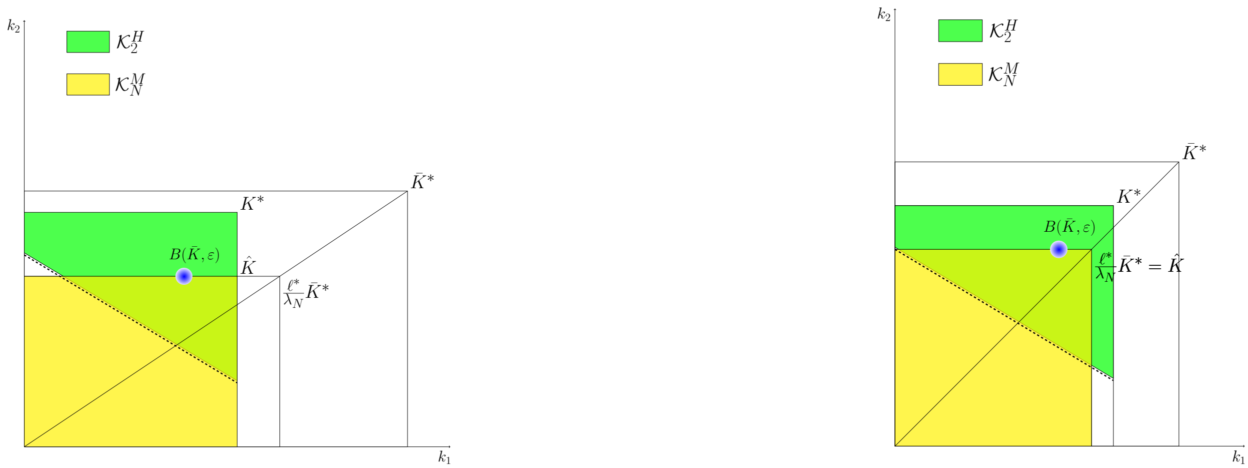

Figure 1. Sets $\mathcal{K}_2^H$ and $\mathcal{K}_N^M$ for the case $n = 2$. On the left, we consider the case when $\bar{K}^*\frac{\ell^*}{\lambda_N}$ is neither greater nor smaller than K^*. On the right, the case $\bar{K}^*\frac{\ell^*}{\lambda_N} < K^*$.

$A + BK$ Metzler and Hurwitz, where A is a Metzler matrix and B a positive matrix. Such a parametrization could be adapted to this specific case, keeping in mind that we have not a single pair (A, B), but $N-1$ pairs $(A, \lambda_i B), i \in [2, N]$, and that we must take into account the additional constraint $0 \leq K \leq K^*$. We will make use of the aforementioned parametrization later in the paper.

Remark 4. *To conclude the section, we would like to remark another reason why the case when the set $\mathcal{K}^{MH}$ is non-empty is of particular interest. If $K \in \mathcal{K}^{MH}$, all matrices $A - \lambda_i BK, i \in [2, N]$, are not only Metzler and Hurwitz, but they also admit a common Linear Copositive Lyapunov function [17], [24], [26], namely a function $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, with $\mathbf{v} \gg 0$, such that $\dot{V}_i(\mathbf{x}) = \mathbf{v}^\top(A - \lambda_i BK)\mathbf{x} < 0$ for every $\mathbf{x} > 0$ and every $i \in [2, N]$. Indeed, for every choice of n indices $i_1, i_2, \dots, i_n \in [2, N]$, the Metzler matrix $[\mathrm{col}_1(A - \lambda_{i_1} BK)\ \mathrm{col}_2(A - \lambda_{i_2} BK)\ \dots\ \mathrm{col}_n(A - \lambda_{i_n} BK)]$ is upper bounded by the Metzler Hurwitz matrix $A - \lambda_2 BK$, and hence is Hurwitz, in turn. This ensures [17], [24] the existence of a common Linear Copositive Lyapunov function. Even more, every Linear Copositive Lyapunov function for the matrix $A - \lambda_2 BK$ is necessarily a Linear Copositive Lyapunov function for all the matrices $A - \lambda_i BK, i \in [2, N]$.*

VII. SPECIAL CASES

In the present section we consider some special cases, namely special classes of agents' models or special communication graphs (and hence Laplacian matrices), for which necessary and sufficient conditions for the solvability of the positive consensus problem can be derived.

A. The case when B is a canonical vector

We consider the case when B is a monomial vector, namely $B = b_i \mathbf{e}_i$, for some $b_i > 0$ and $i \in [1, n]$. The interest in this case comes from the fact that a good number of physical systems that can be modelled through positive or compartmental state-space models have an input-to-state matrix B which is canonical. This happens every time the control input directly affects only one of the state variables (e.g., the gene expression model, some thermal or fluid network models, some chemical reaction networks). As an additional example, in the vehicle model used in [34], [35] to investigate the distributed multi-vehicle coordination problem, the matrix A is Metlzer and unstable, while the matrix B is a canonical vector. In this situation, it entails no loss of generality assuming $B = \mathbf{e}_1$, since we can always reduce ourselves to this situation by resorting to a suitable permutation and to a scaling factor that modify the numeric values of the possible solutions, but do not affect the problem solvability. Accordingly, we can express A as

$$A = \begin{bmatrix} a_{11} & \mathbf{r}^\top \\ \mathbf{c} & A_{22} \end{bmatrix}, \tag{8}$$

where $a_{11} \in \mathbb{R}$, $\mathbf{r}, \mathbf{c} \in \mathbb{R}_+^{n-1}$ are nonnegative vectors, and $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a Metzler matrix. Therefore for every $K = [k_j] \in \mathbb{R}^{1\times n}$ and λ_i we have

$$A - \lambda_i BK = \begin{bmatrix} a_{11} - \lambda_i k_1 & a_{12} - \lambda_i k_2 & \dots & a_{1n} - \lambda_i k_n \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda_i k_1 & \mathbf{r}^\top - \lambda_i [k_2 \ \dots \ k_n] \\ \mathbf{c} & A_{22} \end{bmatrix}. \tag{9}$$

As previously remarked, there is no upper bound on the first entry of the matrix $\bar{K}^*$, and hence $\bar{K}^* = \begin{bmatrix} \bar{k}_1^* & \bar{k}_2^* & \dots & \bar{k}_n^* \end{bmatrix} = \begin{bmatrix} +\infty & \frac{a_{12}}{\ell^*} & \dots & \frac{a_{1n}}{\ell^*} \end{bmatrix} = \begin{bmatrix} +\infty & \frac{1}{\ell^*}\mathbf{r}^\top \end{bmatrix}$. We consider first the case when also the first entry of K^* is $+\infty$. Note that if we consider the block form of the pair (A, B) (see (4)), this situation arises only if $B_i = 0$ for every $i \in [1, r-1]$, and hence B_r is the only nonzero block in B.

Proposition 7. *Assume that A is an irreducible Metzler non-Hurwitz matrix described as in (8), $B = \mathbf{e}_1$ and $K^* = \begin{bmatrix} +\infty & k_2^* & \dots & k_n^* \end{bmatrix}$. The positive consensus problem is solvable if and only if A_{22} is (Metzler and) Hurwitz.*

Proof. [Sufficiency] Let $\alpha_0, \dots, \alpha_{n-2}$ and $\alpha_{n-1} = 1$ be the coefficients of the characteristic polynomial of A_{22}, namely $\det(sI_{n-1} - A_{22}) = s^{n-1} + \alpha_{n-2}s^{n-2} + \dots + \alpha_1 s + \alpha_0$, and notice that if A_{22} is (Metzler and) Hurwitz, then $\alpha_i > 0$ for every $i \in [0, n-1]$. We now prove that, if this is the case, there always exists a feedback matrix $K = \begin{bmatrix} k_1 & 0 & \dots & 0 \end{bmatrix}$, with $0 < K \leq K^*$ (in practice, $k_1 > 0$, since there are no bounds on k_1^*), that solves the positive consensus problem. To this aim, notice that for this choice of K the Metzler matrix $A - \lambda_2 BK$ takes the form:

$$A - \lambda_2 BK = \begin{bmatrix} a_{11} - \lambda_2 k_1 & \mathbf{r}^\top \\ \mathbf{c} & A_{22} \end{bmatrix}.$$

So, if we define $\bar{a} := -a_{11} + \lambda_2 k_1$, and set $\mathbf{r}^\top \mathrm{adj}(sI_{n-1} - A_{22})\mathbf{c} =: \beta_{n-2}s^{n-2} + \dots + \beta_1 s + \beta_0$, the characteristic polynomial of $A - \lambda_2 BK$ can be expressed as:

$$\begin{aligned}
&\det(sI_n - A + \lambda_2 BK) = \det(sI_{n-1} - A_{22})(s + \bar{a}) - \mathbf{r}^\top \mathrm{adj}(sI - A_{22})\mathbf{c} \\
&= (s^{n-1} + \alpha_{n-2}s^{n-2} + \dots + \alpha_1 s + \alpha_0)(s + \bar{a}) - (\beta_{n-2}s^{n-2} + \dots + \beta_1 s + \beta_0) \\
&= s^n + (\alpha_{n-2} + \bar{a})s^{n-1} + (\alpha_{n-3} + \bar{a}\alpha_{n-2} - \beta_{n-2})s^{n-2} + \dots + (\alpha_0 + \bar{a}\alpha_1 - \beta_1)s + (\bar{a}\alpha_0 - \beta_0).
\end{aligned}$$

Therefore, if we take $k_1 > 0$ large enough so that $\bar{a} > \max_{i=0,\dots,n-1} \frac{\beta_i - \alpha_{i-1}}{\alpha_i}$, where we set $\alpha_{-1} = \beta_{n-1} = 0$, then the Metzler matrix $A - \lambda_2 BK$ is Hurwitz since all the coefficients of its characteristic polynomial are positive [14]. Moreover, for every $i \in [3, N]$, the Metzler matrix $A - \lambda_i BK$ is such that $A - \lambda_i BK \leq A - \lambda_2 BK$ and, by the monotonicity property of the spectral abscissa, it follows that K solves the positive consensus problem.

[Necessity] If the consensus problem is solvable, then $\hat{A} := A - \lambda_2 BK^*$ is Metzler and Hurwitz, and a necessary condition for this to happen is that its principal submatrix $\hat{A}_{22}$, obtained by deleting the first row and the first column in $\hat{A}$, is (Metzler and) Hurwitz. As $\hat{A}_{22} = A_{22}$, the result follows. $\square$

Example 5. *Consider the positive single-input agent*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i(t).$$

Notice that A is an irreducible Metzler and non-Hurwitz matrix and that the pair (A, B) is stabilizable. Consider $N = 3$ agents and the same adjacency matrix as in Example 1. The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0$ and $\lambda_2 = 1 < \lambda_3 = 3$. The matrix A_{11}, obtained by deleting the third row and the third column of A, is Hurwitz, and indeed by choosing $K = \begin{bmatrix} 0 & 0 & 7 \end{bmatrix}$ we get that both matrices $A - \lambda_2 BK$ and $A - \lambda_3 BK$ are Metzler and Hurwitz.

On the other hand, if we assume as vector B the canonical vector $B = \mathbf{e}_2$, it is easily seen that the positive consensus problem is not solvable. ♣

It is worth underlying that the necessary condition given in Proposition 7 is independent of the fact that the first entry of K^* is infinite or finite. Indeed, when $B = \mathbf{e}_1$, a necessary condition for the solvability of the positive consensus problem is that A_{22} is Metzler and Hurwitz. However, when $k_1^* < +\infty$, this is no longer sufficient. Indeed, the possibility of resorting to a feedback matrix K whose unique nonzero entry is the first one works if and only if

$$k_1^* \geq k_1 > \lambda_2^{-1} \left(a_{11} + \max_{i=0,\dots,n-1} \frac{\beta_i - \alpha_{i-1}}{\alpha_i} \right).$$

Differently, the characteristic polynomial of the matrices $A - \lambda_i BK, i \in [2, N]$, would not have positive coefficients, thus ruling out the Hurwitz property of these matrices. Therefore the study of the conditions that ensure the positive consensus when $k_1^* < +\infty$ requires, in the general case, a completely different analysis, that keeps into account the specific values taken by the matrix K^*, and is still an open problem.

B. Second-order agents

Consensus among agents described by second order models has been the subject of a good number of papers (see e.g. [50], and references therein). In this subsection we investigate the case when each agent is modelled by a second-order (positive) linear system, i.e.

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_i(t), \tag{10}$$

with a_{12}, a_{21}, b_1 and b_2 nonnegative real numbers. Note that we steadily assume that A is Metzler, non-Hurwitz and non-singular. Recalling that any matrix $M \in \mathbb{R}^{2\times 2}$ is Hurwitz if and only if $\mathrm{tr}(M) < 0$ and $\det(M) > 0$, after elementary manipulations it can be seen that for every $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^2$ and $K \in \mathbb{R}^{1\times 2}$, the matrix $M := A - \lambda BK$ is Hurwitz if and only if

$$\begin{cases} \lambda KB > \mathrm{tr}(A); \\ \lambda K \,\mathrm{adj}(A) B < \det(A). \end{cases} \tag{11}$$

This simple observation allows to prove the following Lemma.

Lemma 5. *Given $A \in \mathbb{R}^{2\times 2}$ and non-singular, $B \in \mathbb{R}^2$ and $K \in \mathbb{R}^{1\times 2}$, for every choice of the $N-1$ positive real numbers $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$, the following facts are equivalent:*

i) *$A - \lambda BK$ is Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$;*

ii) *$A - \lambda_i BK$ is Hurwitz for every $i \in [2, N]$;*

iii) *$A - \lambda_i BK$ is Hurwitz for $i = 2, N$.*

Proof. i) $\Rightarrow$ ii) $\Rightarrow$ iii) are obvious. Assume that iii) holds. If $A - \lambda_2 BK$ and $A - \lambda_N BK$ are Hurwitz matrices, inequalities (11) hold true for $\lambda = \lambda_2$ and $\lambda = \lambda_N$, but then such inequalities are obviously verified also for every $\lambda \in [\lambda_2, \lambda_N]$. This ensures that $A - \lambda BK$ is Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$, namely i) holds. □

From the previous lemma it follows that, when dealing with second-order agents, checking whether a candidate feedback matrix $K \in \mathbb{R}_+^{1\times 2}$, $0 \leq K \leq K^*$, solves the positive consensus problem amounts to checking whether $A - \lambda_2 BK$ and $A - \lambda_N BK$ are both Hurwitz.

The following proposition provides necessary and sufficient conditions for the solvability of the positive consensus problem when dealing with two-dimensional agents.

Proposition 8. *Assume that each agent is described by a second order positive state-space model (10), with A Metzler, non-singular and non-Hurwitz. Then the positive consensus problem is solvable if and only if*

i) *$A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix;*

ii) *$\sigma(A) = (\lambda_{\max}(A), \mu)$, with $\lambda_{\max}(A) > 0$ and $\mu < 0$.*

Moreover, when conditions i) and ii) hold, then $K^* \in \mathcal{K}^H$ and

$$\mathcal{K}^H = \left\{ K \in \mathbb{R}_+^{1\times 2} : 0 \leq K \leq K^*, KA^{-1}B > \frac{1}{\lambda_2} \right\} = \mathcal{K}_2^H. \tag{12}$$

Proof. [Necessity] If the positive consensus problem is solvable, then i) and ii) follow immediately from Proposition 3.

[Sufficiency] We first show that, under assumption ii), if inequalities (11) hold for $i = 2$ and some $K \geq 0$, then they hold also for $i = N$ and the same $K \geq 0$. As $KB \geq 0$, $\lambda_N KB \geq \lambda_2 KB > \mathrm{tr}(A)$. On the other hand, condition ii) implies that $\det(A) < 0$ and therefore $\lambda_2 K\mathrm{adj}(A)B < \det(A)$ implies $K\mathrm{adj}(A)B < 0$. This also ensures that $\lambda_N K\mathrm{adj}(A)B < \lambda_2 K\mathrm{adj}(A)B < \det(A)$. By Lemma 5, we have shown that $K \in \mathcal{K}^H$ if and only if K satisfies inequalities (11) for $i = 2$, namely $A - \lambda_2 BK$ is Hurwitz, which amounts to saying that $K \in \mathcal{K}_2^H$. As a result, condition i) ensures that $K^* \in \mathcal{K}^H$, and $\mathcal{K}^H = \mathcal{K}_2^H$ can be described as in (12). $\square$

Example 6. *Consider the positive single-input agent*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_i(t)$$

Assume that there are $N = 3$ agents and that the Laplacian matrix of the communication graph is as in Example 1, so that $\ell^* = 2$ and the eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0 < \lambda_2 = 1 < \lambda_3 = 3$. Clearly the spectrum of A satisfies condition ii) of Proposition 8. Also, we assume $K^* = \begin{bmatrix} 3/2 & 1/2 \end{bmatrix}$. Therefore $A - BK^*$ is Metzler and Hurwitz. So, the positive consensus problem is solvable and K^* is a solution (note, however, that $A - \lambda_3 BK^*$ is Hurwitz but not Metzler). ♣

C. Communication graph whose Laplacian satisfies special conditions

We consider now the case when the eigenvalues of the Laplacian $\mathcal{L}$ satisfy some algebraic condition. Before proceeding let us introduce the Metzler part of a matrix: given any matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$, we define [21] the *Metzler part of M*, and denote it by $\mathcal{M}(M)$, the matrix

$$[\mathcal{M}(M)]_{ij} := \begin{cases} m_{ii}, & \text{if } i = j; \\ |m_{ij}|, & \text{if } i \neq j. \end{cases}$$

In [21], Hinrichsen and Plischke proved the following monotonicity result (see [13], by Fang and coauthors, for an alternative proof): let $N \in \mathbb{R}^{n \times n}$ be a Metzler matrix, then for any $M \in \mathbb{R}^{n \times n}$ such that $\mathcal{M}(M) \leq N$ the following inequalities hold: $\lambda_{\max}(M) \leq \lambda_{\max}(\mathcal{M}(M)) \leq \lambda_{\max}(N)$. The previous result allows us to easily derive the following condition.

Proposition 9. *Assume that the eigenvalues of the Laplacian $\mathcal{L}$ are such that $\lambda_2 + \lambda_N \leq 2\ell^*$ and all entries of K^* are finite*[7]. *The positive consensus problem is solvable if and only if $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix. When so, $K = K^*$ is a possible solution.*

Proof. We already know (see Proposition 3) that the fact that $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix is a necessary condition for the problem solvability. So, we want to prove

[7] This is always the case if B has at least two nonzero entries, but it also applies to the case when $B = \mathbf{e}_1$ and $k_1^* < +\infty$, as previously discussed.

that when $\lambda_2 + \lambda_N \leq 2\ell^*$, it becomes also sufficient. For $k \in [3, N]$, consider the matrix $A_k^* := A - \lambda_k BK^*$. Its Metzler part is equivalently defined as

$$[\mathcal{M}(A_k^*)]_{ij} = \begin{cases} \lambda_k b_i k_j^* - a_{ij}, & \text{if } i \neq j \text{ and } a_{ij} - \lambda_k b_i k_j^* < 0; \\ a_{ij} - \lambda_k b_i k_j^*, & \text{otherwise.} \end{cases}$$

Note that condition $a_{ij} - \lambda_k b_i k_j^* < 0$ necessarily implies $b_i > 0$. We want to show that $\mathcal{M}(A_k^*) \leq A - \lambda_2 BK^*$. Clearly, for all pairs (i, j) such that $[\mathcal{M}(A_k^*)]_{ij} = [A_k^*]_{ij}$ this is true because $a_{ij} - \lambda_k b_i k_j^* \leq a_{ij} - \lambda_2 b_i k_j^*$. On the other hand, if condition $\lambda_2 + \lambda_k \leq 2\ell^*$ holds, then for every pair of indices $i, j \in [1, n]$, with $i \neq j$,

$$\frac{a_{ij}}{b_i} \frac{2}{\lambda_k + \lambda_2} \geq \frac{a_{ij}}{b_i} \frac{1}{\ell^*} \geq \min_{\substack{i=1,2,\ldots,n \\ i \neq j}} \frac{a_{ij}}{b_i} \frac{1}{\ell^*} = \bar{k}_j^* \geq k_j^*,$$

and this implies that $\lambda_k b_i k_j^* - a_{ij} \leq a_{ij} - \lambda_2 b_i k_j^*$. So, condition $A_k^* \leq \mathcal{M}(A_k^*) \leq A - \lambda_2 BK^*$ and the Hurwitz property of $A - \lambda_2 BK^*$ ensure that $\lambda_{\max}(A_k^*) < 0$, and hence all matrices $A - \lambda_k BK^*, k \in [2, N]$, are Hurwitz. $\square$

Condition $\lambda_2 + \lambda_N \leq 2\ell^*$ in Proposition 9 only depends on the interconnection topology among the agents. The interest in this condition comes from the fact that several meaningful unweighted graphs satisfy it. Among them it is worth mentioning (connected) k-regular graphs, complete bipartite graphs $K_{p,q}$ with $p \geq 2q$ (see Theorem 2.21 in [27]), any tree with a unique vertex of degree ℓ^* (see Theorem 8 in [3], Theorem 2.1 in [8]). As a further example, consider the case of $N \geq 3$ agents whose interconnection topology is described by a star graph, by this meaning that there is an internal node (say node 1) communicating with the remaining $N-1$ nodes, and there is no other interaction among the agents. The Laplacian matrix is given by:

$$\mathcal{L} = \begin{bmatrix} N-1 & -1 & \ldots & -1 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \ldots & 1. \end{bmatrix}$$

In this case $\ell^* = N - 1$, and since $\sum_{i=1}^N \lambda_i = \sum_{i=1}^N \ell_{ii} = 2N - 2$, it follows that $\lambda_2 + \lambda_N \leq 2(N-1) = 2\ell^*$. In addition, if we replace any ray of the star graph with a complete (sub)graph of arbitrary dimension, the algebraic condition $\lambda_2 + \lambda_N \leq 2\ell^*$ still holds (see Theorem 4.4 in [4]). Notice that this communication topology describes quite a realistic situation: the existence of an agent (playing the role of a coordinator) that communicates with all the other agents, and the partition of the remaining agents of the network into groups (not necessarily of the same dimension) such that every agent communicates with all (and only) the agents belonging to its own group.

On the other hand, consider the m-dimensional hypercube defined in [20], [38], namely the graph whose vertex set $\mathcal{V}$ consists of the $N := 2^m$ m-tuples with binary coordinates 0 or 1 and where two vertices are adjacent whenever their corresponding vectors differ in exactly one entry. In this case $\ell^* = \ell_{ii} = m$ for every $i \in [1, N]$ and the Laplacian matrix has (distinct) eigenvalues $\tilde{\lambda}_k = 2k$ with multiplicity $\binom{m}{k}$ for $k \in [0, m]$. Consequently, it is always true that $\lambda_2 + \lambda_N = 2 + 2m > 2m = 2\ell^*$.

VIII. Sufficient conditions for problem solvability

In this section we present some sufficient conditions for the problem solvability that rely on the theory of robust stability of positive systems and on the theory of robust stability of polynomials, and in general lead to matrices $A - \lambda_i BK$ that are Hurwitz but not necessarily Metzler. Since we have already remarked that the set of solutions $\mathcal{K}^H$ is a subset of the set $\mathcal{K}_2^H = \mathcal{K}_2^{MH} = \{K : 0 \leq K \leq K^* \text{ and } A - \lambda_2 BK \text{ is Metzler and Hurwitz}\}$, the key idea is to start from some $K \in \mathcal{K}_2^H$ and to determine sufficient conditions that make such a solution "robust", in the sense that it does not hold only for $\lambda = \lambda_2$ but for every $\lambda \in [\lambda_2, \lambda_N]$.

A. Sufficient conditions from robust stability of positive systems

In this subsection some results on the robust stability of positive linear systems are exploited to derive sufficient conditions for the solvability of the positive consensus problem. In particular, Proposition 10 below provides a sufficient condition for a matrix $K \in \mathbb{R}_+^{1\times n}$ to solve the positive consensus problem expressed as an LMI.

Proposition 10. *Given a state feedback matrix $K \in \mathcal{K}_2^H$, if any of the following two equivalent conditions holds:*

i) $|K\,(A - \lambda_2 BK)^{-1}\,B| < \frac{1}{\lambda_N - \lambda_2}$;
ii) $\exists \mathbf{p} \in \mathbb{R}_+^n$, $\mathbf{p} \gg 0$, *that solves the linear program*

$$\begin{cases} \mathbf{p}^\top (A - \lambda_2 BK) + K \ll 0; \\ \mathbf{p}^\top B < \frac{1}{\lambda_N - \lambda_2}; \end{cases} \tag{13}$$

then $K \in \mathcal{K}^H$. *This implies that*

$$\left\{K \in \mathcal{K}_2^H \colon |K\left(A - \lambda_2 BK\right)^{-1} B| < \frac{1}{\lambda_N - \lambda_2}\right\} \subseteq \mathcal{K}^H.$$

Proof. We first prove the equivalence between conditions i) and ii), which relies on some L_1-gain characterization for positive systems reported in [6]. Consider the positive system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t) \\ y(t) &= K\mathbf{x}(t) \end{aligned} \tag{14}$$

and assume that the input u obeys the output feedback control law $u(t) = -\lambda_2 y(t) + v(t)$. The resulting closed-loop system is described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (A - \lambda_2 BK)\mathbf{x}(t) + Bv(t) \\ y(t) &= K\mathbf{x}(t) \end{aligned} \tag{15}$$

and its transfer function is given by $W(s) = K(sI - A + \lambda_2 BK)^{-1}B$. Since $K \in \mathcal{K}_2^H$, the positive system (15) is asymptotically stable, and by Proposition 2 in [6] its L_1-gain g can be expressed in terms of its transfer function as $g = W(0) = -K\left(A - \lambda_2 BK\right)^{-1} B = |K(A - \lambda_2 BK)^{-1}B|$.

Condition i) amounts to saying that the L_1-gain of the positive system (15) is smaller than $(\lambda_N - \lambda_2)^{-1}$, and by Lemma 1 in [6] this is true if and only if the linear program (13) is feasible.

To prove that condition i) ensures that K solves the positive consensus problem, namely $K \in \mathcal{K}^H$, we make use of a result on robust stability of positive systems by Son and Hinrichsen (see [41]). Set $\bar{A} := A - \lambda_2 BK$, $\Delta := \lambda_2 - \lambda$ and notice that for every $\lambda \in [\lambda_2, \lambda_N]$

$$A - \lambda BK = \bar{A} + \Delta BK, \tag{16}$$

where $\bar{A}$ is Metzler and Hurwitz and ΔBK is a perturbation matrix. Specifically, the matrix BK gives the structure of the perturbation, while Δ can be regarded as an unknown scalar disturbance that gives the size of the perturbation.

By Theorem 5 in [41], the stability radius of the positive system $\dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t)$ with respect to perturbations described as in (16), namely $r(\bar{A}; B, K) := \inf\{|\Delta| : \lambda_{\max}(\bar{A} + \Delta BK) \geq 0\}$, can be computed as $|K(A - \lambda_2 BK)^{-1}B|^{-1}$. So, if condition i) holds, then

$$r(\bar{A}; B, K) = \frac{1}{|K(A - \lambda_2 BK)^{-1}B|} > \lambda_N - \lambda_2,$$

and this ensures that $A - \lambda BK$ is Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$. Therefore, K solves the positive consensus problem and the final statement follows. $\square$

Remark 5. *By following up on Remark 3, we can observe that the parametrization given by Ait Rami and Tadeo in [1] can be used to provide an alternative statement of the previous conditions i) and ii). Indeed, we first note that condition $K \in \mathcal{K}_2^H$ is equivalent to saying that*

$$K = \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix} \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}^{-1} =: \mathbf{z}^\top D^{-1},$$

where the n-dimensional vectors $\mathbf{z} > 0, \mathbf{d} := D\mathbf{1}_n \gg 0$, satisfy

$$\begin{aligned} A\mathbf{d} + B\mathbf{1}_n^\top \mathbf{z} &< 0, \\ a_{ij} d_j + b_i z_j &\geq 0, \\ \mathbf{z}^\top &\leq K^* D. \end{aligned}$$

By keeping in mind that $|K\left(A - \lambda_2 BK\right)^{-1} B| = -K\left(A - \lambda_2 BK\right)^{-1} B$ (see the proof of Proposition 10), we can rewrite condition i) in Proposition 10 as:

$$-\mathbf{z}^\top \left(AD - \lambda_2 B\mathbf{z}^\top\right)^{-1} B < \frac{1}{\lambda_N - \lambda_2}$$

and condition ii) as "$\exists \mathbf{p} \in \mathbb{R}_+^n$, $\mathbf{p} \gg 0$, that solves the linear program

$$\begin{cases} \mathbf{p}^\top \left(AD - \lambda_2 B\mathbf{z}^\top\right) + \mathbf{z}^\top \ll 0; \\ \mathbf{p}^\top B < \frac{1}{\lambda_N - \lambda_2}. \end{cases}$$

Lemma 6. *Assume that $A \in \mathbb{R}^{n\times n}$ is a non-singular and non-Hurwitz Metzler matrix. Assume also that $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix. Then, the sufficient condition given in Proposition 10 holds for some $K \in \mathcal{K}_2^H$ if and only if it holds for $K = K^*$.*

Proof. Sufficiency is obvious. To prove necessity, we consider condition i) of Proposition 10 and prove that for every $K \in \mathcal{K}_2^H$ it holds

$$|K(A - \lambda_2 BK)^{-1} B| \geq |K^*(A - \lambda_2 BK^*)^{-1} B|. \tag{17}$$

To this aim, notice that $K(A - \lambda_2 BK)^{-1} B = K(I_n - \lambda_2 A^{-1} BK)^{-1} A^{-1} B = (KA^{-1}B)(1 - \lambda_2 KA^{-1}B)^{-1}$. Recalling that $KA^{-1}B > \frac{1}{\lambda_2}$ for every $K \in \mathcal{K}_2^H$, condition (17) can be rewritten as

$$\frac{KA^{-1}B}{\lambda_2(KA^{-1}B) - 1} \geq \frac{K^*A^{-1}B}{\lambda_2(K^*A^{-1}B) - 1}. \tag{18}$$

Set $x(K) := KA^{-1}B$ and notice that, since $A^{-1}B$ is a positive vector (see Proposition 3, part iii)), it holds $x(K^*) \geq x(K)$ for every $K \in \mathcal{K}_2^H$. Now, consider the function $f(x) := \frac{x}{\lambda_2 x - 1}$, $x > \frac{1}{\lambda_2}$. It is easy to see that $f(x)$ is a monotone and strictly decreasing function since $f'(x) = -\frac{1}{(\lambda_2 x-1)^2} < 0$. But then, $f(x(K)) \geq f(x(K^*))$ for every $K \in \mathcal{K}_2^H$, namely condition (18) holds. □

As an immediate corollary of Lemma 6, we can rewrite the sufficient condition given in Proposition 10 as follows.

Corollary 3. *Assume that $A \in \mathbb{R}^{n\times n}$ is a non-singular and non-Hurwitz Metzler matrix. If $A - \lambda_2 BK^*$ is (Metzler and) Hurwitz and any of the following two equivalent conditions holds:*

i) $|K^* (A - \lambda_2 BK^*)^{-1} B| < \frac{1}{\lambda_N - \lambda_2}$;

ii) $\exists \mathbf{p} \in \mathbb{R}^n_+$, $\mathbf{p} \gg 0$, *s.t.* $\mathbf{p}^\top (A - \lambda_2 BK^*) + K^* \ll 0$, *and* $\mathbf{p}^\top B < \frac{1}{\lambda_N - \lambda_2}$,

then K^* solves the positive consensus problem.

B. A sufficient condition from the theory of robust stability for polynomials

Another sufficient condition for a matrix $K \in \mathbb{R}^{1\times n}_+$ to solve the positive consensus problem can be derived from a criterion of robust stability of polynomials, i.e. stability of polynomials with uncertain coefficients. Before proceeding, we introduce the *Hurwitz matrix* associated with a given polynomial [5].

Definition 2. *[5] Consider the polynomial $d(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \in \mathbb{R}[s]$ of degree n ($a_n \neq 0$). The* Hurwitz matrix *associated with $d(s)$ is the $n \times n$ real matrix*

$$H_{d,n} := \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & \dots & 0 \\ 0 & a_n & a_{n-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & a_0 \end{bmatrix}$$

For the sake of clarity, the matrix $H_{d,n}$ has the following structure: the first and the second rows coincide with the second and the first row of the Routh table, respectively, completed with zeros; every couple of consecutive rows is obtained by the previous couple of rows by means of a one-step shift to the right (and the insertion of a 0 on the left). Note that if $a_0 \neq 0$, then $H_{d,n}$ is non-singular [5].

Proposition 11. *Denote by $d(s)$ the characteristic polynomial of $A \in \mathbb{R}^{n\times n}$, i.e. $d(s) := \det(sI_n - A)$, and let $\bar\lambda \in [\lambda_2, \lambda_N]$ and $K \in \mathbb{R}_+^{1\times n}$, $0 \leq K \leq K^*$, be such that $A - \bar\lambda BK$ is Hurwitz. Define the polynomial, of degree (at most) $n-1$, $q(s) := K\,\mathrm{adj}(sI_n - A)B$, and denote by $H_{q,n} \in \mathbb{R}^{n\times n}$ the Hurwitz matrix associated with $q(s)$ regarded as a polynomial of degree n. Define also the matrix pencil*

$$H_{p,n}(\lambda) := H_{d,n} + \lambda H_{q,n}, \quad \lambda \in \mathbb{R}, \tag{19}$$

and denote by $0 \leq \mu_1 < \cdots < \mu_k$ the nonnegative, real, distinct eigenvalues of $H_{p,n}(\lambda)$, namely the nonnegative, real, distinct values for which the matrix pencil $H_{p,n}(\lambda)$ becomes a singular matrix. If there exists $j \in [1, k]$ such that $[\lambda_2, \lambda_N] \subset (\mu_j, \mu_{j+1})$, where $\mu_{k+1} = +\infty$, then for every $i \in [2, N]$ the matrix $A - \lambda_i BK$ is Hurwitz, i.e. K solves the positive consensus problem.

Before proceeding with the proof of Proposition 11 we need to state the following Lemma on the robust stability of polynomials.

Lemma 7. *Let $d(s)$ and $q(s)$ be two polynomials such that $\deg d(s) = n > \deg q(s)$. Consider the family of polynomials parametrized by λ, $p(s, \lambda) = d(s) + \lambda q(s)$, where $\lambda \in [\lambda_-, \lambda_+]$ and $\lambda_- > 0$. Assume that there exists $\bar\lambda \in [\lambda_-, \lambda_+]$ such that $p(s, \bar\lambda)$ is Hurwitz. Then, the polynomial $p(s, \lambda)$ is Hurwitz for every $\lambda \in [\lambda_-, \lambda_+]$ if and only if the Hurwitz matrix associated with $p(s, \lambda)$, namely $H_{p,n}(\lambda)$, is nonsingular for every $\lambda \in [\lambda_-, \lambda_+]$.*

The proof of Lemma 7 above follows from a simple application of Lemma 4.8.3 in [5] once we notice that for every $\lambda \in [\lambda_-, \lambda_+]$ the polynomial $p(s, \lambda)$ can be written as $p(s, \lambda) = p(s, \bar\lambda) + (\lambda - \bar\lambda)q(s)$. For the sake of brevity, the proof is omitted.

We can now prove Proposition 11.

Proof. (Proposition 11) Introduce the characteristic polynomial $p(s, \lambda)$ of the matrix $A - \lambda BK$

$$p(s, \lambda) := \det(sI_n - A + \lambda BK) = d(s) + \lambda q(s),$$

and note that the Hurwitz matrix associated with $p(s, \lambda)$ is the matrix pencil $H_{p,n}(\lambda)$ defined in (19). Since by hypothesis $p(s, \bar\lambda)$ is Hurwitz and there exists $j \in [1, k]$ such that $H_{p,n}(\lambda)$ is nonsingular for every $\lambda \in (\mu_j, \mu_{j+1}) \supset [\lambda_2, \lambda_N]$, then it follows from Lemma 7 that $p(s, \lambda)$ is Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$, and hence K solves the positive consensus problem. $\square$

Example 7. *Consider the positive single-input agent:*

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0.5 & 2 & -3 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u_i(t)$$

Assume that there are $N = 5$ agents and that the Laplacian matrix of the communication graph is the following one:

$$\mathcal{L} = \begin{bmatrix} 1.05 & -0.8 & -0.25 & 0 & 0 \\ -0.8 & 1.05 & 0 & -0.25 & 0 \\ -0.25 & 0 & 1 & -0.25 & -0.5 \\ 0 & -0.25 & -0.25 & 1 & -0.5 \\ 0 & 0 & -0.5 & -0.5 & 1 \end{bmatrix}.$$

In this case $\ell^* = 1.05$, $\lambda_2 = 0.3876$, $\lambda_5 = 1.9405$ and $\bar{K}^* = \begin{bmatrix} 0 & 1.9048 & 0.4762 \end{bmatrix}$. We assume $K^* = \bar{K}^*$. Notice that the necessary condition of Proposition 3 is satisfied since $\lambda_{\max}(A) = 1.4901$ is the only nonnegative real eigenvalue of $\sigma(A)$ and $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix. Now, since $k_1^* = 0$, for every $K \in \mathbb{R}_+^{1\times 3}$, $0 \leq K \leq K^*$, the matrix $H_{q,n}$ takes the following form:

$$H_{q,n} = \begin{bmatrix} 2k_2 + k_3 & 7k_2 + 4k_3 & 0 \\ 0 & 9k_2 + 4k_3 & 0 \\ 0 & 2k_2 + k_3 & 7k_2 + 4k_3 \end{bmatrix},$$

and the matrix pencil defined in (19) results

$$H_{p,n}(\lambda) = \begin{bmatrix} 3 & -5.5 & 0 \\ 1 & -3 & 0 \\ 0 & 3 & -5.5 \end{bmatrix} + \lambda \begin{bmatrix} 2k_2 + k_3 & 7k_2 + 4k_3 & 0 \\ 0 & 9k_2 + 4k_3 & 0 \\ 0 & 2k_2 + k_3 & 7k_2 + 4k_3 \end{bmatrix}.$$

Then, it is easy to verify that the eigenvalues of the matrix pencil are $\tilde{\mu}_1 = \frac{5.5}{7k_2+4k_3}$ and the zeros of the polynomial $g(\tilde{\mu}) = \tilde{\mu}^2(2k_2 + k_3)(9k_2 + 4k_3) + \tilde{\mu}(14k_2 + 5k_3) - 3.5$. If we choose $K = K^*$ we have $\mu_1 = 0.0951 < \mu_2 = 0.3609$, and hence $[\lambda_2, \lambda_5] \subset (\mu_2, +\infty)$ and by Proposition 11 K^* solves the positive consensus problem. ♣

IX. Concluding remarks

In this paper we have investigated the positive consensus problem for homogeneous multi-agent systems, described by a single-input positive state-space model, by assuming that the agents' interactions are cooperative, and distributed control is achieved through the classic DeGroot's feedback control law. Preliminary analysis allowed us to focus only on the case when the agents' state matrix is an irreducible Metzler matrix. Necessary or sufficient conditions for problem solvability have been derived. Equivalent conditions for problem solvability have been derived by introducing special assumptions either on the agents' description or on the communication graph. The practical relevance of those assumptions has been commented upon in the corresponding subsections. As a general solution is still missing, in Section VI we have provided a complete analysis of the cases when stronger versions of the original problem are addressed: namely when we search for a (single) feedback matrix K that makes all matrices $A - \lambda_i BK, i \in [2, N]$, Metzler and Hurwitz, and the case when all the solutions of the positive consensus problem make the aforementioned matrices Metzler and Hurwitz. Future research aims at determining weaker sufficient conditions for the problem solution, and on focusing on special classes of communication graphs for which a complete solution is available.

References

[1] M. Ait Rami and F. Tadeo. Control synthesis for positive linear systems with bounded control. *IEEE Trans. Circ. Sys. - Part II*, 54, no. 2:151–155, 2007.

[2] J. Back and A. Astolfi. Design of positive linear observers for positive linear systems via coordinate transformations and positive realizations. *SIAM J. Contr. Opt.*, 47 (1):345–373, 2008.

[3] R.B. Bapat. The Laplacian matrix of a graph. *The Mathematics Student*, 65:214–223, 1996.

[4] R.B. Bapat, A.K. Lal, and S. Pati. On algebraic connettivity of graphs with at most two points of articulation in each block. *Lin. Multilinear Alg.*, 60:415–432, 2012.

[5] B. R. Barmish. *New tools for robustness of linear systems*. Macmillan, New York, 1994.

[6] C. Briat. Robust stability and stabilization of uncertain linear positive systems via integral linear constraints - L_1- and L_∞-gains characterizations. *Int. J. Robust Nonlinear Control*, 23:1932–1954, 2013.

[7] D.D. Chaudhary, S.P. Nayse, and L.M. Waghmare. Application of wireless sensor networks for greenhouse parameter control in precision agriculture. *Int. J. Wireless & Mobile Networks*, 3 (1), 2011.

[8] K.ch. Das. An improved upper bound for Laplacian graph eigenvalues. *Lin. Alg. Appl.*, 368:269–278, 2003.

[9] M.H. DeGroot. Reaching a consensus. *J. Amer. Statist. Assoc.*, 69, no. 345:118–121, 1974.

[10] Y. Ebihara, D. Peaucelle, and D. Arzelier. Decentralized control of positive systems using L_1-induced norm characterization. In *Proc. 2012 IEEE CDC*, pages 6653–6658, Maui, HI, USA, 2012.

[11] Y. Ebihara, D. Peaucelle, and D. Arzelier. Stability and persistence analysis of large scale interconnected positive systems. In *Proc. 2013 ECC*, Zurich, Switzerland, 2013.

[12] Y. Ebihara, D. Peaucelle, and D. Arzelier. Efficient convergence rate analysis of multi-agent positive systems under formation control. In *Proc. 2014 IFAC World Congress*, Cape Town, South Africa, 2014.

[13] Y. Fang, K. A. Loparo, and X. Feng. Sufficient conditions for the stability of interval matrices. *Int. J. Control*, 58(4):969–977, 1993.

[14] L. Farina and S. Rinaldi. *Positive linear systems: theory and applications*. Wiley-Interscience, Series on Pure and Applied Mathematics, New York, 2000.

[15] J.A. Fax and R.M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Trans. Aut. Contr.*, 49 (9):1465–1476, 2004.

[16] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23:298–305, 1973.

[17] E. Fornasini and M.E. Valcher. Linear copositive Lyapunov functions for continuous-time positive switched systems. *IEEE Trans. Aut. Contr.*, 55 (8):1933–1937, 2010.

[18] E. Fu and T. Markham. On the eigenvalues and diagonal entries of a hermitian matrix. *Lin. Alg. Appl.*, 179:7–10, 1993.

[19] F.R. Gantmacher. *The theory of matrices*. Chelsea Pub. Co., 1960.

[20] F. Harary, J.P. Hayes, and H.-J. Wu. A survey of the theory of hypercube graphs. *Compu. & Math. Appl.*, 15(4):277–289, 1988.

[21] D. Hinrichsen and E. Plischke. Robust stability and transient behaviour of positive linear systems. *Vietnam J. Math.*, 35:429–462, 2007.

[22] Q. Hui and W.M. Haddad. Distributed nonlinear control algorithms for network consensus. *Automatica*, 44 (9):237581, 2008.

[23] F. Knorn, M.J. Corless, and R.N. Shorten. A result on implicit consensus with application to emissions control. In *Proc. 2011 CDC-ECC*, pages 1299 – 1304, Orlando, FL, 2011.

[24] F. Knorn, O. Mason, and R.N. Shorten. On linear co-positive Lyapunov functions for sets of linear positive systems. *Automatica*, 45 (8):1943–1947, August 2009.

[25] J. Lin, A.S. Morse, and B.D.O. Anderson. The multi-agent rendezvous problem. In *Proc. 42nd IEEE CDC*, pages 1508 –1513, Maui, Hawaii, 2003.

[26] O. Mason and R.N. Shorten. On linear copositive Lyapunov functions and the stability of switched positive linear systems. *IEEE Trans. Aut. Contr.*, 52, no. 7:1346 – 1349, 2007.

[27] R. Merris. Laplacian Matrices of Graphs: A Survey. *Lin. Alg. Appl.*, 197-198:143–176, 1994.

[28] H. Minc. *Nonnegative Matrices*. J.Wiley & Sons, New York, 1988.

[29] B. Mohar. The Laplacian spectrum of graphs. *Graph Theory, Combinatorics, and Applications*, 2:871–898, 1991.

[30] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proc. IEEE*, 95, no. 1:215–233, 2007.

[31] S. Pejic. *Algebraic Graph Theory in the Analysis of Frequency Assignment Problems*. PhD thesis, London School of Economics and Political Science, 2008.

[32] A.V. Proskurnikov, A. Matveev, and M. Cao. Opinion dynamics in social networks with hostile camps: Consensus vs. polarization. *IEEE Trans. Aut. Contr.*, 61, (6):1524 – 1536, 2016.

[33] A. Rantzer. Distributed control of positive systems. In *Proc. 2011 IEEE CDC-ECC*, pages 6608 – 6611, Orlando, FL, 2011.

[34] W. Ren. On consensus algorithms for double-integrator dynamics. In *Proc. 2007 IEEE CDC*, pages 2295 - 2300, New Orleans, LA, USA, 2007.

[35] W. Ren and E.M. Atkins. Distributed multi-vehicle coordinated control via local information exchange. *Int. J. Robust Nonlinear Control*, 17, (10-11):1002–1033, 2007.

[36] W. Ren and R.W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Aut. Contr.*, 50 (5):655–661, 2005.

[37] W. Ren, R.W. Beard, and E.M. Atkins. Information consensus in multivehicle cooperative control. *IEEE Contr. Sys. Mag.*, 27 (2):71–82, 2007.

[38] Y. Saad and M.H. Schultz. Topological properties of hypercubes. *IEEE Trans. Compu.*, 37(7):867–872, 1988.

[39] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45, no. 11:2557–2562, 2009.

[40] H.L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. AMS, Mathematical Surveys and Monographs, Providence, RI, vol. 41, 1995.

[41] N.K. Son and D. Hinrichsen. Robust stability of positive continuous time systems. *Numer. Funct. Anal. Optimiz.*, 17 (5 & 6):649–659, 1996.

[42] Y. Su and J. Huang. Stability of a class of linear switching systems with applications to two consensus problems. *IEEE Trans. Aut. Contr.*, 57 (6):1420–1430, June 2012.

[43] J.N. Tsitsiklis. *Problems in Decentralized Decision Making and Computation*. PhD thesis, Department of EECS, MIT, 1984.

[44] M. E. Valcher and P. Misra. On the stabilizability and consensus of positive homogeneous multi-agent dynamical systems. *IEEE Trans. Aut. Contr.*, 59(7):1936–1941, 2013.

[45] M.E. Valcher and I. Zorzan. New results on the solution of the positive consensus problem. In *Proc. 55th IEEE CDC*, pages 5251–5256, Las Vegas, Nevada, 2016.

[46] M.E. Valcher and I. Zorzan. On the consensus problem with positivity constraints. In *Proc. 2016 American Control Conference*, pages 2846–2851, Boston, MA, 2016.

[47] P. Wieland, J.-S. Kim, and F. Allgower. On topology and dynamics of consensus among linear high-order agents. *Int. J. Sys. Sci.*, 42, no. 10:1831–1842, 2011.

[48] P. Wieland, J.-S. Kim, H. Scheu, and F. Allgower. On consensus in multi-agent systems with linear high- order agents. In *Proc. 17th IFAC World Congress*, page 1541–1546, Seoul, Korea, 2008.

[49] W. Xia, M. Cao, and K.H. Johansson. Structural balance and opinion separation in trust-mistrust social networks. *IEEE Trans. Contr. Netw. Sys.*, 3 (1):46–56, 2016.

[50] X. Zhang and X. Liu. Further results on consensus of second-order multi-agent systems with exogenous disturbance. *IEEE Trans. Circ. Sys. I: Regular Papers*, 60, (12):3215–3226, 2013.