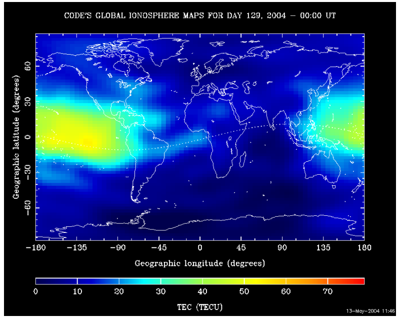
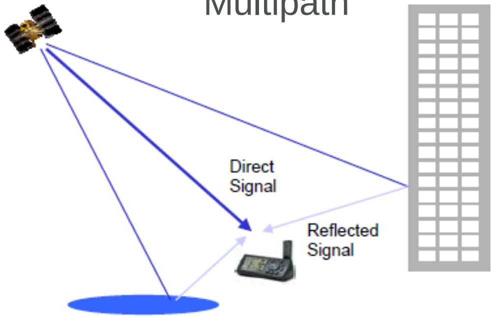
Part 4A – Error Sources and Accuracy

GNSS Error Sources

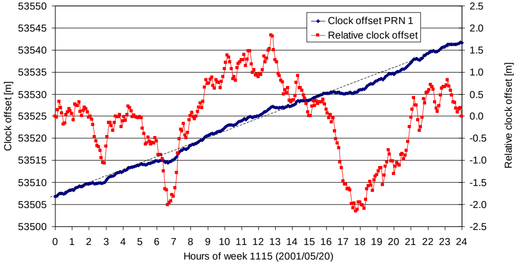
Troposphere and Ionosphere



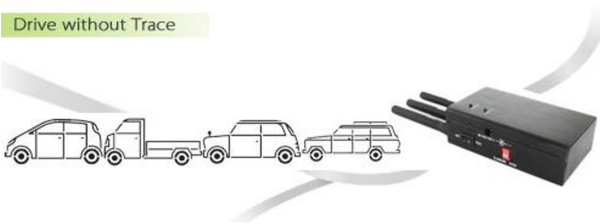
Multipath



Orbit and Clock



Jamming/Spoofing



When is an "Error" an Error?

- Idealized pseudo-range measurements reflect the distance and clock offsets
  $p = \rho + c(dt_{\mathrm{rcv}} - dt^{\mathrm{sat}})$
- Real measurements include numerous contributions, part of which are included in more elaborate measurement models
- An "error" occurs, if a contribution is not (rigorously) considered in the model, e.g.
  - Difference between broadcast ephemerides and real satellite orbit/clock
  - Difference between real and modelled tropospheric delay
  - Higher-order ionospheric delays not compensated by a dual-frequency combination
  - Receiver noise and multipath
- Some parameters can be adjusted (leaving only an estimation error)
  - Receiver clock offset
  - Wet troposphere delay

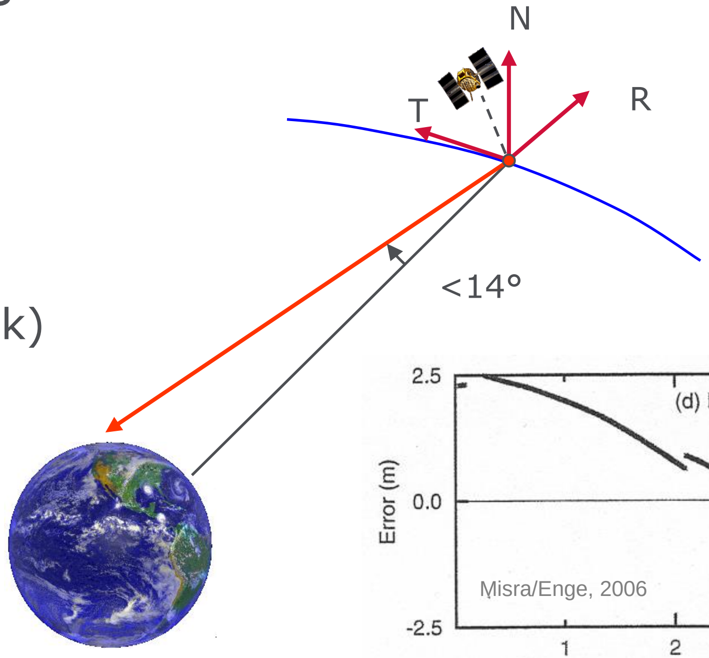
GNSS Measurement Model

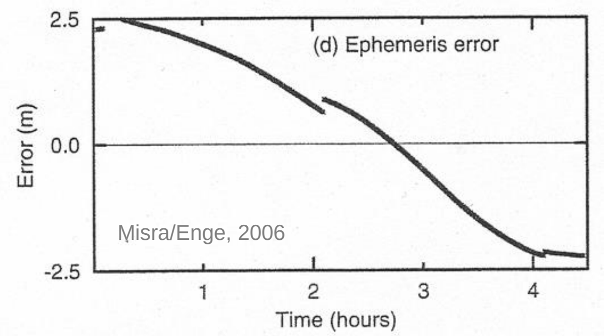
$$\begin{Bmatrix} p \\ \lambda\phi \end{Bmatrix} = |\boldsymbol{r} - \boldsymbol{r}_{\mathrm{GNSS}}| + c \cdot (\delta t - \delta t_{\mathrm{GNSS}}) + \begin{Bmatrix} \\ N\lambda \end{Bmatrix} + T \pm I + b + \varepsilon$$

Measurement = Geometric distance satellite–receiver ($|\boldsymbol{r} - \boldsymbol{r}_{\mathrm{GNSS}}|$)
+ receiver clock offset (δt)
– satellite clock offset (δt_{GNSS})
+ carrier phase ambiguity ($N\lambda$)
+ tropospheric delay (T)
+ ionospheric delay (I)
+ internal delays / biases (b)
+ noise and multipath errors (ε)

GNSS Satellite Position Error

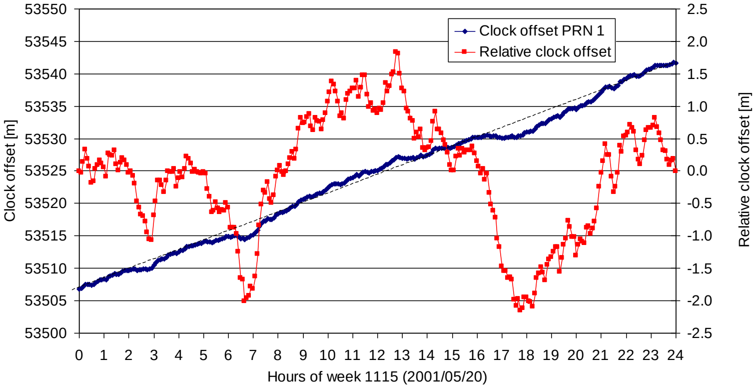
- Contribution to pseudorange error
  - radial up to 100%
  - tangential/normal up to 25%
- Broadcast Ephemerides
  - few m (3D)
  - PR error 0.5 m rms (incl. clock)
- Precise ephemerides
  - 2 cm (3D, IGS rapid)





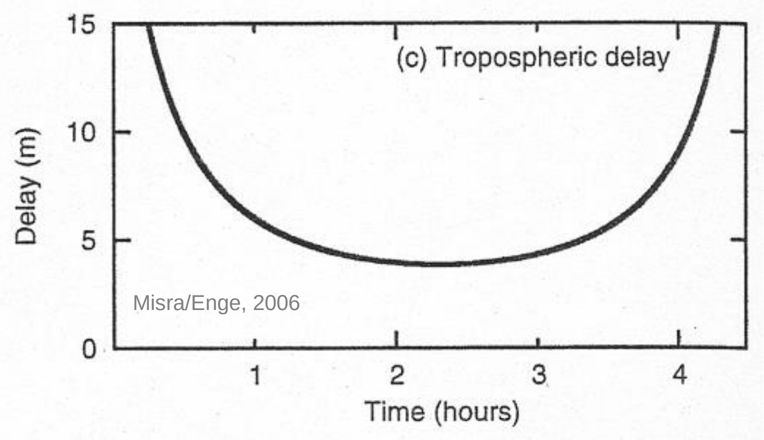
GNSS Clock Offset Error

- Broadcast ephemerides
  - Mean drift and offset are known and predictable
  - Stochastic variation depend on clock type (Cs has larger scatter than Rb/H)
  - Deviation from clock polynomial 0.5-2 ns (0.2-0.6 m PR error)
- Precise ephemerides
  - Clock uncertainty few 10s ps (3-10 mm)
  - 0.1-10 cm interpolation error depending on grid (30 s to 15 min) and clock stability



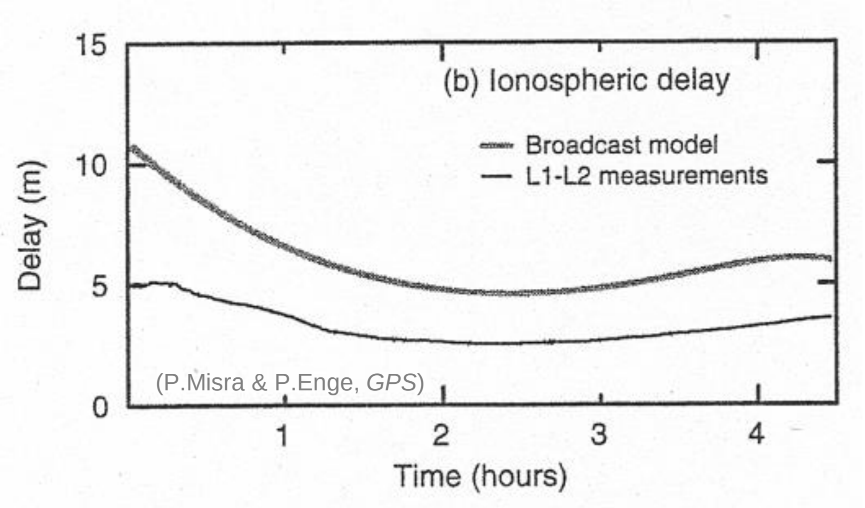
Tropospheric Delay Error

- Vertical delay
  - ~2.4m (at sea level)
  - Uncertainty
    0.3m (dry) + 0.8m (wet)
- Pronounced elevation dependence
  - $m(E) \sim 1/\sin(E)$ (10x at 5°!)
- Error
  - 5-10 cm offline modeling error
  - 0.5-5 m real-time error
    (lack of meteo data)



Ionospheric Delay Error

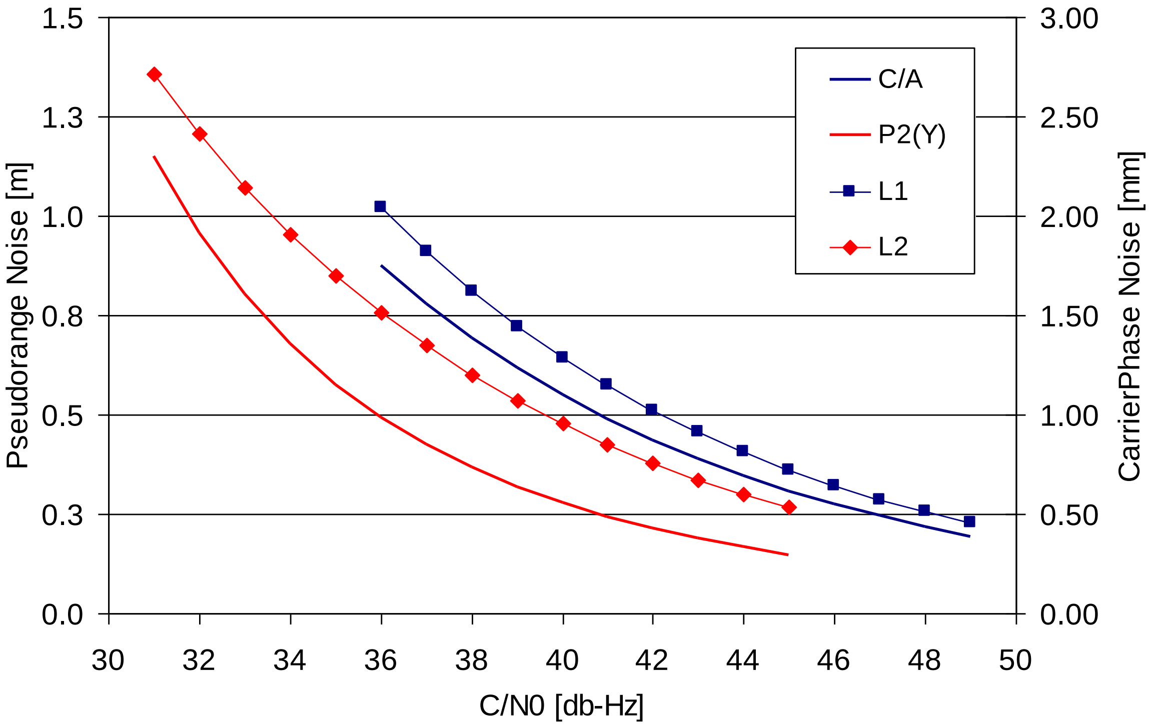
- Error of vertical delay
  - Klobuchar model: up to 50% (1-10s m at L1)
  - Dual-frequency: <1cm
- Weak elevation dependence (e.g. ~3x at 5°!) due to high altitude of electron density maximum



Measurement Noise

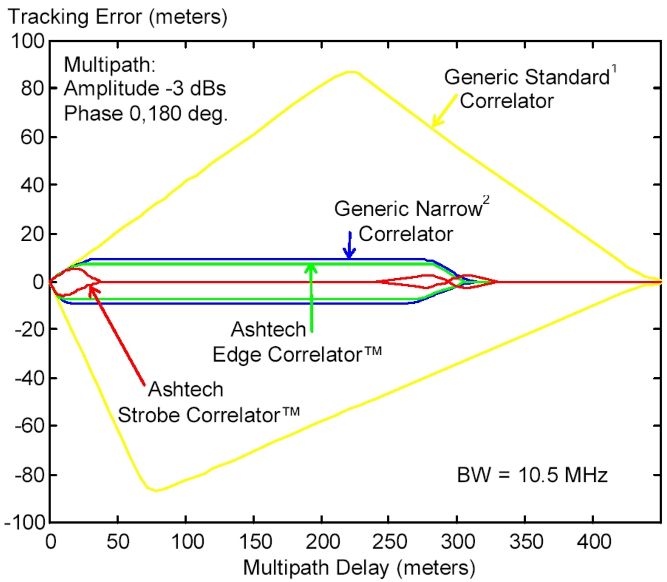
- Depends on signal quality (C/N_0)
  - Antenna
  - Elevation
- Depends on receiver design
  - Tracking loops (tolerable dynamics)
  - Correlation (e.g., narrow correlator, semi-codeless)
  - Smoothing
- Pseudorange
  - C/A-code 0.1–3m (0.1–1% of chip length)
  - Semi-codeless P(Y)-code 0.1–1m (~1% of chip length)
  - Galileo: down to 2 cm (E5 AltBOC signal)
- Carrier phase
  - 0.5–3.0 mm (~1% of wavelength)

Measurement Noise (Example)



Multipath Errors

- Pseudorange errors in case of strong reflections
  - C/A-code 50-100 m
  - C/A-code with narrow correlator 5-10 m
  - P-code 5-10 m
- Typical pseudorange errors
  - 1-5 m
- Phase errors
  - Maximum $\lambda/4$ (5-6 cm)
  - Typical 0.5-1 cm
- Obvious from signal strength variations

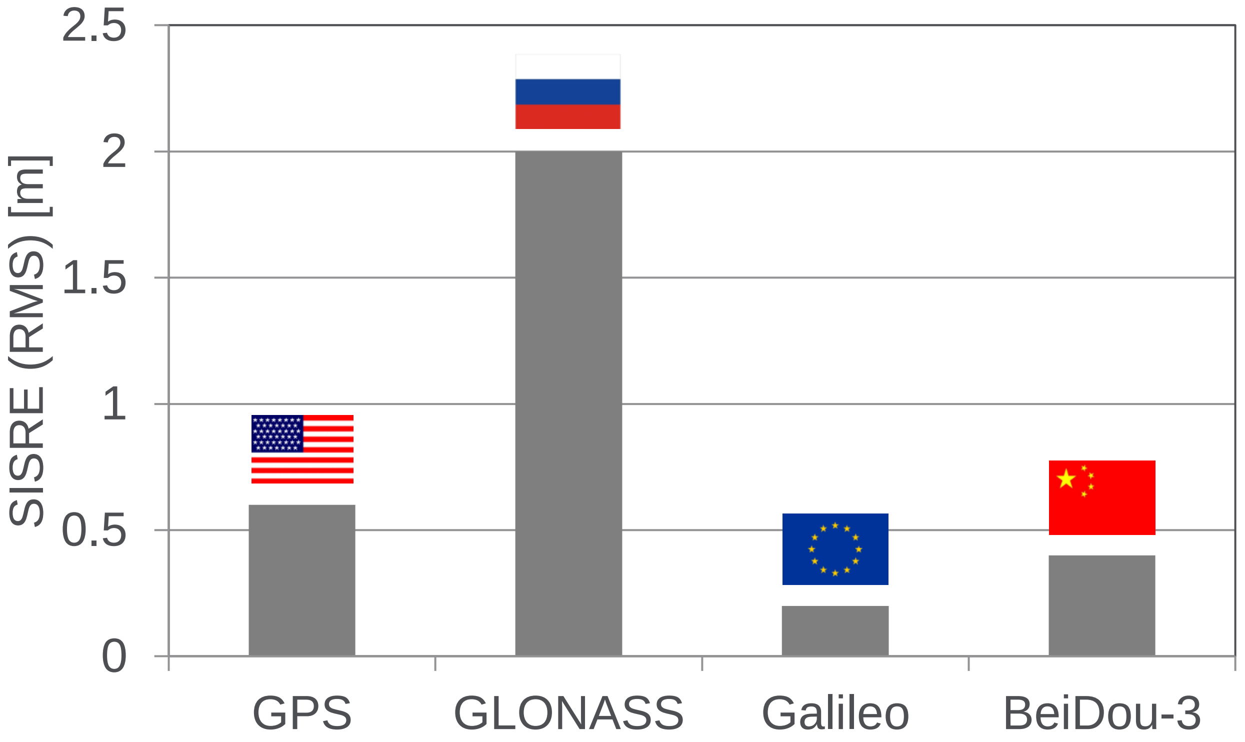


User Equivalent Range Error (UERE)

- Combines contributions of measurement and modelling errors
- Signal in Space Range Error (SISRE)
  - Space/control-segment related errors (satellite clock and ephemeris, biases)
- User Equipment Error (UEE)
  - Noise & Multipath
  - Propagation (modeling of atmospheric delays)
- For statistically independent errors

$$\sigma_{\mathrm{UERE}} = \sqrt{\sigma^2_{\mathrm{SISRE}} + \sigma^2_{\mathrm{UEE}}}$$

Signal-In-Space Range Error



UERE Examples

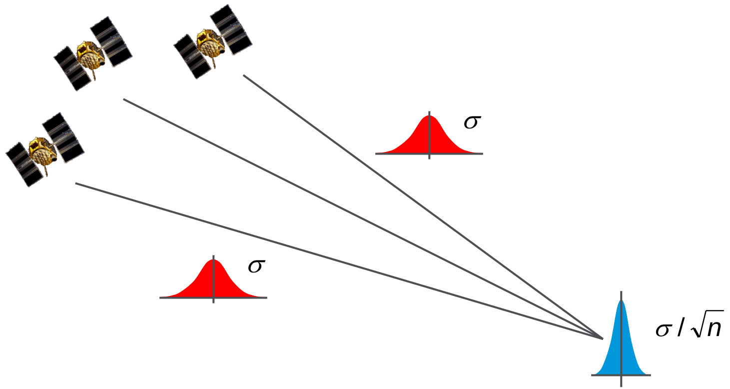
UERE Contribution	Standard Positioning Service	Geodetic Positioning
Satellite clock & ephemerides	0.5 m	0.05 m
Atmospheric delay modeling	1-20 m	0.01 m
Noise and multipath	1-5 m	0.01 m
Total	2-7 m	0.05 m

Just order of magnitude!

Impact of Measurement Errors on Positioning

Number of measurements

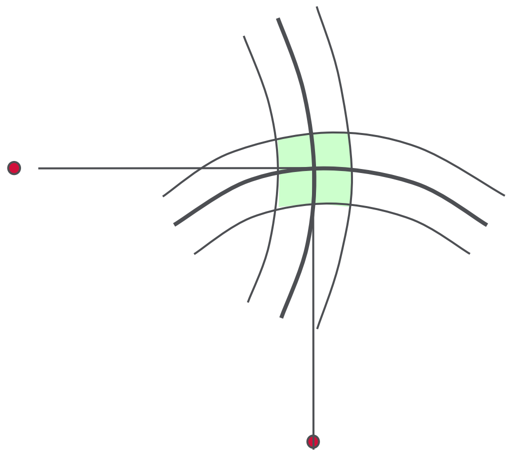
- Statistical errors average
- Systematic errors remain!



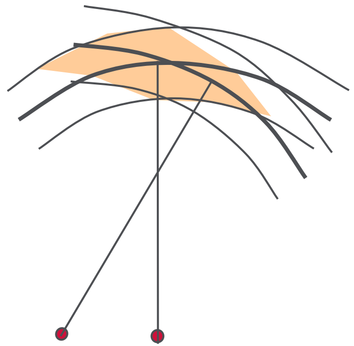
Impact of Measurement Errors on Positioning

Geometry (distribution of line-of-sight directions)

Favorable:
positioning error ~ range-error



Unfavorable:
positioning error can grow largly



Dilution of Precision (DOP)

- Ratio of RMS position and range measurement error
- Depends only on geometry and number of measurements
- Horizontal (HDOP), vertical (VDOP), timing (TDOP)

$$\sigma(\text{pos}) = \text{DOP} \cdot \text{UERE}$$

Statistical position error

Geometry factor

Error of individual range measurement

DOP Dilution of Precision
UERE User Equivalent Range Error

Accuracy of the Least-Squares Solution

Least-squares solution is a linear function of the observations

$$\Delta \boldsymbol{x} = (\boldsymbol{A}^T \boldsymbol{A})^{-1} \boldsymbol{A}^T \Delta \boldsymbol{z}$$

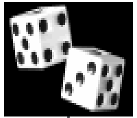
with

$\Delta \boldsymbol{x} = (\Delta \boldsymbol{r}\, ; \Delta\, c\delta t)$ Position & clock correction

$\boldsymbol{A} = \begin{pmatrix} -\mathbf{e}_1^T & +1 \\ \vdots & \vdots \\ -\mathbf{e}_n^T & +1 \end{pmatrix}$ Partial derivatives (design matrix, geometry matrix)

$\Delta \boldsymbol{z} = \boldsymbol{z} - \boldsymbol{g}(\boldsymbol{x})$ Difference measured-modeled observations

Accuracy of the Least-Squares Solution



random measurement errors (uncorrelated)

$\mathrm{E}(\mathbf{x}) = ?$

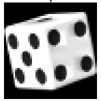
$\mathrm{Cov}(\mathbf{x}) = ?$

$$\Delta\boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}(\boldsymbol{A}^T\Delta\boldsymbol{z})$$

$\mathrm{E}(\mathbf{z}) = \bar{\mathbf{z}}$

$\mathrm{Var}(z_i) = \sigma^2_{\mathrm{UERE}}$

randomly distributed position & clock errors (correlated)



Result

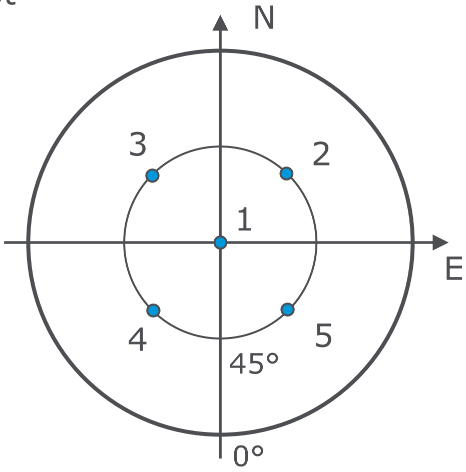
- Least-Squares Solution is unbiased
- $\mathrm{Cov}(\Delta\boldsymbol{x}) = (\boldsymbol{A}^T\boldsymbol{A})^{-1} \cdot \sigma^2_{\mathrm{UERE}}$

Example of DOP Computation

Satellite #1: zenith (E=90°)
Satellites #2-5: E=45° and A=45°, 315°,225°, 135°

$$A = \begin{pmatrix} 0.0 & 0.0 & -1.00 & +1.0 \\ -0.5 & -0.5 & -0.71 & +1.0 \\ +0.5 & -0.5 & -0.71 & +1.0 \\ +0.5 & +0.5 & -0.71 & +1.0 \\ -0.5 & +0.5 & -0.71 & +1.0 \end{pmatrix}$$

„Skyplot"



$$(A^T A)^{-1} = \begin{pmatrix} 1.0 & 0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 14.6 & 11.2 \\ 0.0 & 0.0 & 11.2 & 8.7 \end{pmatrix}$$

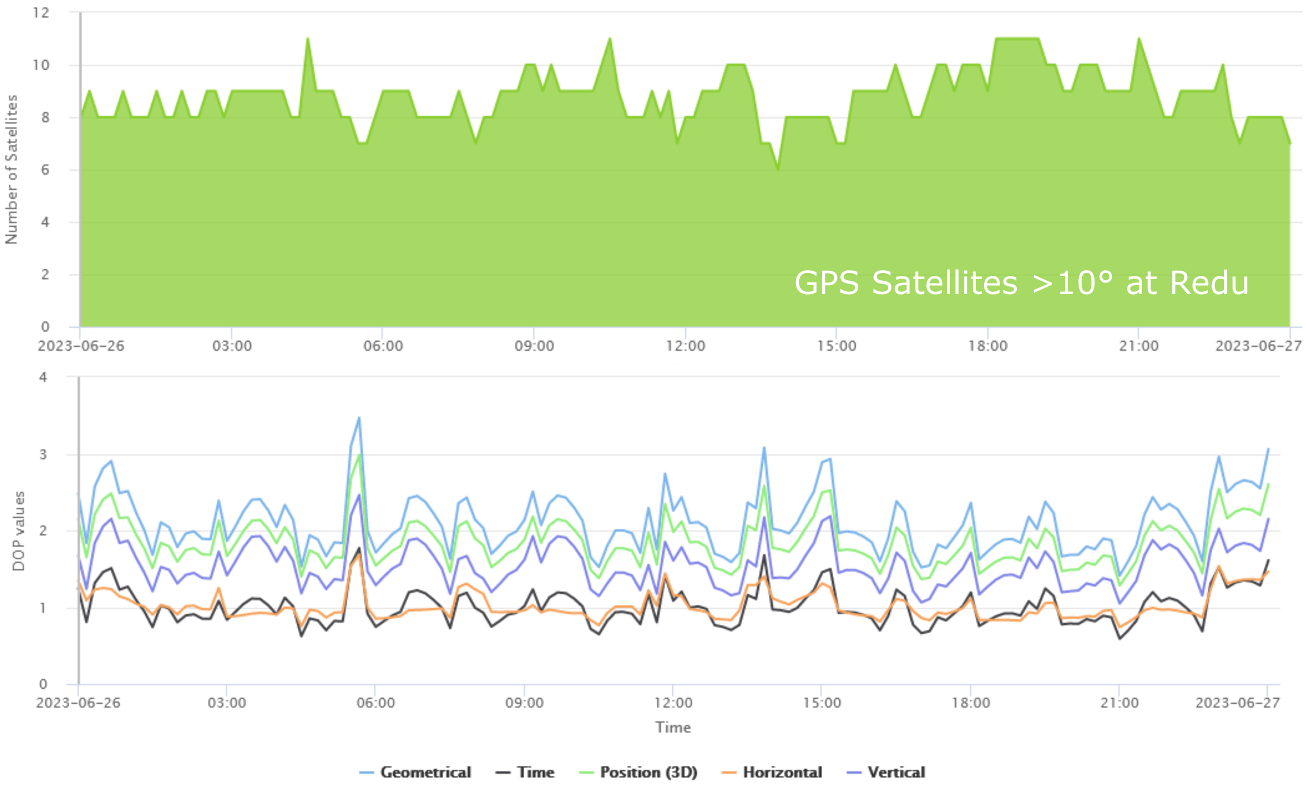
Why so bad?

EDOP = NDOP = 1.0
VDOP = 3.8, TDOP = 3.0
PDOP = 4.1

Notes

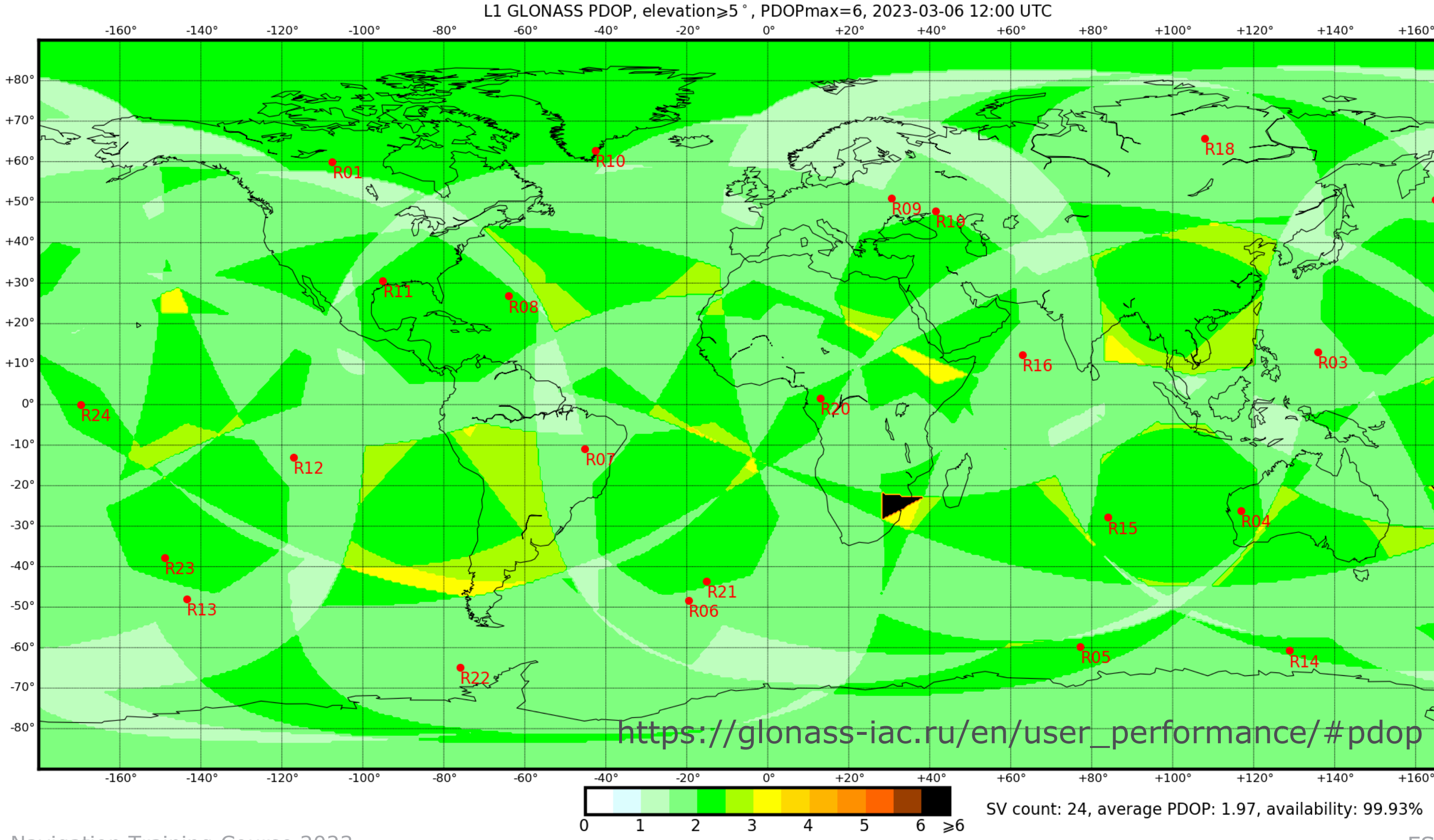
- DOP computation depends only on line-of-sight directions
- Measurement and modeling errors (ephemerides, clock offsets, atmosphere, multipath, noise) have *no* impact on the DOP and need not be known for DOP predictions
- Simple models of the GNSS orbits (e.g., almanac) are sufficient for DOP analyses

Satellite Visibility and DOP

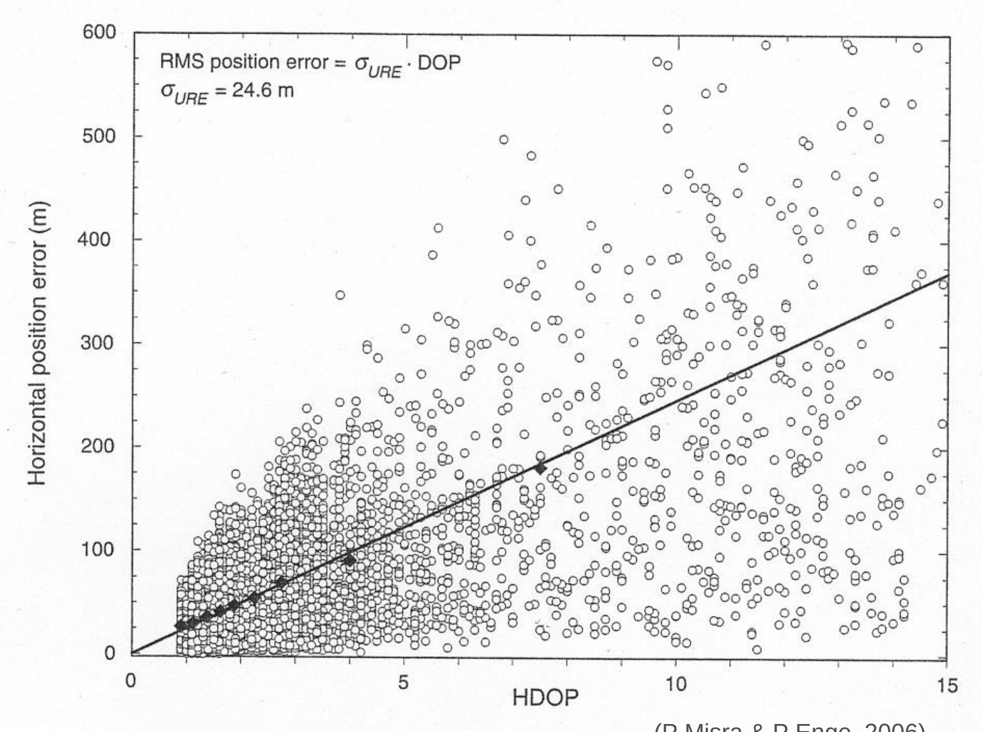


Trimble
(www.gnssplanning.com)

DOP Map


L1 GLONASS PDOP, elevation⩾5 ° , PDOPmax=6, 2023-03-06 12:00 UTC
https://glonass-iac.ru/en/user_performance/#pdop
R01
R02
R03
R04
R05
R06
R07
R08
R09
R10
R11
R12
R13
R14
R15
R16
R17
R18
R19
R20
R21
R22
R23
R24
0 1 2 3 4 5 6 ⩾6
SV count: 24, average PDOP: 1.97, availability: 99.93%

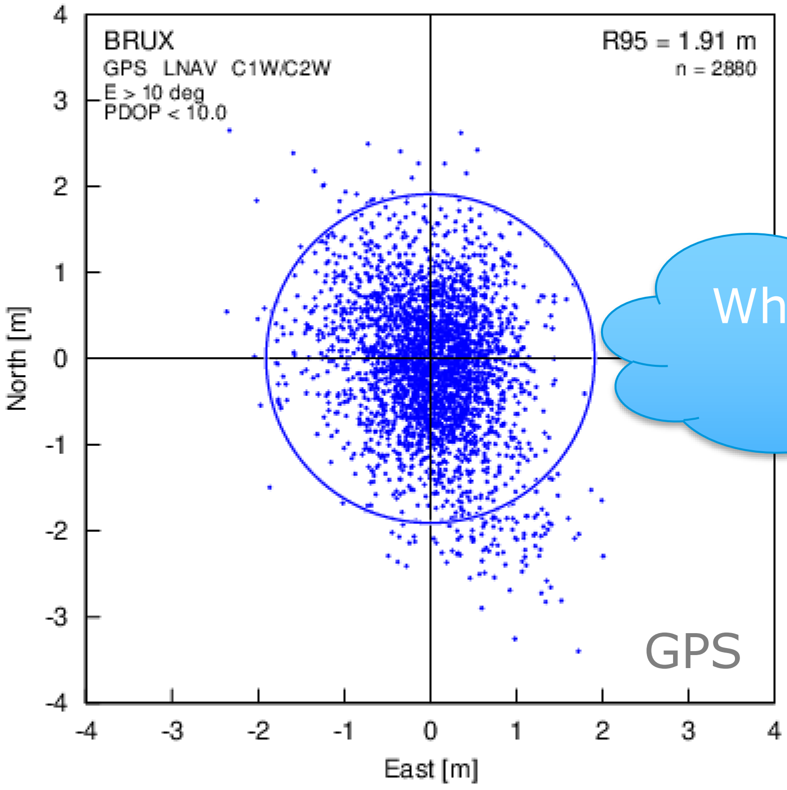
DOP, UERE and Individual Errors

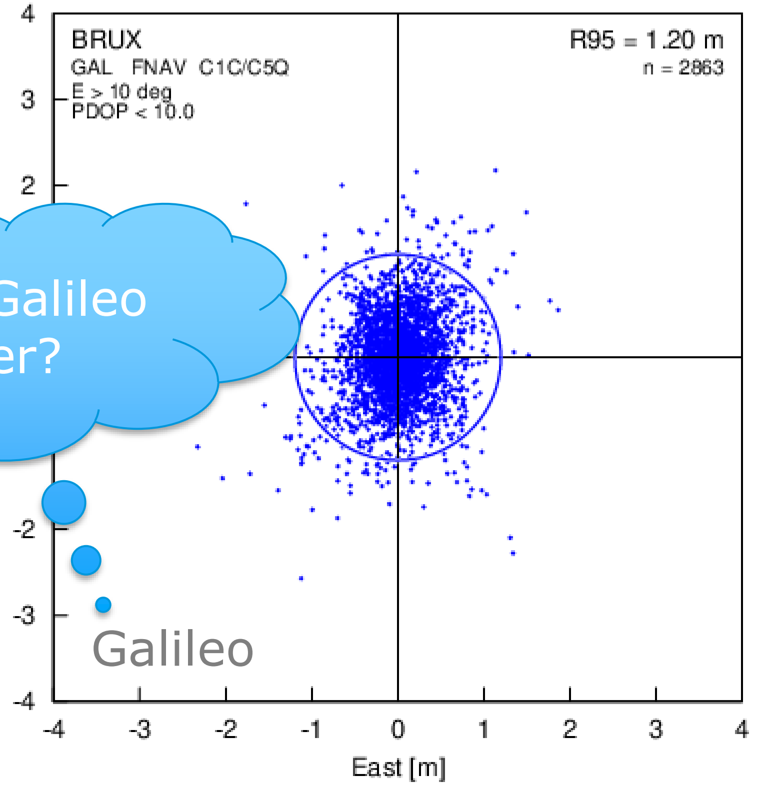


(P.Misra & P.Enge, 2006)

- DOP and UERE are statistical values
- They don’t tell us about individual errors

Single Point Positioning Results (Dual-Frequ.)





Why is Galileo better?

Take Home Messages

- Measurement and modeling errors
  - GNSS ephemerides and clocks (well known in postprocessing)
  - Troposphere (well modeled) und ionosphere (eliminated with 2-frequ.)
  - Multipath and noise (depends on receiver quality)
  - UERE ~2-7m
- Dilution of Precision
  - Diagonal elements of (normalized) covariance matrix
  - Depends on distribution and number of observed satellites
    - Typical PDOP~1-3 for full constellation
    - High PDOPs (10++) may occur in case of unfavorable visibility („*urban canyon*")
  - VDOP ~2·HDOP
- Resulting single-point positioning accuracy σ~2-15m

Pseudorange Equation

$$\rho_j = \sqrt{\left(x_j - x_u\right)^2 + \left(y_j - y_u\right)^2 + \left(z_j - z_u\right)^2} + ct_u$$
$$= f(x_u, y_u, z_u, t_u)$$

Pseudorange Equation

$$\rho_j = \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + ct_u$$
$$= f(x_u, y_u, z_u, t_u)$$

$$x_u = \hat{x}_u + \Delta x_u$$
$$y_u = \hat{y}_u + \Delta y_u$$
$$z_u = \hat{z}_u + \Delta z_u$$
$$t_u = \hat{t}_u + \Delta t_u$$

$$f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, \hat{t}_u + \Delta t_u) = f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)$$
$$+ \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{x}_u} \Delta x_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{y}_u} \Delta y_u$$
$$+ \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{z}_u} \Delta z_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{t}_u} \Delta t_u + \ldots$$

Pseudorange Partial Derivatives

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{x}_u} = -\frac{x_j - \hat{x}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{y}_u} = -\frac{y_j - \hat{y}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{z}_u} = -\frac{z_j - \hat{z}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{t}_u} = c$$

Linearized Pseudorange Equation

$$\hat{\rho}_j - \rho_j = \frac{x_j - \hat{x}_u}{\hat{r}_j}\Delta x_u + \frac{y_j - \hat{y}_u}{\hat{r}_j}\Delta y_u + \frac{z_j - \hat{z}_u}{\hat{r}_j}\Delta z_u - ct_u$$

Linearized Pseudorange Equation

$$\hat{\rho}_j - \rho_j = \frac{x_j - \hat{x}_u}{\hat{r}_j}\Delta x_u + \frac{y_j - \hat{y}_u}{\hat{r}_j}\Delta y_u + \frac{z_j - \hat{z}_u}{\hat{r}_j}\Delta z_u - ct_u$$

$$\Delta\rho_j = \hat{\rho}_j - \rho_j$$

$$a_{xj} = \frac{x_j - \hat{x}_u}{\hat{r}_j}$$

$$a_{yj} = \frac{y_j - \hat{y}_u}{\hat{r}_j}$$

$$a_{zj} = \frac{z_j - \hat{z}_u}{\hat{r}_j}$$

$$\Delta\rho_j = a_{xj}\Delta x_u + a_{yj}\Delta y_u + a_{zj}\Delta z_u - ct_u$$

Pseudorange Matrix Equation

$$\Delta\rho_1 = a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - c\Delta t_u$$

$$\Delta\rho_2 = a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - c\Delta t_u$$

$$\vdots$$

$$\Delta\rho_n = a_{xn}\Delta x_u + a_{yn}\Delta y_u + a_{zn}\Delta z_u - c\Delta t_u$$

Pseudorange Matrix Equation

$$\Delta\rho_1 = a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - c\Delta t_u$$

$$\Delta\rho_2 = a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - c\Delta t_u$$

$$\vdots$$

$$\Delta\rho_n = a_{xn}\Delta x_u + a_{yn}\Delta y_u + a_{zn}\Delta z_u - c\Delta t_u$$

$$\Delta\boldsymbol{\rho} = \boldsymbol{H}\,\Delta\boldsymbol{x}$$

$$\Delta\boldsymbol{\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \vdots \\ \Delta\rho_n \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{xn} & a_{xn} & 1 \end{bmatrix} \qquad \Delta\boldsymbol{x} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ -\mathrm{c}\Delta t_u \end{bmatrix}$$

Least Squares Theory

$$\boldsymbol{y} = \boldsymbol{H}\,\boldsymbol{x} + \boldsymbol{n}$$

- $\boldsymbol{x}$ vector of M unknown parameters
- $\boldsymbol{y}$ vector of N noisy measurements
- $\boldsymbol{n}$ vector of the measurement errors

The maximum likelihood estimate of $\boldsymbol{x}$, denoted by $\hat{\boldsymbol{x}}$, is:

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{x}}{\operatorname{argmax}}\, p(\boldsymbol{y}|\boldsymbol{x})$$

Least Squares Theory

$$\boldsymbol{y} = \boldsymbol{H}\,\boldsymbol{x} + \boldsymbol{n}$$

- If n_i for $i = 1, \dots n$ are independent with distribution $N(0, \sigma^2)$ then:

$$\begin{aligned}\hat{\boldsymbol{x}} &= \underset{\boldsymbol{x}}{\operatorname{argmax}} \frac{1}{(2\pi\sigma)^{N/2}} e^{-\frac{1}{2\sigma^2}\|\boldsymbol{y}-\boldsymbol{H}\boldsymbol{x}\|^2} \\ &= \underset{\boldsymbol{x}}{\operatorname{argmin}} \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|^2\end{aligned}$$

Least Squares Solution

$$\boldsymbol{y} = \boldsymbol{H}\,\boldsymbol{x} + \boldsymbol{n}$$

- Solution can be found by setting to 0 the derivative of $||\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}||^2$ with respect to $\hat{\boldsymbol{x}}$:

$$\frac{d}{d\hat{\boldsymbol{x}}}||\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}||^2 = 2\boldsymbol{H}^T\boldsymbol{H}\hat{\boldsymbol{x}} - 2\boldsymbol{H}^T\boldsymbol{y} = 0$$

$$\hat{\boldsymbol{x}} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{y}$$

Weighted Least Squares

$$\boldsymbol{y} = \boldsymbol{H}\,\boldsymbol{x} + \boldsymbol{n}$$

- If n_i for $i = 1, \dots n$ are Gaussian distributed with zero mean but not necessarily with the same variance or independent, then:

$$\hat{\boldsymbol{x}} = \operatorname*{argmax}_{\boldsymbol{x}} \frac{1}{(2\pi)^{N/2}|\boldsymbol{R_n}|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{Hx})^T \boldsymbol{R_n}^{-1}(\boldsymbol{y}-\boldsymbol{Hx})}$$

$$= \operatorname*{argmin}_{\boldsymbol{x}} (\boldsymbol{y} - \boldsymbol{Hx})^T \boldsymbol{R_n}^{-1} (\boldsymbol{y} - \boldsymbol{Hx})$$

Weighted Least Squares

$$\boldsymbol{y} = \boldsymbol{H}\,\boldsymbol{x} + \boldsymbol{n}$$

- If n_i for $i = 1, \dots n$ are Gaussian distributed with zero mean but not necessarily with the same variance or independent, then:

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{x}}{\operatorname{argmax}} \frac{1}{(2\pi)^{N/2}|\boldsymbol{R_n}|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{Hx})^T \boldsymbol{R_n^{-1}}(\boldsymbol{y}-\boldsymbol{Hx})}$$

$$= \underset{\boldsymbol{x}}{\operatorname{argmin}} (\boldsymbol{y} - \boldsymbol{Hx})^T \boldsymbol{R_n^{-1}} (\boldsymbol{y} - \boldsymbol{Hx})$$

$$\Rightarrow \quad \hat{\boldsymbol{x}} = (\boldsymbol{H}^T \boldsymbol{R_n^{-1}} \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{R_n^{-1}} \boldsymbol{y}$$

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Position Solution

$$\Delta\boldsymbol{\rho} = \boldsymbol{H}\,\Delta\boldsymbol{x} + \boldsymbol{n}$$

$$\Delta\boldsymbol{x} = \left(\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\boldsymbol{H}^T\Delta\boldsymbol{\rho}$$

$$\Delta\boldsymbol{\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \vdots \\ \Delta\rho_n \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{xn} & a_{xn} & 1 \end{bmatrix} \qquad \Delta\boldsymbol{x} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ -\mathrm{c}\Delta t_u \end{bmatrix}$$

Iterative Least Squares

- Iterative approach:
  1. Start with an initial estimate $\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u$
  2. Compute the deviations $\Delta\hat{x}_u, \Delta\hat{y}_u, \Delta\hat{z}_u, \Delta\hat{t}_u$ through Least Square
  3. Get the new estimate
  4. Restart from 1 until convergence

Error Propagation

$$\Delta \boldsymbol{x} = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \boldsymbol{H}^T \Delta \boldsymbol{\rho}$$

- Pseudorange errors $d\boldsymbol{\rho}$ leads to a solution error $d\boldsymbol{x}$ of:

$$d\boldsymbol{x} = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \boldsymbol{H}^T d\boldsymbol{\rho}$$

Error Statistics

$$d\boldsymbol{x} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T d\boldsymbol{\rho}$$

$$\begin{aligned}\mathrm{cov}(d\boldsymbol{x}) &= \boldsymbol{E}[d\boldsymbol{x}\, d\boldsymbol{x}^T] \\ &= (\boldsymbol{H}^T\boldsymbol{H})^{-1}\, \boldsymbol{H}^T \mathrm{cov}(d\boldsymbol{\rho})\, \boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{H})^{-1}\end{aligned}$$

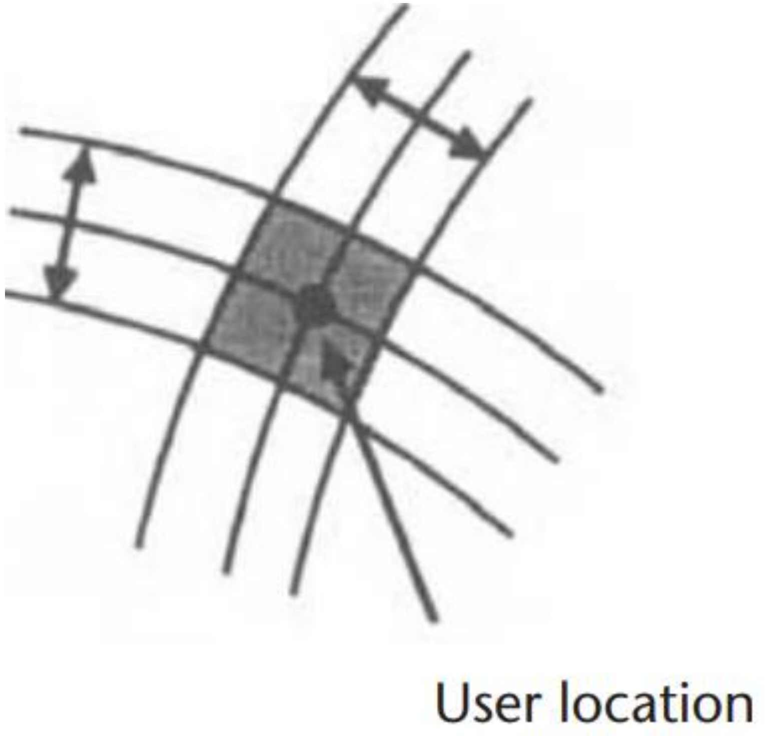
- If $d\rho_i$ for $i = 1, \dots n$ are i.i.d. with std σ_{UERE} then $\mathrm{cov}(d\boldsymbol{\rho}) = \sigma_{UERE}^2\, \boldsymbol{I}_\mathrm{n}$ and:

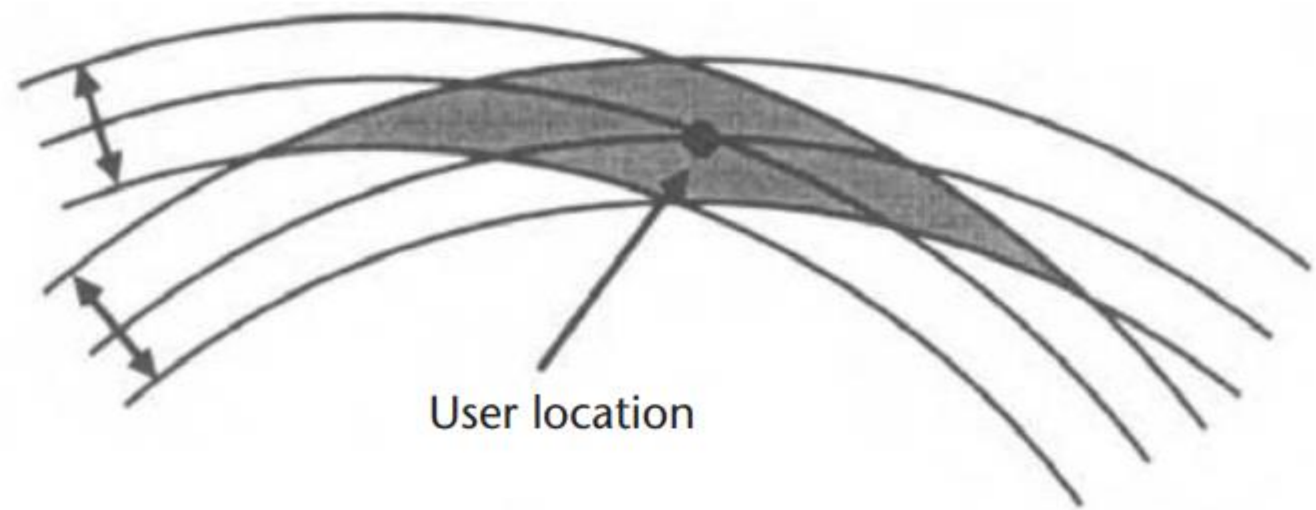
$$\mathrm{cov}(d\boldsymbol{x}) = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\sigma_{UERE}^2$$

Dilution Of Precision (DOP)

- The components of the matrix $(\boldsymbol{H}^T\boldsymbol{H})^{-1}$ quantify how pseudorange errors translate into components of the covariance of $d\boldsymbol{x}$

$$\mathrm{cov}(d\boldsymbol{x}) = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\sigma_{UERE}^2$$





Dilution Of Precision (DOP)

- The components of the matrix $(\boldsymbol{H}^T\boldsymbol{H})^{-1}$ quantify how pseudorange errors translate into components of the covariance of $\boldsymbol{dx}$

$$(\mathrm{H}^T\mathrm{H})^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix}$$

$$\mathrm{GDOP} = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}} \Rightarrow \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + \sigma_{ct_b}^2} = \mathrm{GDOP} \times \sigma_{UERE}$$

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$$\mathrm{PDOP} = \sqrt{D_{11} + D_{22} + D_{33}} \qquad \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2} = \mathrm{PDOP} \times \sigma_{UERE}$$

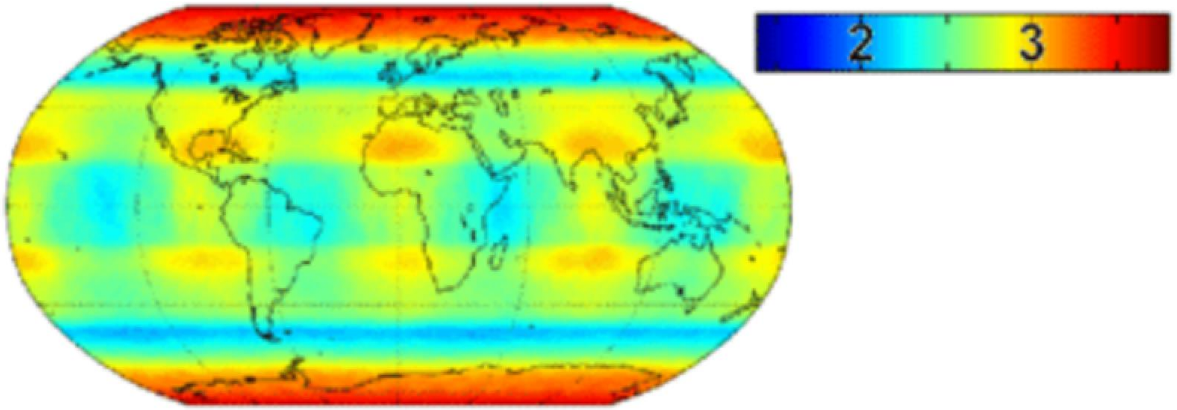
$$\mathrm{HDOP} = \sqrt{D_{11} + D_{22}} \qquad \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2} = \mathrm{HDOP} \times \sigma_{UERE}$$

$$\mathrm{VDOP} = \sqrt{D_{33}} \qquad \sigma_{z_u} = \mathrm{VDOP} \times \sigma_{UERE}$$

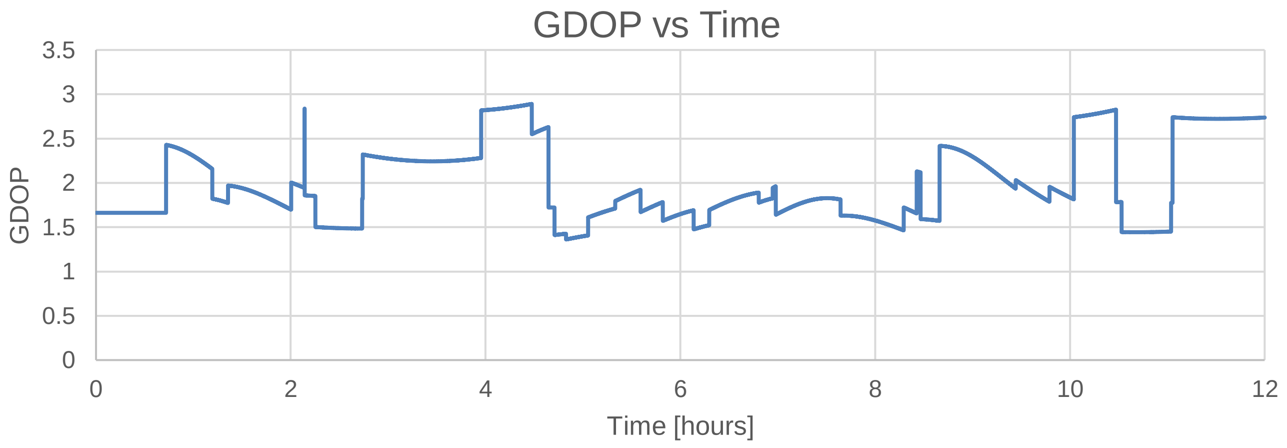
$$\mathrm{TDOP} = \sqrt{D_{44}}/c \qquad \sigma_{t_b} = \mathrm{TDOP} \times \sigma_{UERE}$$

Galileo DOP Values

- Average PDOP vs. position for Galileo with masking angle of 5°



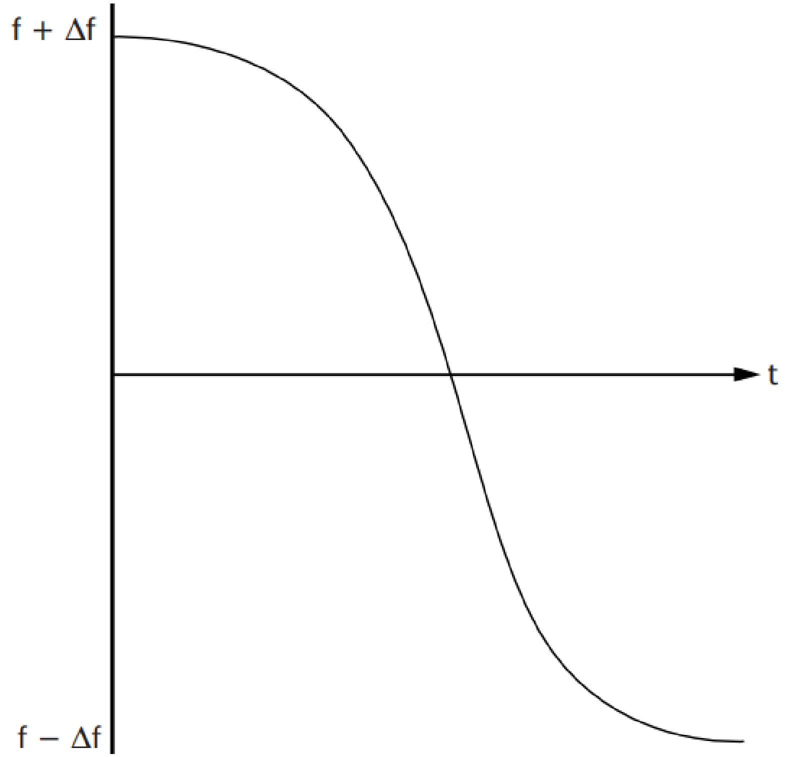
- GDOP vs. time for Galileo with masking angle of 5°



Doppler Effect

$$f_R = f_T\left[1 - \frac{\mathbf{v}_r \cdot \boldsymbol{a}_j}{c}\right]$$

- f_R is the received frequency
- f_T is the transmitted frequency
- $\mathbf{v}_r$ is the satellite to user relative velocity
- $\boldsymbol{a}$ user to the satellite unit vector
- c is the speed of light



Max Δf ≈ 4 KHz for static user and GPS L1

Doppler Measurements

$$f_{R_j} = f_{T_j}\left[1 - \frac{(\mathbf{v}_j - \dot{\boldsymbol{u}}) \cdot \boldsymbol{a}_j}{c}\right]$$

Doppler Measurements

$$f_{R_j} = f_{T_j}\left[1 - \frac{\left(\mathbf{v}_j - \dot{\boldsymbol{u}}\right) \cdot \boldsymbol{a}_j}{c}\right]$$

$$f_{R_j} = f_j(1 + \dot{t}_u)$$

Doppler Measurements

$$\frac{c\left(f_j - f_{Tj}\right)}{f_{Tj}} + v_{xj}a_{xj} + v_{yj}a_{yj} + v_{zj}a_{zj} = \dot{x}_u a_{xj} + \dot{y}_u a_{yj} + \dot{z}_u a_{zj} - \frac{cf_j \dot{t}_u}{f_{Tj}}$$

- $\mathbf{v}_j = \left[v_{xj}, v_{yj}, v_{zj}\right]$
- $\boldsymbol{a}_{\boldsymbol{j}} = \left[a_{xj}, a_{yj}, a_{zj}\right]$
- $\dot{\boldsymbol{u}} = [\dot{x}_u, \dot{y}_u, \dot{z}_u]$

Doppler Measurements

$$d_j = \dot{x}_u a_{xj} + \dot{y}_u a_{yj} + \dot{z}_u a_{zj} - c\dot{t}_u$$

- $\mathbf{v}_j = [v_{xj}, v_{yj}, v_{zj}]$
- $\boldsymbol{a}_j = [a_{xj}, a_{yj}, a_{zj}]$
- $\dot{\boldsymbol{u}} = [\dot{x}_u, \dot{y}_u, \dot{z}_u]$

$$d_j = \frac{c(f_j - f_{Tj})}{f_{Tj}} + v_{xj} a_{xj} + v_{yj} a_{yj} + v_{zj} a_{zj}$$

$$\frac{f_j}{f_{T_j}} \approx 1$$

Velocity Solution

$$\boldsymbol{d} = \boldsymbol{H}\,\boldsymbol{g} + \boldsymbol{n}$$

$$\boldsymbol{g} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{d}$$

$$\boldsymbol{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{xn} & a_{xn} & 1 \end{bmatrix} \qquad \boldsymbol{g} = \begin{bmatrix} \dot{x}_u \\ \dot{x}_u \\ \dot{x}_u \\ -c\dot{t}_u \end{bmatrix}$$

Error Statistics and DOP

- If d_i for $i = 1, \ldots n$ are i.i.d. with std σ_d then:

$$\mathrm{cov}(\boldsymbol{g}) = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \sigma_d^2$$

$$(\mathrm{H}^T \mathrm{H})^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix}$$

$$\mathrm{GDOP} = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

$$\mathrm{PDOP} = \sqrt{D_{11} + D_{22} + D_{33}}$$

$$\mathrm{HDOP} = \sqrt{D_{11} + D_{22}}$$

$$\mathrm{VDOP} = \sqrt{D_{33}}$$

$$\mathrm{TDOP} = \sqrt{D_{44}} / c$$

Pseudorange Measurement

$$\rho = r + c(dt_u - dt_s) + T + \alpha_f STEC + K_{P,u} - K_{P,s} + M + \epsilon_u$$

- r is the geometric range between the satellite and receiver antenna phase centres at emission and reception time, respectively
- dt_u and dt_s are the receiver and satellite clock offsets from the GNSS time scale, including the relativistic satellite clock correction
- T is the tropospheric delay, which is non-dispersive
- $\alpha_f STEC$ is a frequency-dependent ionospheric delay term, where α_f is the conversion factor between the integrated electron density along the ray path $STEC$, and the signal delay at frequency f
- $K_{P,u}$ and $K_{P,s}$ are the receiver and satellite instrumental delays
- M represents the effect of multipath
- ϵ_u is the receiver estimation noise

Pseudorange Measurement