Tutorial

Integer Programming for Constraint Programmers

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Mathematics for key technologies

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ZIB – Zuse Institute Berlin

- Non-university research institute of the state of Berlin (Germany)
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    - Numerical analysis and modeling
    - Visualization and data analysis
  - Discrete Mathematics
    - **Optimization**
    - Scientific Information
  - Computer Science
    - Parallel and Distributed Systems
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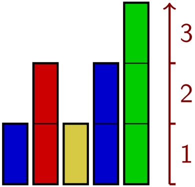
May 28 - June 1, 2012
9th International Conference on Integration of Artificial Intelligence and Operations Research Techniques in Constraint Programming for Combinatorial Optimization Problems

Integer Programming for Constraint Programmers

1. Introduction
2. Linear programming
3. Integer (linear) programming
4. Summary
5. Discussion

Definition

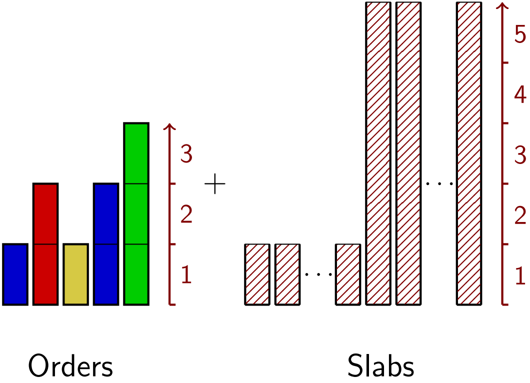
The steel mill slab problem consists of assigning colored, sized orders to slabs of certain different capacities such that the total loss is minimized and at most two different colors are present in each slab.



Orders

Definition

The steel mill slab problem consists of assigning colored, sized orders to slabs of certain different capacities such that the total loss is minimized and at most two different colors are present in each slab.



Definition

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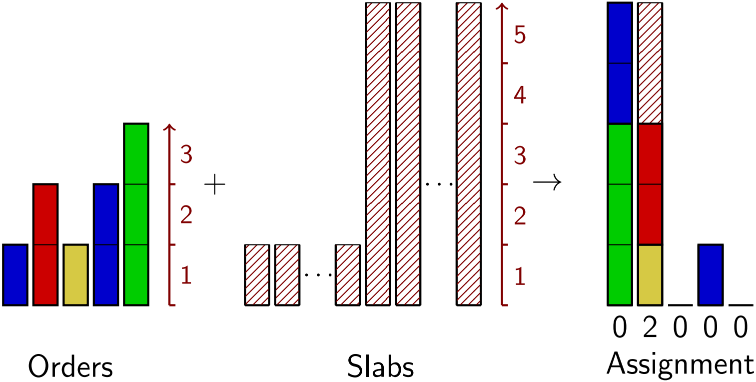

▷ Problem number 38 of the CSPLib (`http://www.csplib.org/`)

(A) Constraint programming formulation

Given

- ▷ $\mathcal{K}$ set of possible capacities for the slabs
- ▷ $\mathcal{C}$ set of colors
- ▷ $\mathcal{O}$ set of orders, $|\mathcal{O}| = n$
  - s_i size of order i
  - c_i color of order i

Binary variables

- ▷ $y_{ij} = 1$ if order i is assigned to slab j
- ▷ $z_{cj} = 1$ if color c is used in slab j

Observation

- ▷ We need at most n slabs
- ▷ Let $\mathcal{S}$ be the set of slabs

- Array storing the leftover depending on the load

$$\mathcal{L}[i] = \text{argmin}\{k - i \mid k \in \mathcal{K} \text{ and } k \geq i\}$$

  for $i = 0, \ldots, \mathcal{K}_{\max}$
- $\mathcal{K}_{\max} := \max\{k \mid k \in \mathcal{K}\}$

Leftover array

- Array storing the leftover depending on the load

$$\mathcal{L}[i] = \text{argmin}\{k - i \mid k \in \mathcal{K} \text{ and } k \geq i\}$$

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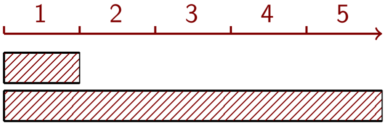
Example



$$\mathcal{L} = \boxed{0}\,\boxed{}\,\boxed{}\,\boxed{}\,\boxed{}\,\boxed{}$$

Leftover array

- Array storing the leftover depending on the load

$$\mathcal{L}[i] = \operatorname{argmin}\{k - i \mid k \in \mathcal{K} \text{ and } k \geq i\}$$

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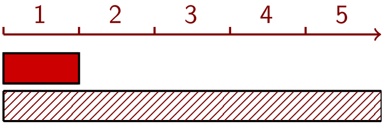


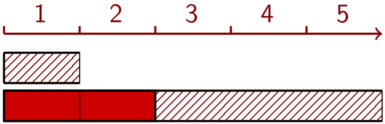
$$\mathcal{L} = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & & & & \\ \hline \end{array}$$

- Array storing the leftover depending on the load

$$\mathcal{L}[i] = \text{argmin}\{k - i \mid k \in \mathcal{K} \text{ and } k \geq i\}$$

  for $i = 0, \ldots, \mathcal{K}_{\max}$
- $\mathcal{K}_{\max} := \max\{k \mid k \in \mathcal{K}\}$

Example



$\mathcal{L} =$

0	0	3			

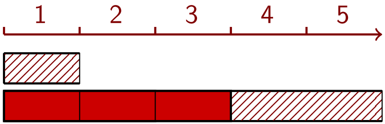
Leftover array

- Array storing the leftover depending on the load

$$\mathcal{L}[i] = \text{argmin}\{k - i \mid k \in \mathcal{K} \text{ and } k \geq i\}$$

  for $i = 0, \ldots, \mathcal{K}_{\max}$
- $\mathcal{K}_{\max} := \max\{k \mid k \in \mathcal{K}\}$

Example



$\mathcal{L} =$

0	0	3	2		

Leftover array

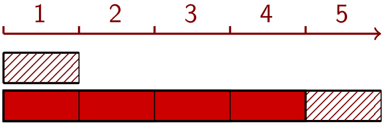
- Array storing the leftover depending on the load

$$\mathcal{L}[i] = \text{argmin}\{k - i \mid k \in \mathcal{K} \text{ and } k \geq i\}$$

  for $i = 0, \ldots, \mathcal{K}_{\max}$

- $\mathcal{K}_{\max} := \max\{k \mid k \in \mathcal{K}\}$

Example



$$\mathcal{L} = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & 3 & 2 & 1 & \\ \hline\end{array}$$

Leftover array

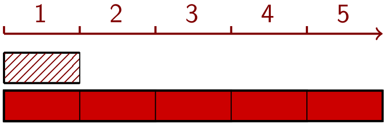
- Array storing the leftover depending on the load

$$\mathcal{L}[i] = \operatorname{argmin}\{k - i \mid k \in \mathcal{K} \text{ and } k \geq i\}$$

  for $i = 0, \ldots, \mathcal{K}_{\max}$

- $\mathcal{K}_{\max} := \max\{k \mid k \in \mathcal{K}\}$

Example



$$\mathcal{L} = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 3 & 2 & 1 & 0 \\ \hline \end{array}$$

(A) Constraint programming formulation

$$
\begin{array}{lll}
\min & \displaystyle\sum_{j\in\mathcal{S}} \mathcal{L}\left[\sum_{i\in\mathcal{O}} s_i\, y_{ij}\right] & \\
\text{subject to} & \displaystyle\sum_{j\in\mathcal{S}} y_{ij} = 1 & \forall i \in \mathcal{O} \\
& \displaystyle\sum_{i\in\mathcal{O}} s_i\, y_{ij} \leq \mathcal{K}_{\max} & \forall j \in \mathcal{S} \\
& y_{ij} \leq z_{c_i j} & \forall i \in \mathcal{O}\ \forall j \in \mathcal{S} \\
& \displaystyle\sum_{c\in\mathcal{C}} z_{cj} \leq 2 & \forall j \in \mathcal{S} \\
& y_{ij}, z_{cj} \in \{0,1\} & \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S}
\end{array}
$$

(A) Constraint programming formulation

$$
\begin{array}{llll}
\min & \displaystyle\sum_{j\in\mathcal{S}} \mathcal{L}\left[\sum_{i\in\mathcal{O}} s_i\, y_{ij}\right] & & \\
\text{subject to} & \displaystyle\sum_{j\in\mathcal{S}} y_{ij} = 1 & \forall i \in \mathcal{O} & \text{Assignment} \\
& \displaystyle\sum_{i\in\mathcal{O}} s_i\, y_{ij} \le \mathcal{K}_{\max} & \forall j \in \mathcal{S} & \\
& y_{ij} \le z_{c_i j} & \forall i \in \mathcal{O}\ \forall j \in \mathcal{S} & \\
& \displaystyle\sum_{c\in\mathcal{C}} z_{cj} \le 2 & \forall j \in \mathcal{S} & \\
& y_{ij}, z_{cj} \in \{0,1\} & \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S} &
\end{array}
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(A) Constraint programming formulation

$$
\begin{array}{llll}
\min & \sum_{j \in \mathcal{S}} \mathcal{L}\left[\sum_{i \in \mathcal{O}} s_i \, y_{ij}\right] & & \\
\text{subject to} & \sum_{j \in \mathcal{S}} y_{ij} = 1 & \forall i \in \mathcal{O} & \\
& \sum_{i \in \mathcal{O}} s_i \, y_{ij} \le \mathcal{K}_{\max} & \forall j \in \mathcal{S} & \text{Capacity} \\
& y_{ij} \le z_{c_i j} & \forall i \in \mathcal{O} \; \forall j \in \mathcal{S} & \\
& \sum_{c \in \mathcal{C}} z_{cj} \le 2 & \forall j \in \mathcal{S} & \\
& y_{ij}, z_{cj} \in \{0, 1\} & \forall i \in \mathcal{O} \; \forall c \in \mathcal{C} \; \forall j \in \mathcal{S} &
\end{array}
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(A) Constraint programming formulation

$$
\begin{aligned}
\min \quad & \sum_{j \in \mathcal{S}} \mathcal{L}\left[\sum_{i \in \mathcal{O}} s_i \, y_{ij}\right] && \\
\text{subject to} \quad & \sum_{j \in \mathcal{S}} y_{ij} = 1 && \forall i \in \mathcal{O} \\
& \sum_{i \in \mathcal{O}} s_i \, y_{ij} \leq \mathcal{K}_{\max} && \forall j \in \mathcal{S} \\
& y_{ij} \leq z_{c_i j} && \forall i \in \mathcal{O} \; \forall j \in \mathcal{S} \quad \text{Coloring} \\
& \sum_{c \in \mathcal{C}} z_{cj} \leq 2 && \forall j \in \mathcal{S} \\
& y_{ij}, z_{cj} \in \{0, 1\} && \forall i \in \mathcal{O} \; \forall c \in \mathcal{C} \; \forall j \in \mathcal{S}
\end{aligned}
$$

(A) Constraint programming formulation

$$\min \quad \sum_{j \in \mathcal{S}} \mathcal{L}\left[\sum_{i \in \mathcal{O}} s_i \, y_{ij}\right] \qquad \text{Leftover}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{j \in \mathcal{S}} y_{ij} = 1 && \forall i \in \mathcal{O} \\
& \sum_{i \in \mathcal{O}} s_i \, y_{ij} \leq \mathcal{K}_{\max} && \forall j \in \mathcal{S} \\
& y_{ij} \leq z_{c_i j} && \forall i \in \mathcal{O} \; \forall j \in \mathcal{S} \\
& \sum_{c \in \mathcal{C}} z_{cj} \leq 2 && \forall j \in \mathcal{S} \\
& y_{ij}, z_{cj} \in \{0, 1\} && \forall i \in \mathcal{O} \; \forall c \in \mathcal{C} \; \forall j \in \mathcal{S}
\end{aligned}$$

- ELEMENT constraint

(An) Integer programming formulation

Given

- $\mathcal{K}$ set of possible capacities for the slabs
- $\mathcal{C}$ set of colors
- $\mathcal{O}$ set of orders, $|\mathcal{O}| = n$
  - s_i size of order i
  - c_i color of order i

Binary variables

- $x_{kj} = 1$ if capacity k is assigned to slab j
- $y_{ij} = 1$ if order i is assigned to slab j
- $z_{cj} = 1$ if color c is used in slab j

Observation

- We need at most n slabs
- Let $\mathcal{S}$ be the set of slabs

(An) Integer programming formulation

$$
\begin{aligned}
\text{min} \quad & \sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{K}} k\, x_{kj} - \sum_{i \in \mathcal{O}} s_i \\
\text{subject to} \quad & \sum_{k \in \mathcal{K}} x_{kj} = 1 && \forall j \in \mathcal{S} \\
& \sum_{j \in \mathcal{S}} y_{ij} = 1 && \forall i \in \mathcal{O} \\
& \sum_{i \in \mathcal{O}} s_i\, y_{ij} \leq \sum_{k \in \mathcal{K}} k\, x_{kj} && \forall j \in \mathcal{S} \\
& y_{ij} \leq z_{c_i j} && \forall i \in \mathcal{O}\ \forall j \in \mathcal{S} \\
& \sum_{c \in \mathcal{C}} z_{cj} \leq 2 && \forall j \in \mathcal{S} \\
& x_{kj}, y_{ij}, z_{cj} \in \{0, 1\} && \forall k \in \mathcal{K}\ \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S}
\end{aligned}
$$

(An) Integer programming formulation

$$
\begin{array}{llll}
\min & \displaystyle\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{K}} k\, x_{kj} - \sum_{i\in\mathcal{O}} s_i & & \\
\text{subject to} & \displaystyle\sum_{k\in\mathcal{K}} x_{kj} = 1 & \forall j \in \mathcal{S} & \text{Assignment} \\
& \displaystyle\sum_{j\in\mathcal{S}} y_{ij} = 1 & \forall i \in \mathcal{O} & \\
& \displaystyle\sum_{i\in\mathcal{O}} s_i\, y_{ij} \le \sum_{k\in\mathcal{K}} k\, x_{kj} & \forall j \in \mathcal{S} & \\
& y_{ij} \le z_{c_i j} & \forall i \in \mathcal{O}\ \forall j \in \mathcal{S} & \\
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& x_{kj}, y_{ij}, z_{cj} \in \{0,1\} & \forall k \in \mathcal{K}\ \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S} &
\end{array}
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\begin{aligned}
\text{min} \quad & \sum_{j\in\mathcal{S}} \sum_{k\in\mathcal{K}} k\, x_{kj} - \sum_{i\in\mathcal{O}} s_i \\
\text{subject to} \quad & \sum_{k\in\mathcal{K}} x_{kj} = 1 && \forall j \in \mathcal{S} \\
& \sum_{j\in\mathcal{S}} y_{ij} = 1 && \forall i \in \mathcal{O} \\
& \sum_{i\in\mathcal{O}} s_i\, y_{ij} \le \sum_{k\in\mathcal{K}} k\, x_{kj} && \forall j \in \mathcal{S} && \text{Capacity} \\
& y_{ij} \le z_{c_i j} && \forall i \in \mathcal{O}\ \forall j \in \mathcal{S} \\
& \sum_{c\in\mathcal{C}} z_{cj} \le 2 && \forall j \in \mathcal{S} \\
& x_{kj}, y_{ij}, z_{cj} \in \{0,1\} && \forall k \in \mathcal{K}\ \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S}
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(An) Integer programming formulation

$$
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\text{min} \quad & \sum_{j\in\mathcal{S}} \sum_{k\in\mathcal{K}} k\, x_{kj} - \sum_{i\in\mathcal{O}} s_i \\
\text{subject to} \quad & \sum_{k\in\mathcal{K}} x_{kj} = 1 && \forall j \in \mathcal{S} \\
& \sum_{j\in\mathcal{S}} y_{ij} = 1 && \forall i \in \mathcal{O} \\
& \sum_{i\in\mathcal{O}} s_i\, y_{ij} \le \sum_{k\in\mathcal{K}} k\, x_{kj} && \forall j \in \mathcal{S} \\
& y_{ij} \le z_{c_i j} && \forall i \in \mathcal{O}\ \forall j \in \mathcal{S} && \text{Coloring} \\
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& x_{kj}, y_{ij}, z_{cj} \in \{0,1\} && \forall k \in \mathcal{K}\ \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S}
\end{aligned}
$$

(An) Integer programming formulation

$$\min \sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{K}} k\, x_{kj} - \sum_{i \in \mathcal{O}} s_i$$

Leftover

$$
\begin{array}{lll}
\text{subject to} & \displaystyle\sum_{k \in \mathcal{K}} x_{kj} = 1 & \forall j \in \mathcal{S} \\
& \displaystyle\sum_{j \in \mathcal{S}} y_{ij} = 1 & \forall i \in \mathcal{O} \\
& \displaystyle\sum_{i \in \mathcal{O}} s_i\, y_{ij} \leq \sum_{k \in \mathcal{K}} k\, x_{kj} & \forall j \in \mathcal{S} \\
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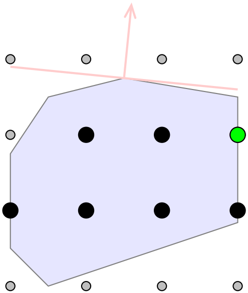
& x_{kj}, y_{ij}, z_{cj} \in \{0, 1\} & \forall k \in \mathcal{K}\ \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S}
\end{array}
$$

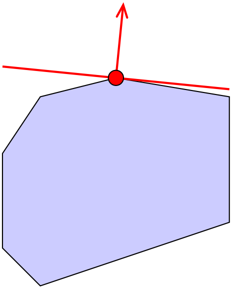
```
###################
#
# data parsing
#
###################

# number of capacities
param ncapacity := read DATAFILE as "1n" use 1;
do print "ncapacity = ", ncapacity;

# number of colors
param ncolors := read  DATAFILE as "1n" skip 1 use 1;
do print "ncolors = ", ncolors;

# number of orderns
param norders := read  DATAFILE as "1n" skip 2 use 1;
do print "norders = ", norders;

# get the capacity array
set tmp := {read DATAFILE as "<n+>" use 1};
set capacities := if card(tmp) == ncapacity then tmp union {0} else tmp union {0} \ {ncapacity} end ;
do check ncapacity == card(capacities)-1;
do print capacities;

# index set for orders
set I := {1..norders};

# index set for colors
set C := {1..ncolors};

# get orders
set orders[<i> in I] := {read DATAFILE as "<1n,2n>" skip 2+i use 1};
#do forall <i> in I do print orders[i];

###################
#
# decision variables
#
###################

# slab variables which capacities is assigned to which slab
var x[I * capacities] binary;

# which order is assigned to which slab
var y[I * I] binary;

# which color is used by which slab
var z[C * I];

###################
#
# objective function
#
###################

minimize leftover:
         sum <s,c> in I * capacities : c * x[s,c] - sum <o,s> in I * I : ord(orders[o],1,1) * y[o,s];

###################
#
# constraints
#
###################

# each slab gets exactly one capacities
subto oneCapacity:
        forall <s> in I : sum <c> in capacities: x[s,c] == 1;

# each order is assigned to exactly one slab
subto oneSlab:
        forall <o> in I : sum <s> in I: y[o,s] == 1;

# each slab is not over loaded
subto Capacity:
        forall <s> in I : sum <c> in capacities: c * x[s,c] - sum <o> in I: ord(orders[o],1,1) * y[o,s] >= 0;

# color linking constraints
subto linking:
        forall <o,s> in I * I : y[o,s] - z[ord(orders[o],1,2),s] <= 0;

# color linking constraints
subto Color:
        forall <s> in I : sum <c> in C : z[c,s] <= 2;
```

Integer program

Linear program relaxation



$\rightarrow$



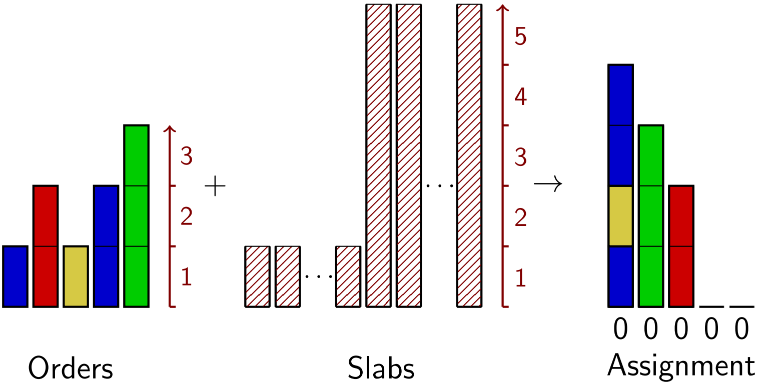
$$\min\{IP\} \geq \min\{LP\}$$

$$\max\{IP\} \leq \max\{LP\}$$

▷ Omit integrality condition

$$
\begin{aligned}
\min \quad & \sum_{j\in\mathcal{S}} \sum_{k\in\mathcal{K}} k\, x_{kj} - \sum_{i\in\mathcal{O}} s_i \\
\text{subject to} \quad & \sum_{k\in\mathcal{K}} x_{kj} = 1 && \forall j \in \mathcal{S} \\
& \sum_{j\in\mathcal{S}} y_{ij} = 1 && \forall i \in \mathcal{O} \\
& \sum_{i\in\mathcal{O}} s_i\, y_{ij} \leq \sum_{k\in\mathcal{K}} k\, x_{kj} && \forall j \in \mathcal{S} \\
& y_{ij} \leq z_{c_i j} && \forall i \in \mathcal{O}\ \forall j \in \mathcal{S} \\
& \sum_{c\in\mathcal{C}} z_{cj} \leq 2 && \forall j \in \mathcal{S} \\
& x_{kj}, y_{ij}, z_{cj} \in [0,1] && \forall k \in \mathcal{K}\ \forall i \in \mathcal{O}\ \forall c \in \mathcal{C}\ \forall j \in \mathcal{S}
\end{aligned}
$$

Root node solution



Root node solution

Capacities

$$x_{51} = 0.8$$
$$x_{01} = 0.2$$
$$x_{52} = 0.6$$
$$x_{02} = 0.4$$
$$x_{53} = 0.4$$
$$x_{03} = 0.6$$
$$x_{04} = 1.0$$
$$x_{06} = 1.0$$

Assignments

$$y_{11} = 1.0$$
$$y_{23} = 1.0$$
$$y_{31} = 1.0$$
$$y_{41} = 1.0$$
$$y_{52} = 1.0$$

Colors

$$z_{11} = 1.0$$
$$z_{23} = 1.0$$
$$z_{31} = 1.0$$
$$z_{32} = 1.0$$
$$z_{33} = 1.0$$
$$z_{34} = 1.0$$
$$z_{35} = 1.0$$
$$z_{42} = 1.0$$

- Remaining decission variables are zero

Capacities

$$x_{51} = 0.8$$
$$x_{01} = 0.2$$
$$x_{52} = 0.6$$
$$x_{02} = 0.4$$
$$x_{53} = 0.4$$
$$x_{03} = 0.6$$
$$x_{04} = 1.0$$
$$x_{06} = 1.0$$

Assignments

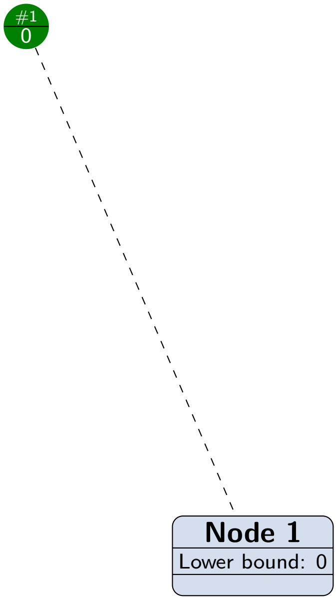
$$y_{11} = 1.0$$
$$y_{23} = 1.0$$
$$y_{31} = 1.0$$
$$y_{41} = 1.0$$
$$y_{52} = 1.0$$

Colors

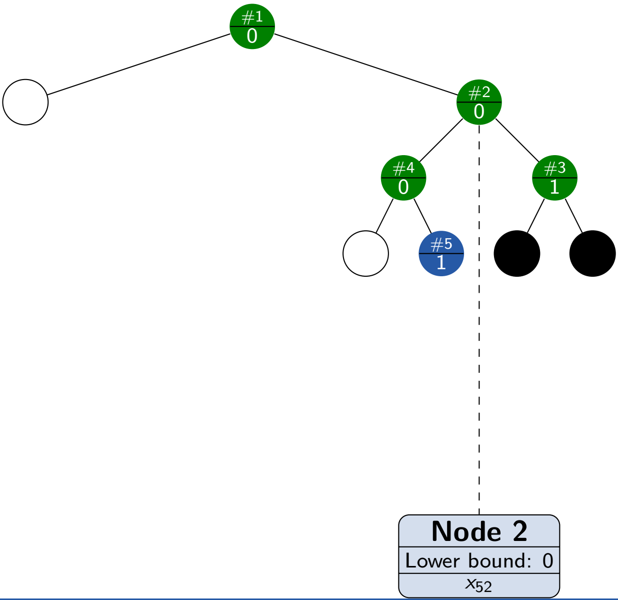
$$z_{11} = 1.0$$
$$z_{23} = 1.0$$
$$z_{31} = 1.0$$
$$z_{32} = 1.0$$
$$z_{33} = 1.0$$
$$z_{34} = 1.0$$
$$z_{35} = 1.0$$
$$z_{42} = 1.0$$

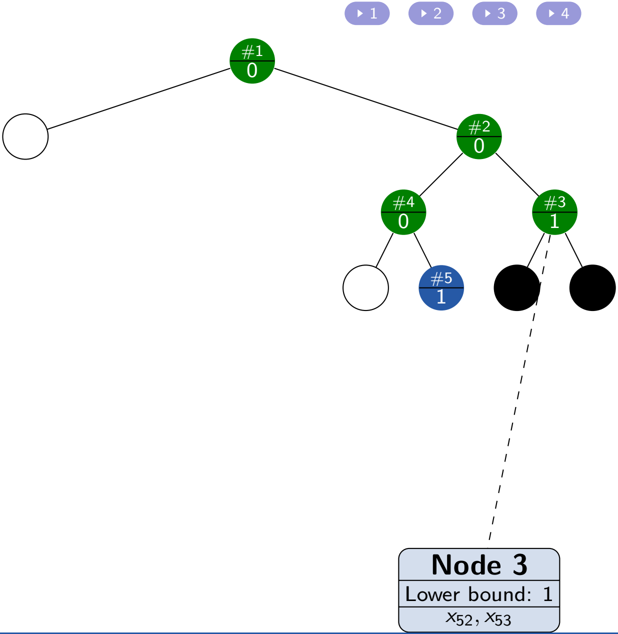
- Remaining decission variables are zero
- Observation: Independently of the problem instance the root LP value for this model is always zero.

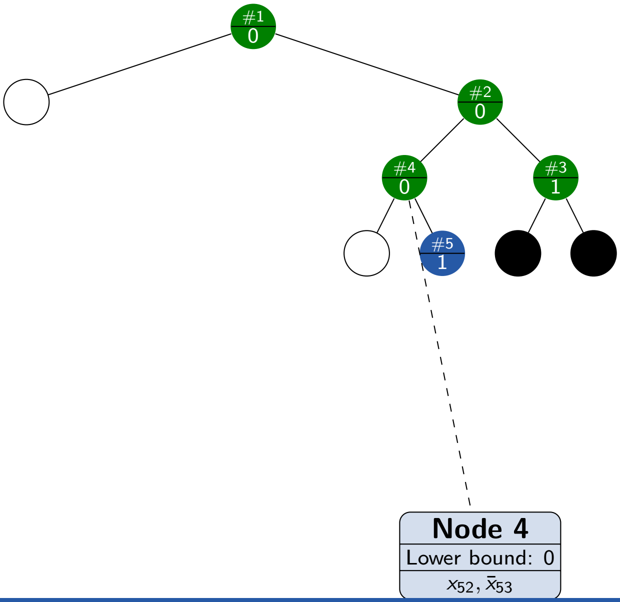
node	left	depth	frac	curdualbound	dualbound	primalbound
1	0	0	6	0.000000e+00	0.000000e+00	--


#1
0
Node 1
Lower bound: 0

▸ 1 ▸ 2 ▸ 3 ▸ 4

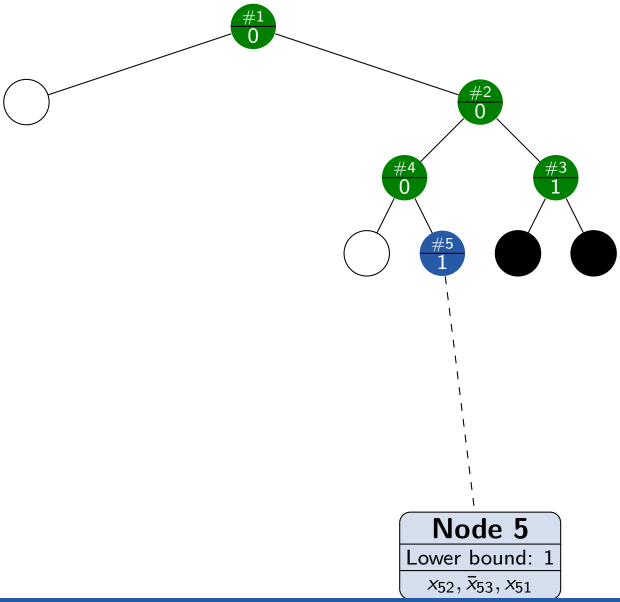



1
2
3
4
#1
0
#2
0
#4
0
#3
1
#5
1
Node 3
Lower bound: 1
x_{52}, x_{53}

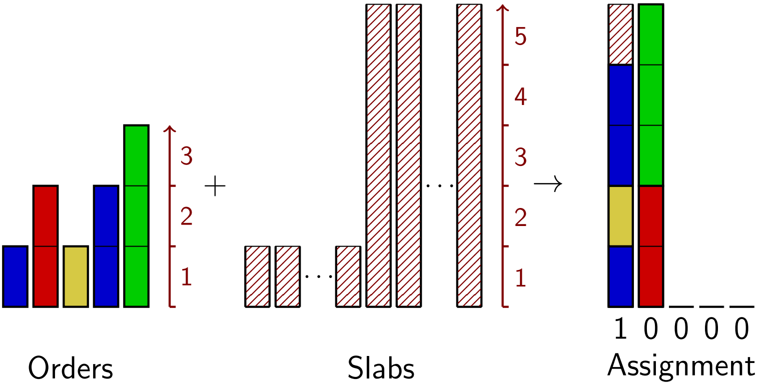

#1
0
#2
0
#4
0
#3
1
#5
1
Node 4
Lower bound: 0
$x_{52}, \bar{x}_{53}$

Search tree

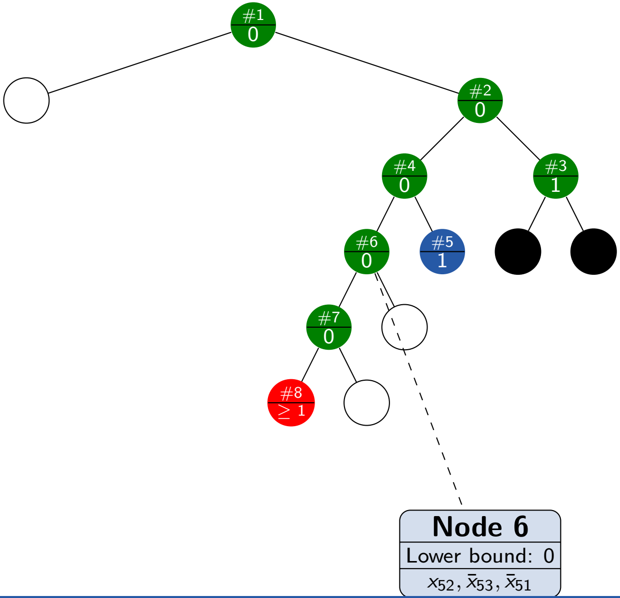
▸ 1 ▸ 2 ▸ 3 ▸ 4



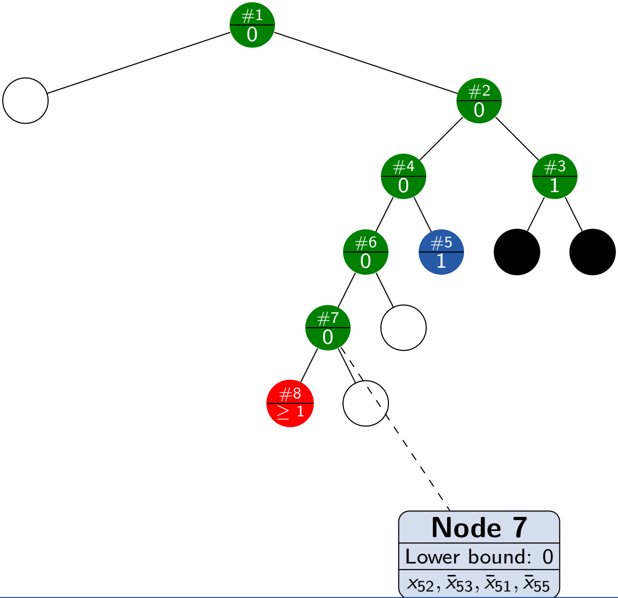
	node	left	depth	frac	curdualbound	dualbound	primalbound
	1	0	0	6	0.000000e+00	0.000000e+00	--
	1	2	0	6	0.000000e+00	0.000000e+00	--
	2	3	1	4	0.000000e+00	0.000000e+00	--
	3	4	2	6	1.000000e+00	0.000000e+00	--
	4	5	2	2	0.000000e+00	0.000000e+00	--
*	5	2	3	-	1.000000e+00	0.000000e+00	1.000000e+00


3
2
1
+
5
4
3
2
1
→
1 0 0 0 0
Orders
Slabs
Assignment

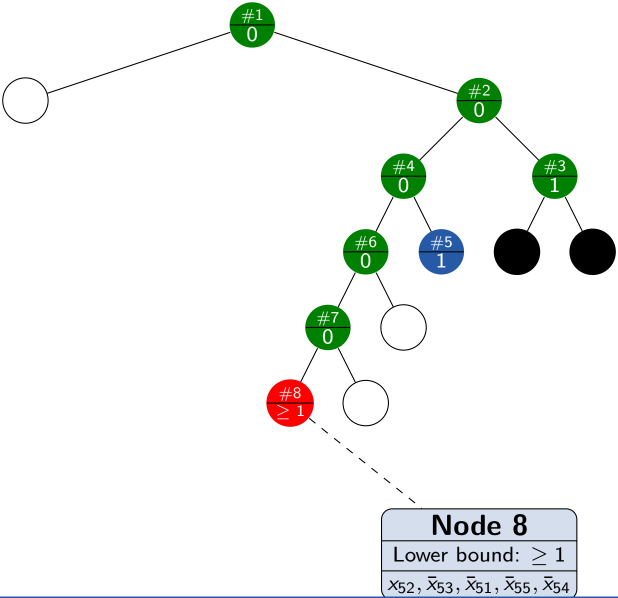
▸ 1 ▸ 2 ▸ 3 ▸ 4



▸ 1 ▸ 2 ▸ 3 ▸ 4

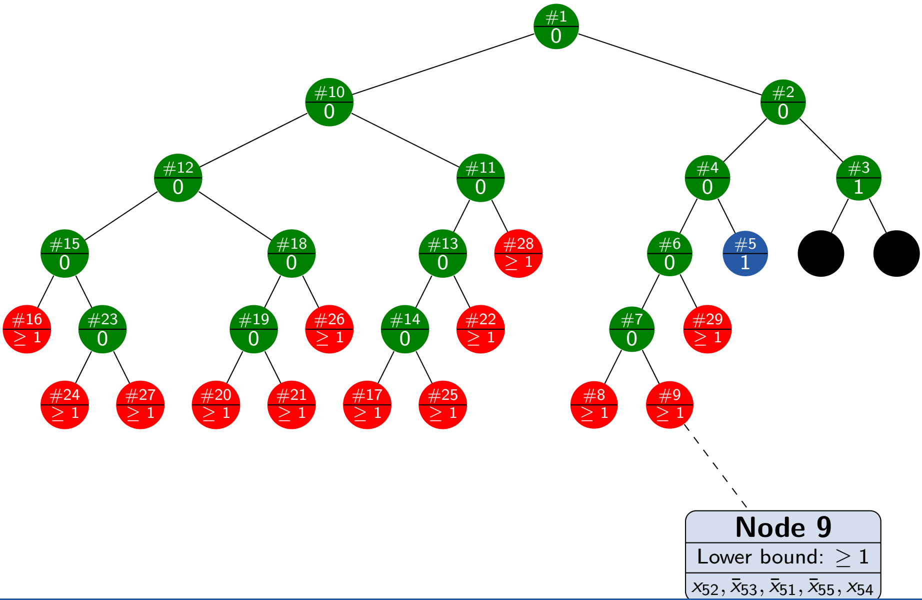

#1
0
#2
0
#4
0
#3
1
#6
0
#5
1
#7
0
#8
≥ 1
Node 7
Lower bound: 0
$x_{52}, \bar{x}_{53}, \bar{x}_{51}, \bar{x}_{55}$

▸ 1 ▸ 2 ▸ 3 ▸ 4

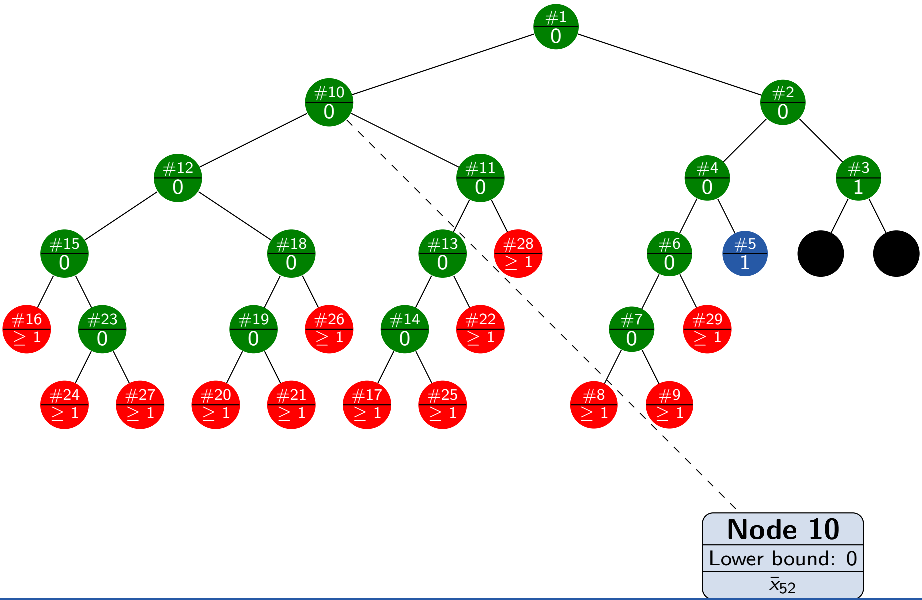

#1
0
#2
0
#4
0
#3
1
#6
0
#5
1
#7
0
#8
≥ 1
Node 8
Lower bound: ≥ 1
$x_{52}, \bar{x}_{53}, \bar{x}_{51}, \bar{x}_{55}, \bar{x}_{54}$

	node	left	depth	frac	curdualbound	dualbound	primalbound
	1	0	0	6	0.000000e+00	0.000000e+00	--
	1	2	0	6	0.000000e+00	0.000000e+00	--
	2	3	1	4	0.000000e+00	0.000000e+00	--
	3	4	2	6	1.000000e+00	0.000000e+00	--
	4	5	2	2	0.000000e+00	0.000000e+00	--
*	5	2	3	-	1.000000e+00	0.000000e+00	1.000000e+00
	6	3	3	2	0.000000e+00	0.000000e+00	1.000000e+00
	7	4	4	2	0.000000e+00	0.000000e+00	1.000000e+00
	8	3	5	-	--	0.000000e+00	1.000000e+00

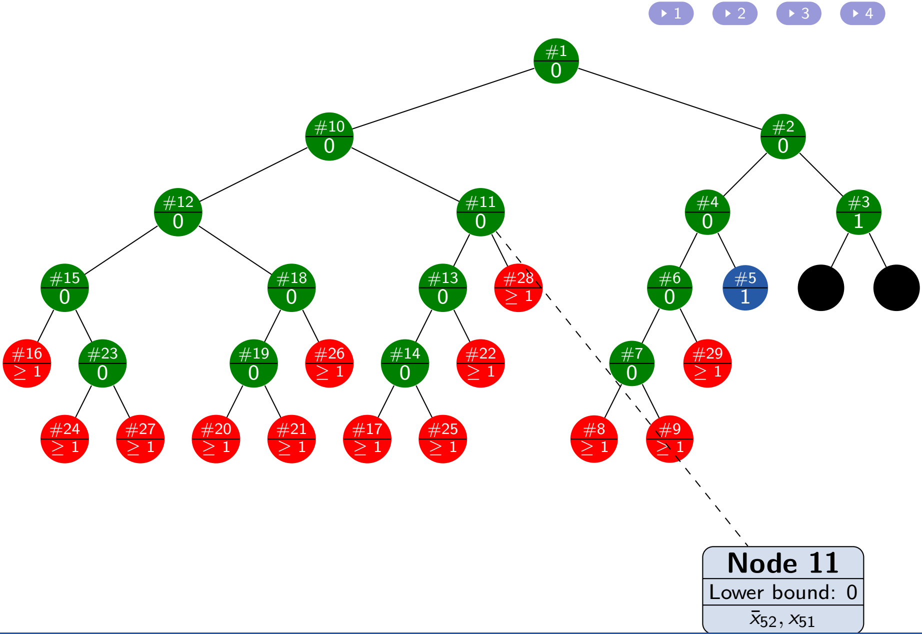
▸ 1 ▸ 2 ▸ 3 ▸ 4


#1
0
#10
0
#2
0
#12
0
#11
0
#4
0
#3
1
#15
0
#18
0
#13
0
#28
≥ 1
#6
0
#5
1
#16
≥ 1
#23
0
#19
0
#26
≥ 1
#14
0
#22
≥ 1
#7
0
#29
≥ 1
#24
≥ 1
#27
≥ 1
#20
≥ 1
#21
≥ 1
#17
≥ 1
#25
≥ 1
#8
≥ 1
#9
≥ 1
Node 9
Lower bound: ≥ 1
$x_{52}, \bar{x}_{53}, \bar{x}_{51}, \bar{x}_{55}, x_{54}$

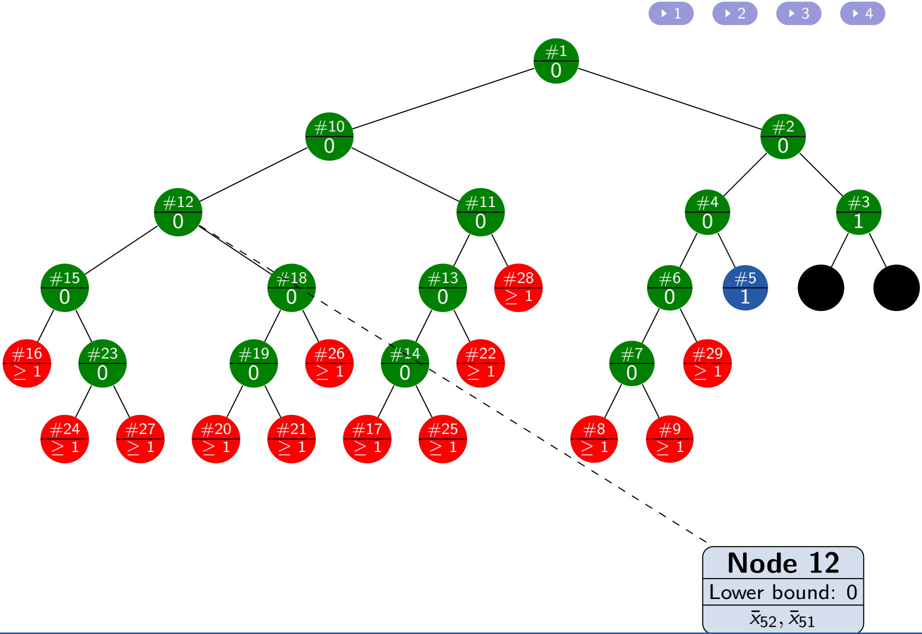
▸ 1 ▸ 2 ▸ 3 ▸ 4



#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 10
Lower bound: 0
$\bar{x}_{52}$


▸ 1
▸ 2
▸ 3
▸ 4
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 11
Lower bound: 0
$\bar{x}_{52}, x_{51}$

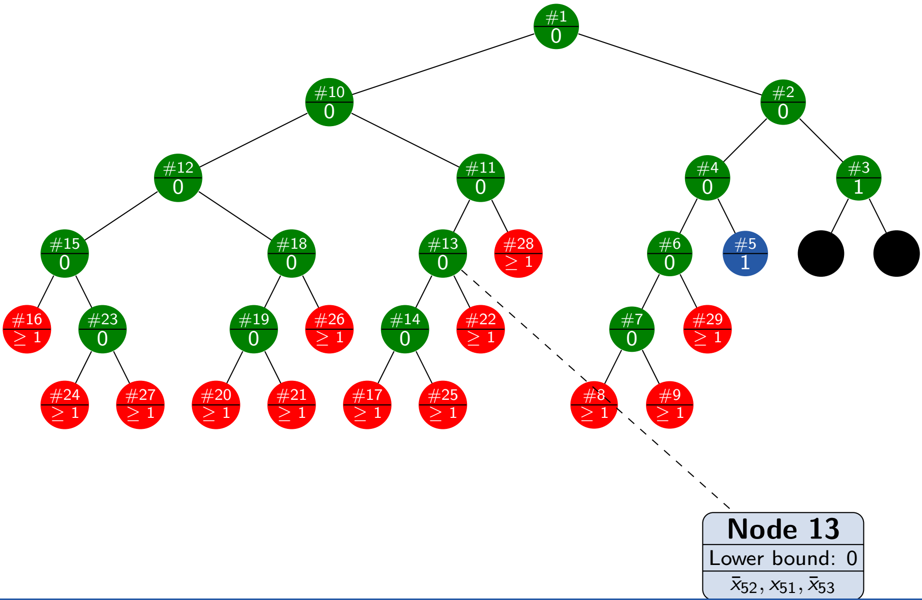
Search tree

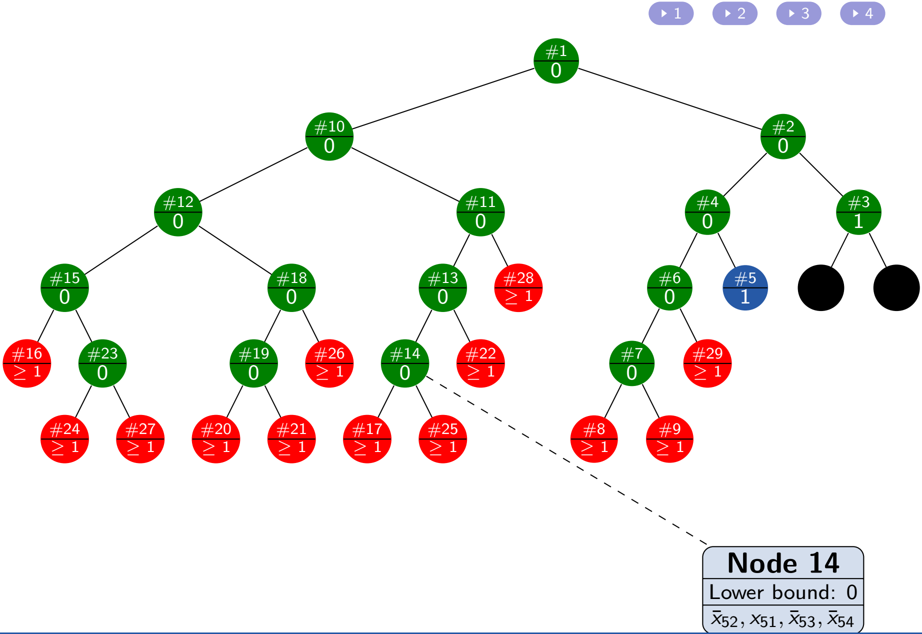


▸ 1
▸ 2
▸ 3
▸ 4
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 12
Lower bound: 0
$\bar{x}_{52}, \bar{x}_{51}$

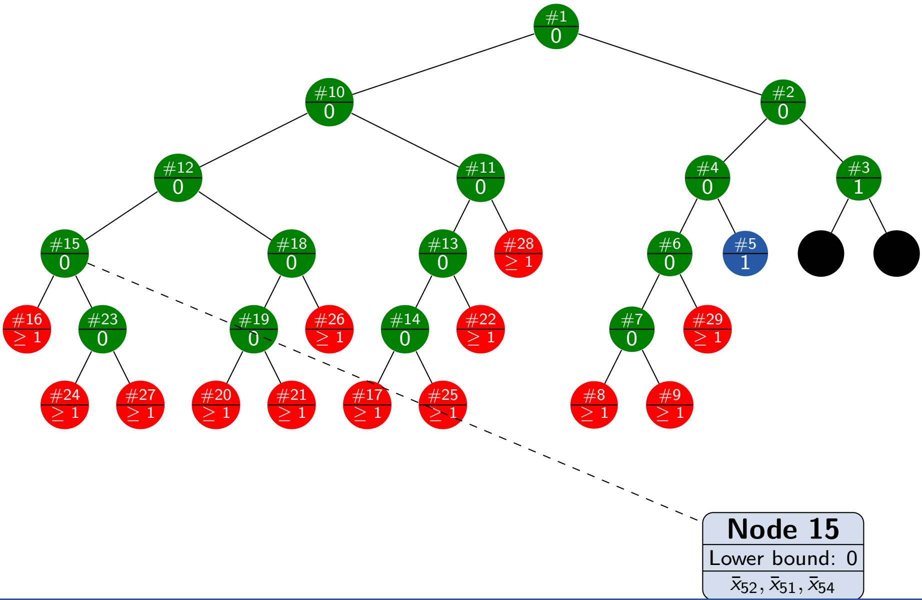
Search tree

1 2 3 4

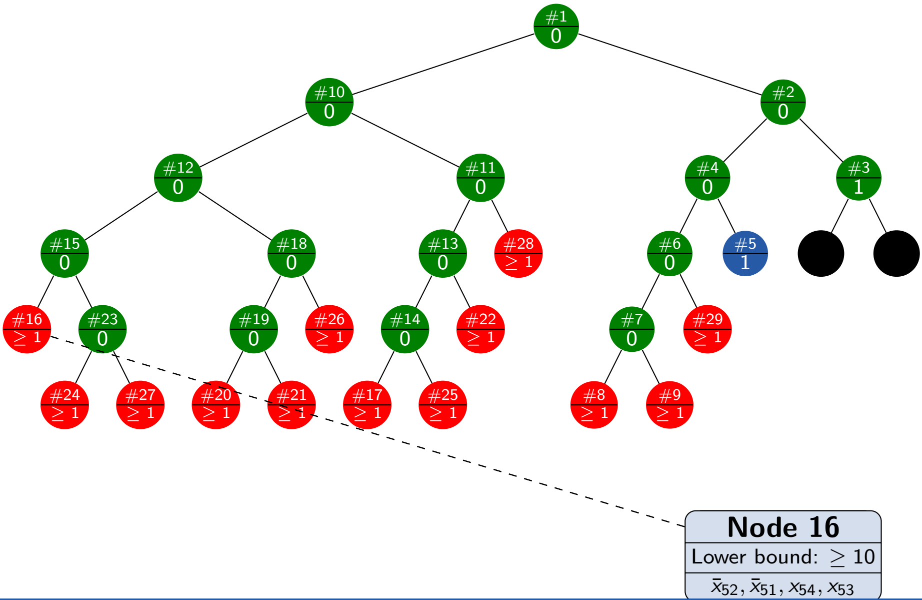



1
2
3
4
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 14
Lower bound: 0
$\bar{x}_{52}, x_{51}, \bar{x}_{53}, \bar{x}_{54}$

▸ 1 ▸ 2 ▸ 3 ▸ 4

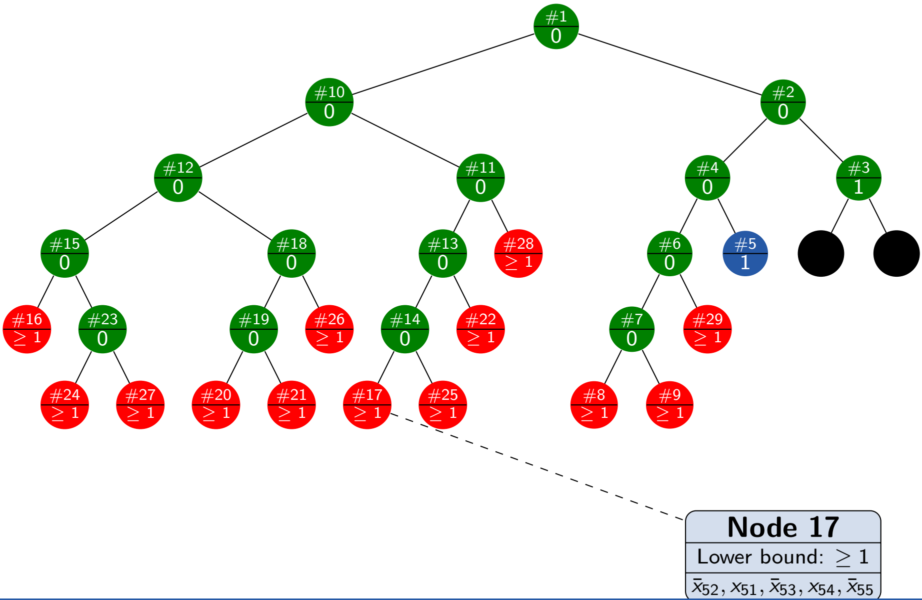

#1
0
#10
0
#2
0
#12
0
#11
0
#4
0
#3
1
#15
0
#18
0
#13
0
#28
≥ 1
#6
0
#5
1
#16
≥ 1
#23
0
#19
0
#26
≥ 1
#14
0
#22
≥ 1
#7
0
#29
≥ 1
#24
≥ 1
#27
≥ 1
#20
≥ 1
#21
≥ 1
#17
≥ 1
#25
≥ 1
#8
≥ 1
#9
≥ 1
Node 15
Lower bound: 0
$\bar{x}_{52}, \bar{x}_{51}, \bar{x}_{54}$

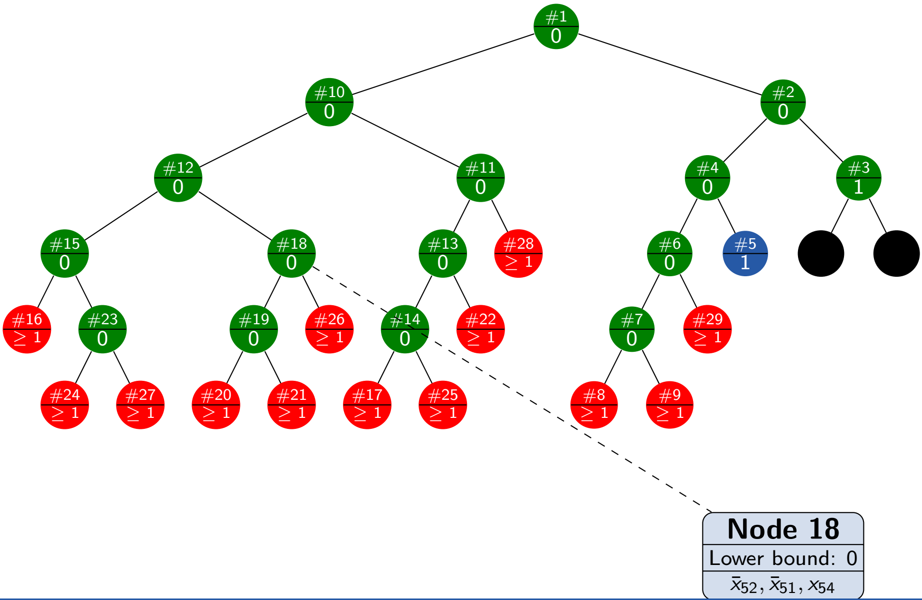
1 2 3 4


#1
0
#10
0
#2
0
#12
0
#11
0
#4
0
#3
1
#15
0
#18
0
#13
0
#28
≥ 1
#6
0
#5
1
#16
≥ 1
#23
0
#19
0
#26
≥ 1
#14
0
#22
≥ 1
#7
0
#29
≥ 1
#24
≥ 1
#27
≥ 1
#20
≥ 1
#21
≥ 1
#17
≥ 1
#25
≥ 1
#8
≥ 1
#9
≥ 1
Node 16
Lower bound: ≥ 10
$\bar{x}_{52}, \bar{x}_{51}, x_{54}, x_{53}$

Search tree

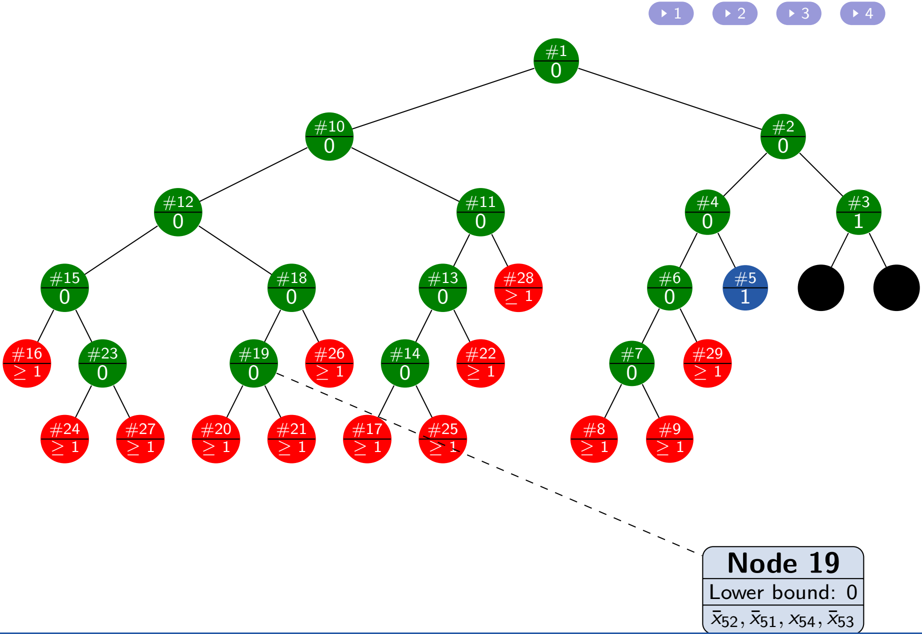
▸ 1 ▸ 2 ▸ 3 ▸ 4



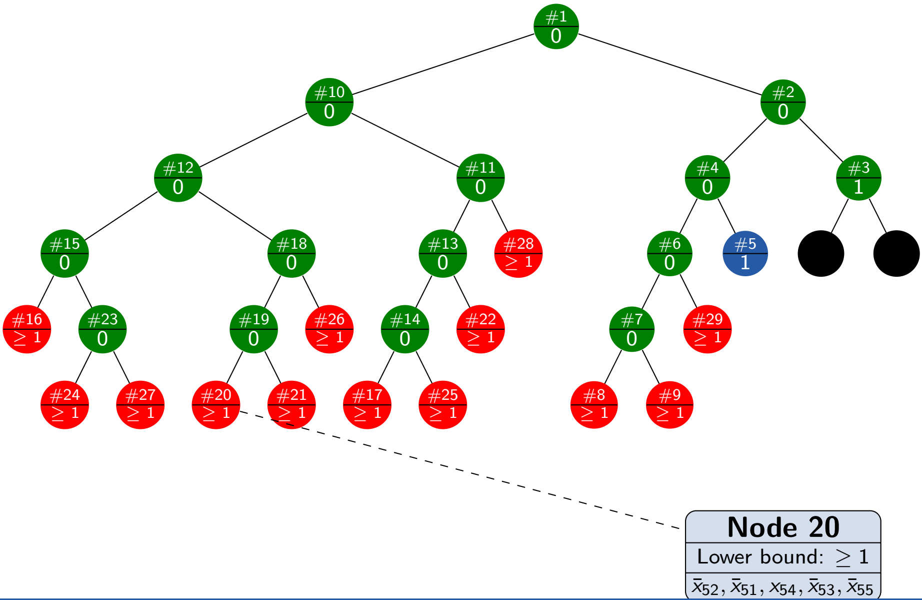


#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 18
Lower bound: 0
$\bar{x}_{52}, \bar{x}_{51}, x_{54}$

Search tree



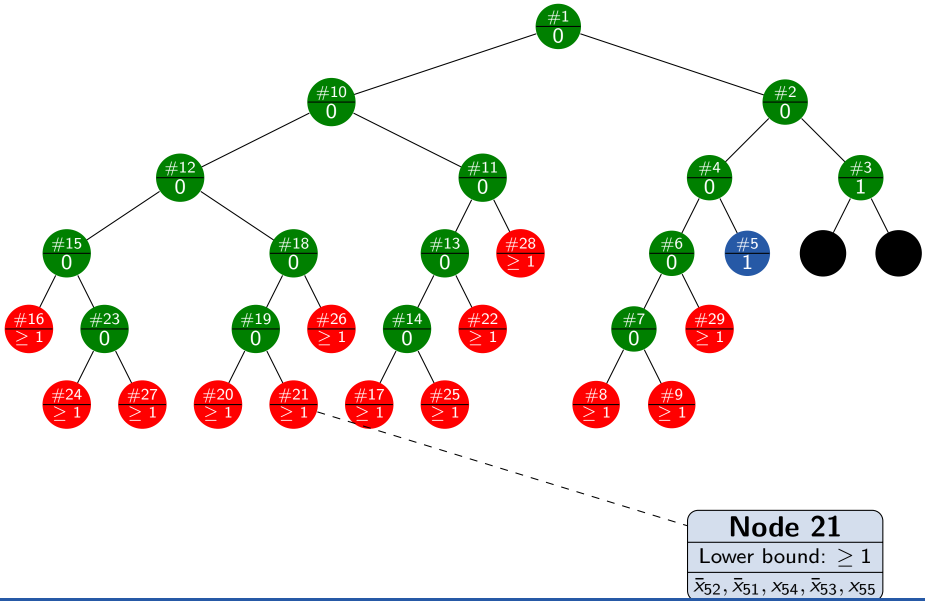
▸ 1 ▸ 2 ▸ 3 ▸ 4
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 19
Lower bound: 0
$\bar{x}_{52}, \bar{x}_{51}, x_{54}, \bar{x}_{53}$



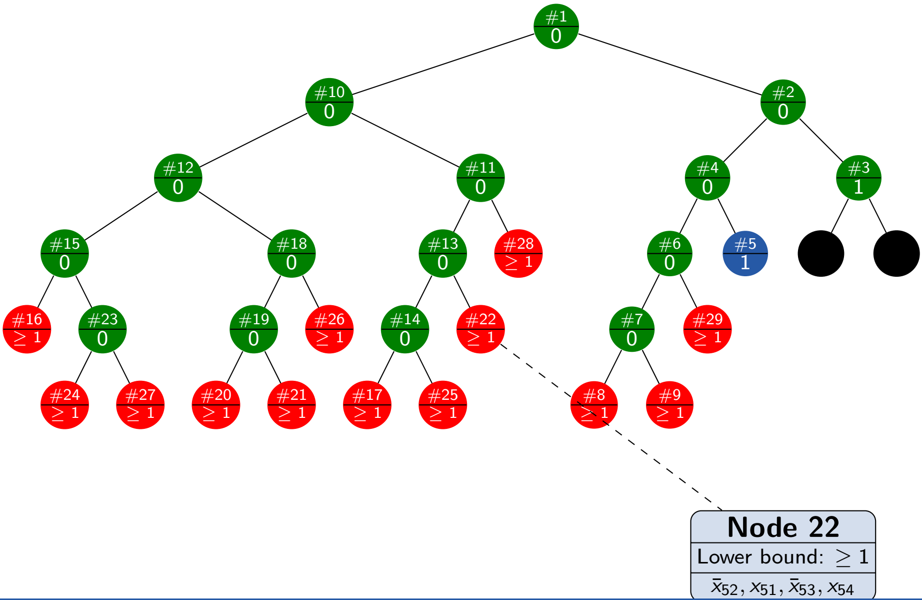
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 20
Lower bound: ≥ 1
$\bar{x}_{52}, \bar{x}_{51}, x_{54}, \bar{x}_{53}, \bar{x}_{55}$

Search tree

▸ 1 ▸ 2 ▸ 3 ▸ 4

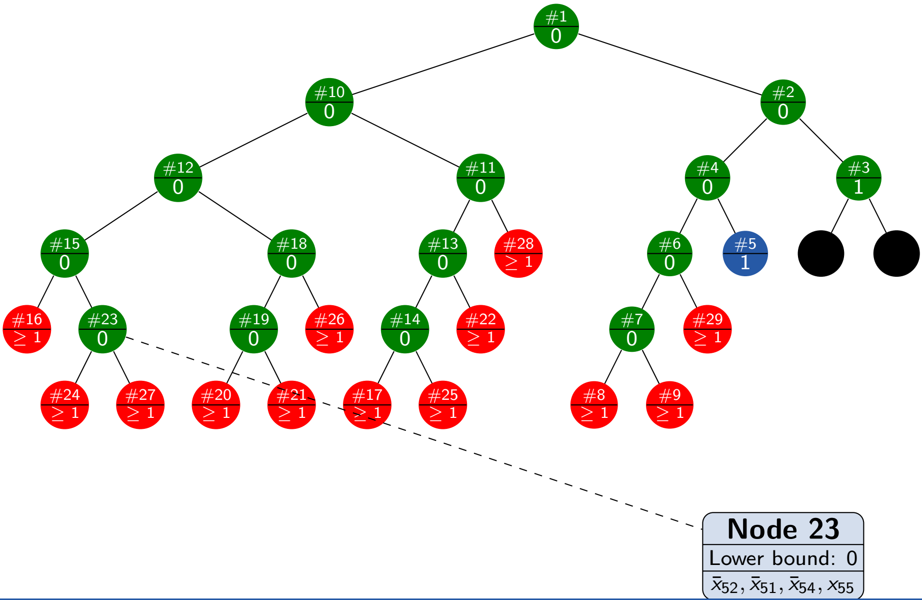


▸ 1 ▸ 2 ▸ 3 ▸ 4

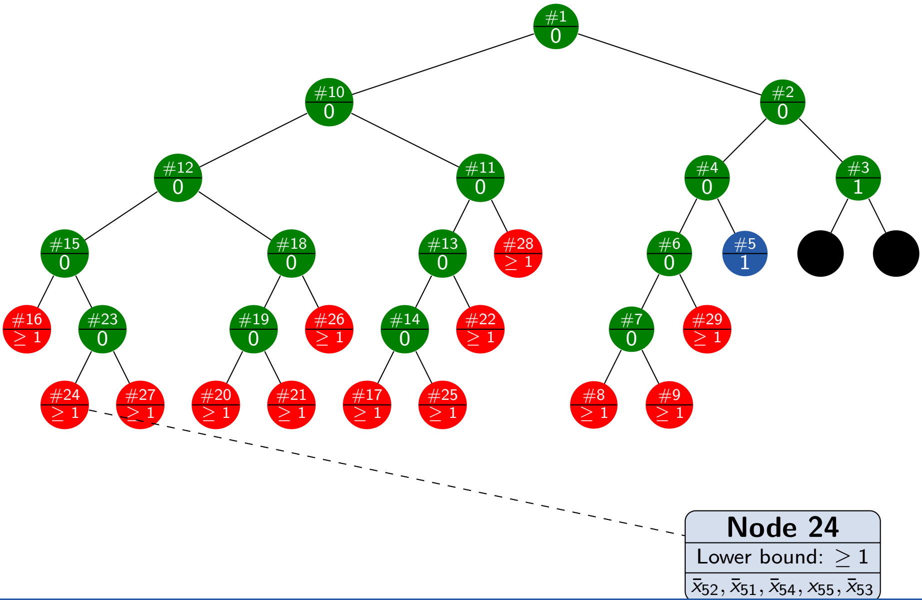


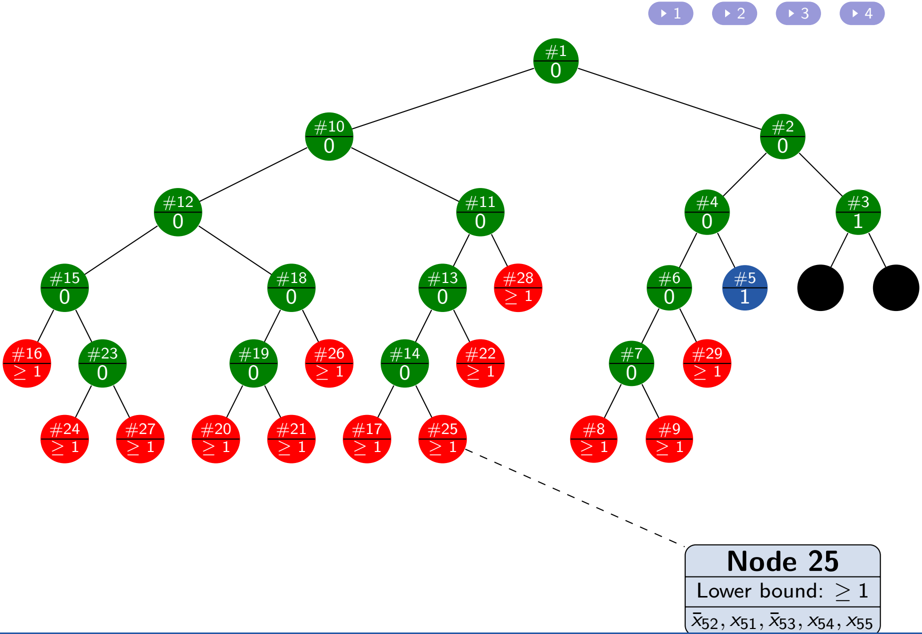
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 22
Lower bound: ≥ 1
$\bar{x}_{52}, x_{51}, \bar{x}_{53}, x_{54}$

▸ 1 ▸ 2 ▸ 3 ▸ 4

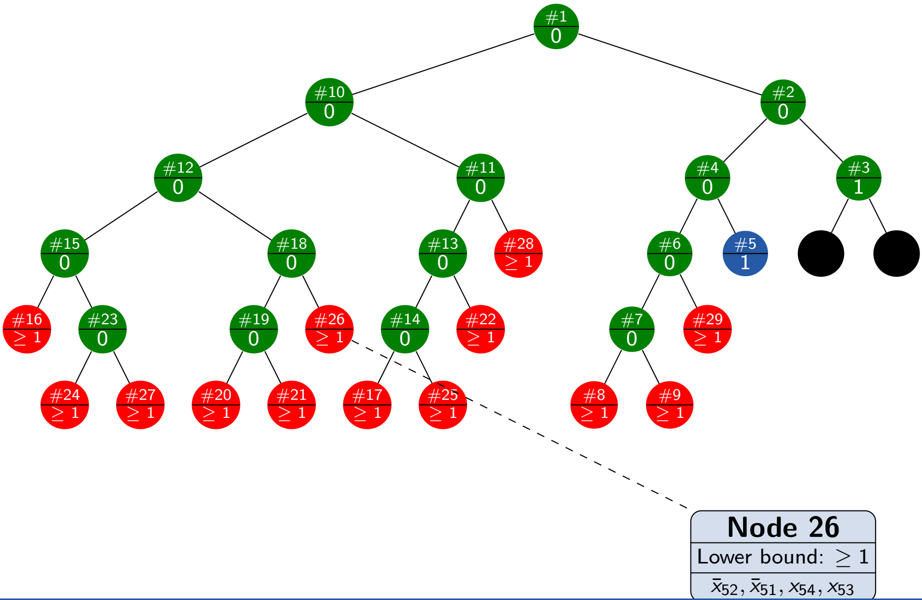

#1
0
#10
0
#2
0
#12
0
#11
0
#4
0
#3
1
#15
0
#18
0
#13
0
#28
≥ 1
#6
0
#5
1
#16
≥ 1
#23
0
#19
0
#26
≥ 1
#14
0
#22
≥ 1
#7
0
#29
≥ 1
#24
≥ 1
#27
≥ 1
#20
≥ 1
#21
≥ 1
#17
≥ 1
#25
≥ 1
#8
≥ 1
#9
≥ 1
Node 23
Lower bound: 0
$\bar{x}_{52}, \bar{x}_{51}, \bar{x}_{54}, x_{55}$

▸ 1 ▸ 2 ▸ 3 ▸ 4


#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 24
Lower bound: ≥ 1
$\bar{x}_{52}, \bar{x}_{51}, \bar{x}_{54}, x_{55}, \bar{x}_{53}$

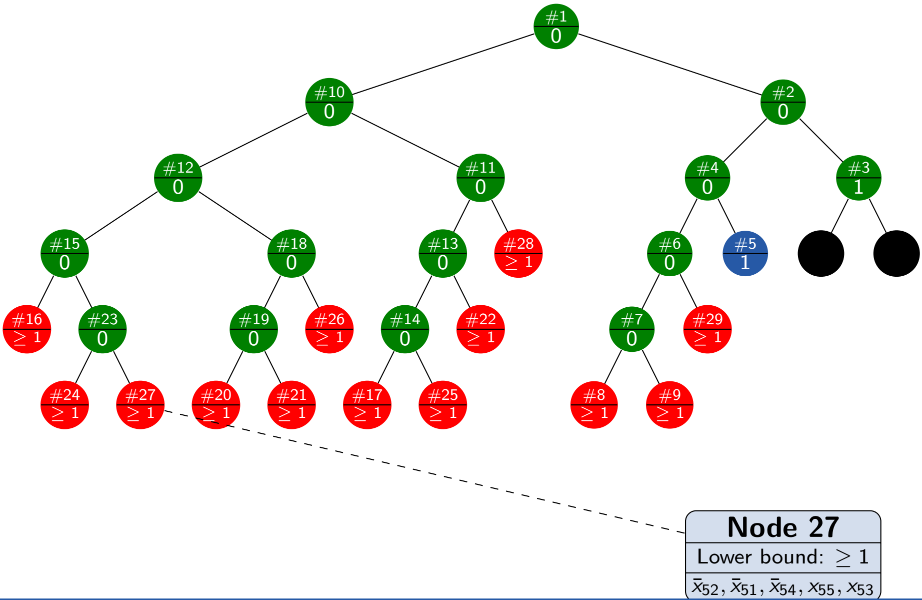

1
2
3
4
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 25
Lower bound: ≥ 1
$\bar{x}_{52}, x_{51}, \bar{x}_{53}, x_{54}, x_{55}$

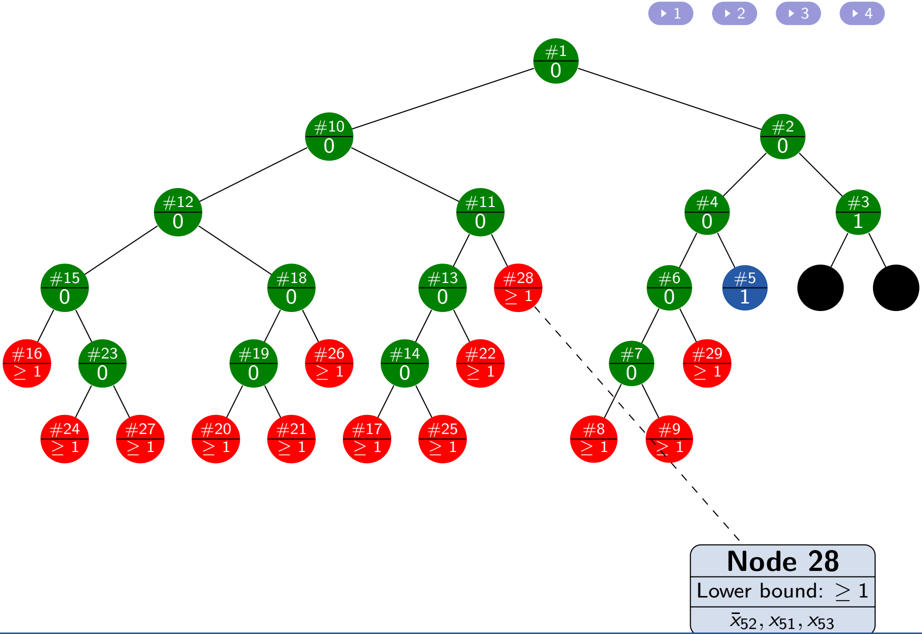
▸ 1 ▸ 2 ▸ 3 ▸ 4


#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 26
Lower bound: ≥ 1
$\bar{x}_{52}, \bar{x}_{51}, x_{54}, x_{53}$

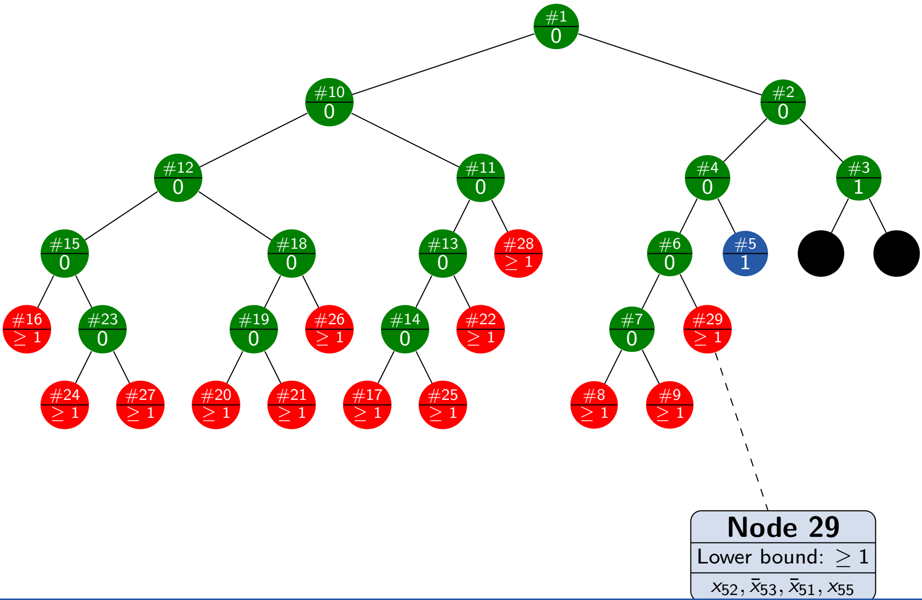
Search tree

▸ 1 ▸ 2 ▸ 3 ▸ 4




1
2
3
4
#1 0
#10 0
#2 0
#12 0
#11 0
#4 0
#3 1
#15 0
#18 0
#13 0
#28 ≥ 1
#6 0
#5 1
#16 ≥ 1
#23 0
#19 0
#26 ≥ 1
#14 0
#22 ≥ 1
#7 0
#29 ≥ 1
#24 ≥ 1
#27 ≥ 1
#20 ≥ 1
#21 ≥ 1
#17 ≥ 1
#25 ≥ 1
#8 ≥ 1
#9 ≥ 1
Node 28
Lower bound: ≥ 1
$\bar{x}_{52}, x_{51}, x_{53}$

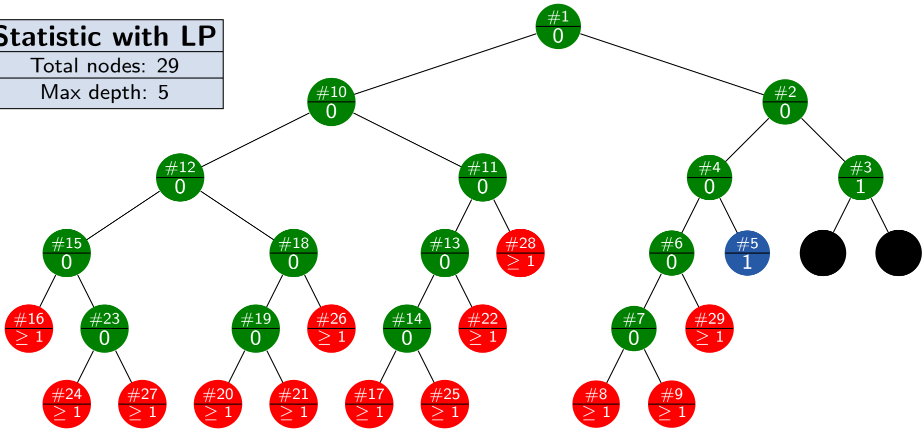
▸ 1 ▸ 2 ▸ 3 ▸ 4


#1
0
#10
0
#2
0
#12
0
#11
0
#4
0
#3
1
#15
0
#18
0
#13
0
#28
≥ 1
#6
0
#5
1
#16
≥ 1
#23
0
#19
0
#26
≥ 1
#14
0
#22
≥ 1
#7
0
#29
≥ 1
#24
≥ 1
#27
≥ 1
#20
≥ 1
#21
≥ 1
#17
≥ 1
#25
≥ 1
#8
≥ 1
#9
≥ 1
Node 29
Lower bound: ≥ 1
$x_{52}, \bar{x}_{53}, \bar{x}_{51}, x_{55}$

	node	left	depth	frac	curdualbound	dualbound	primalbound
	1	0	0	6	0.000000e+00	0.000000e+00	--
	1	2	0	6	0.000000e+00	0.000000e+00	--
	2	3	1	4	0.000000e+00	0.000000e+00	--
	3	4	2	6	1.000000e+00	0.000000e+00	--
	4	5	2	2	0.000000e+00	0.000000e+00	--
*	5	2	3	-	1.000000e+00	0.000000e+00	1.000000e+00
	6	3	3	2	0.000000e+00	0.000000e+00	1.000000e+00
	7	4	4	2	0.000000e+00	0.000000e+00	1.000000e+00
	8	3	5	-	--	0.000000e+00	1.000000e+00
	9	2	5	-	--	0.000000e+00	1.000000e+00
	10	3	1	2	4.440892e-16	0.000000e+00	1.000000e+00
	11	4	2	2	4.440892e-16	0.000000e+00	1.000000e+00
	12	5	2	4	4.440892e-16	0.000000e+00	1.000000e+00
	13	6	3	4	1.776357e-15	0.000000e+00	1.000000e+00
	14	7	4	4	1.776357e-15	0.000000e+00	1.000000e+00
	node	left	depth	frac	curdualbound	dualbound	primalbound
	15	8	3	4	1.184238e-15	0.000000e+00	1.000000e+00
	16	7	4	-	--	0.000000e+00	1.000000e+00
	17	6	5	-	--	0.000000e+00	1.000000e+00
	18	7	3	2	4.440892e-16	0.000000e+00	1.000000e+00
	19	8	4	4	8.881784e-16	0.000000e+00	1.000000e+00
	20	7	5	-	--	0.000000e+00	1.000000e+00
	21	6	5	-	--	0.000000e+00	1.000000e+00
	22	5	4	-	--	0.000000e+00	1.000000e+00
	23	6	4	2	1.184238e-15	0.000000e+00	1.000000e+00
	24	5	5	-	--	0.000000e+00	1.000000e+00
	25	4	5	-	--	0.000000e+00	1.000000e+00
	26	3	4	-	--	0.000000e+00	1.000000e+00
	27	2	5	-	--	0.000000e+00	1.000000e+00
	28	1	3	-	--	0.000000e+00	1.000000e+00

▸ 1 ▸ 2 ▸ 3 ▸ 4

Statistic with LP
Total nodes: 29
Max depth: 5



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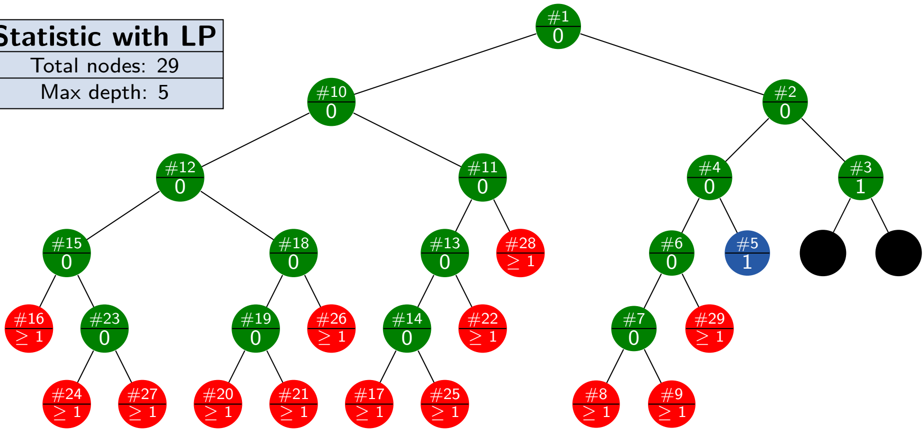

Statistic without LP
Total nodes: 2154
Max depth: 21

Summary

- A solution of a linear relaxation gives a proven dual bound
  - in case of minimization it is a lower bound
  - in case of maximization it is a upper bound
- Linear relaxation is a natural relaxation for an integer program
  - omitting integrality conditions
- Linear relaxation gives a global view w.r.t. all linear constraints
- Linear relaxation guides the search via fractional variables

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Coming up:

- How can a linear program be solved?
- How can an integer program be solved?
- For what is the linear programming relaxation used within an integer programming solver?

Integer Programming for Constraint Programmers

General linear programs (LPs)

Continuous variables:	$x_i \geq 0,\ lb_i \leq x_i \leq ub_i,\ x_i$ free	(columns)
Linear constraints:	$a_1x_1 + \ldots + a_nx_n \overset{\leq}{\underset{\geq}{=}} b$	(rows)
Linear objective:	$c_1x_1 + \ldots + c_nx_n$	$(\rightarrow$ min/max)

General linear programs (LPs)

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Linear objective: $c_1 x_1 + \ldots + c_n x_n$ ($\rightarrow$ min/max)

Computational standard form: $\min \left\{ c'x \mid Ax = b, x \geq 0 \right\}$

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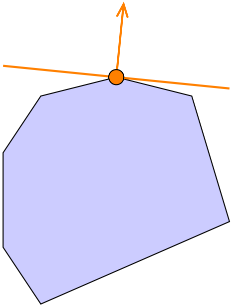
Computational standard form: $\min \left\{ c'x \mid Ax = b, x \geq 0 \right\}$

- feasible region is convex and polyhedral
- problem may be
  - unbounded,
  - infeasible, or
  - optimal
- always optimal vertex solution (if an optimal solution exists)



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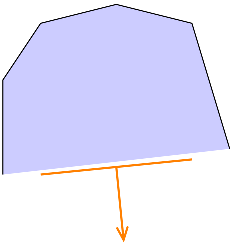

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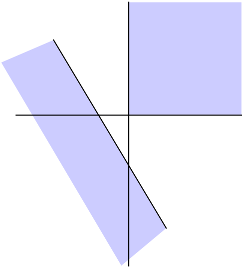

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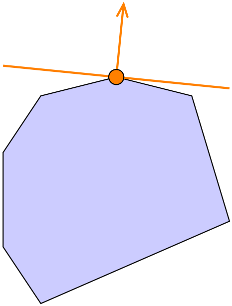

An incomplete history on linear programming

1827 J. Fourier: Variable elimination algorithm ("Fourier-Motzkin")

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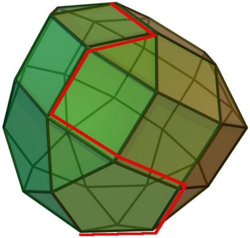
- by far the most used algorithm to solve LPs
- worst case exponential running time



simplex algorithm
[Dantzig 1947]

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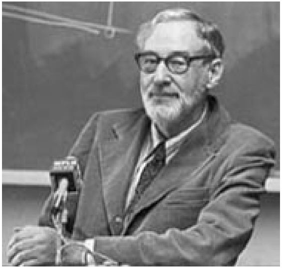

1975: Nobel price in Economic Science

"Optimal allocation of ressources"

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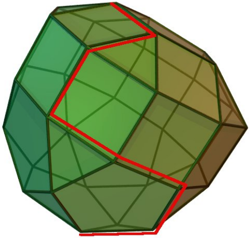
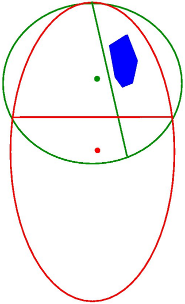
- first polynomial time algorithm

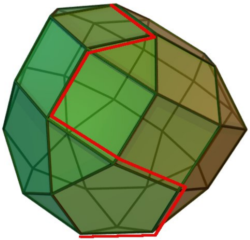




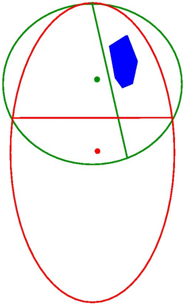
simplex algorithm
[Dantzig 1947]

ellipsoid method
[Khachiyan 1979]

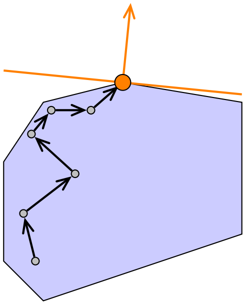
LP algorithms



simplex algorithm
[Dantzig 1947]



ellipsoid method
[Khachiyan 1979]



interior point
[Karmarkar 1984]

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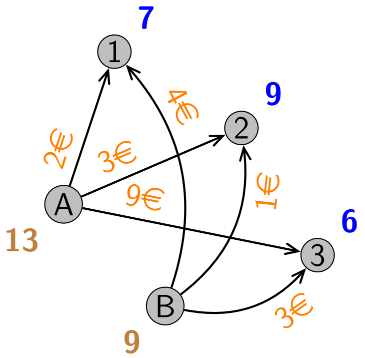
1984 N. Karmarkar: Interior Point Method/Barrier Algorithm

$\geq$ 1987 Primal-Dual Interior Point Algorithms

- basis for state-of-the-art interior point implementations
- for single, sparse LPs often faster than simplex



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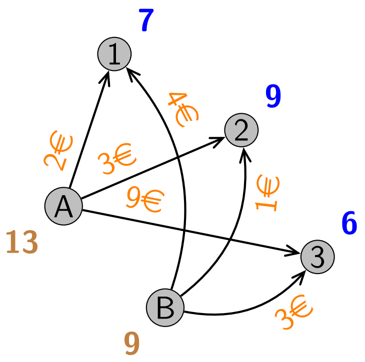
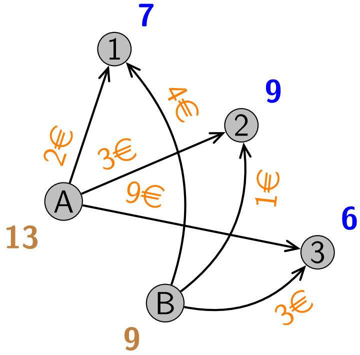


$$
\begin{aligned}
\min \quad & 2x_{A,1} + 3x_{A,2} + 9x_{A,3} + \\
& 4x_{B,1} + 1x_{B,2} + 3x_{B,3} \\
\text{s.t.} \quad & x_{A,1} + x_{A,2} + x_{A,3} = 13 \\
& x_{B,1} + x_{B,2} + x_{B,3} = 9 \\
& x_{A,1} + x_{B,1} = 7 \\
& x_{A,2} + x_{B,2} = 9 \\
& x_{A,3} + x_{B,3} = 6 \\
& x \geq 0
\end{aligned}
$$

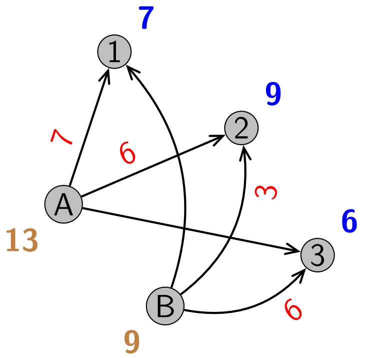


$$\min \quad 2x_{A,1} + 3x_{A,2} + 9x_{A,3} + 4x_{B,1} + 1x_{B,2} + 3x_{B,3}$$

$$\begin{aligned} \text{s.t.} \quad x_{A,1} + x_{A,2} + x_{A,3} &= 13 \\ x_{B,1} + x_{B,2} + x_{B,3} &= 9 \\ x_{A,1} + x_{B,1} &= 7 \\ x_{A,2} + x_{B,2} &= 9 \\ x_{A,3} + x_{B,3} &= 6 \\ x &\geq 0 \end{aligned}$$

Heuristic solution with objective value 53:

$A \to 1 = 7 \qquad A \to 2 = 6 \qquad B \to 2 = 3 \qquad B \to 3 = 6$



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Heuristic solution with objective value 53:

$A \rightarrow 1 = 7$ $\quad A \rightarrow 2 = 6$ $\quad B \rightarrow 2 = 3$ $\quad B \rightarrow 3 = 6$

Dual multipliers: proofing solution quality

$$
\begin{array}{lrrrrrrl}
\min & 2x_{A,1} & +3x_{A,2} & +9x_{A,3} & +4x_{B,1} & +1x_{B,2} & +3x_{B,3} & \\
\text{s.t.} & x_{A,1} & +x_{A,2} & +x_{A,3} & & & & =13 \\
& & & & x_{B,1} & +x_{B,2} & +x_{B,3} & =9 \\
& x_{A,1} & & & +x_{B,1} & & & =7 \\
& & x_{A,2} & & & +x_{B,2} & & =9 \\
& & & x_{A,3} & & & +x_{B,3} & =6 \\
& & & & & & x & \geq 0
\end{array}
$$

Dual multipliers: proofing solution quality

$$
\begin{array}{llcl}
\min & & 2x_{A,1}+3x_{A,2}+9x_{A,3}+4x_{B,1}+1x_{B,2}+3x_{B,3} & \\
\text{s.t.} & \color{red}{2\times} & x_{A,1}+x_{A,2}+x_{A,3} & =13 \\
& \color{red}{1\times} & x_{B,1}+x_{B,2}+x_{B,3} & =9 \\
& & x_{A,1}+x_{B,1} & =7 \\
& & x_{A,2}+x_{B,2} & =9 \\
& \color{red}{2\times} & x_{A,3}+x_{B,3} & =6 \\
& & x & \geq 0 \\
& \color{red}{+} & & \\
\hline
\color{red}{\Rightarrow \min \geq} & & 2x_{A,1}+2x_{A,2}+3x_{A,3}+1x_{B,1}+1x_{B,2}+3x_{B,3} & =\mathbf{47}
\end{array}
$$

$$
\begin{array}{llcccccccl}
\min & & 2x_{A,1} & +3x_{A,2} & +9x_{A,3} & +4x_{B,1} & +1x_{B,2} & +3x_{B,3} & & \\
\text{s.t.} & \color{red}{3\times} & x_{A,1} & +x_{A,2} & +x_{A,3} & & & & = & 13 \\
 & \color{red}{1\times} & & & & x_{B,1} & +x_{B,2} & +x_{B,3} & = & 9 \\
 & \color{red}{-1\times} & x_{A,1} & & & +x_{B,1} & & & = & 7 \\
 & & & x_{A,2} & & & +x_{B,2} & & = & 9 \\
 & \color{red}{2\times} & & & x_{A,3} & & & +x_{B,3} & = & 6 \\
 & & & & & & & x & \geq & 0 \\
 & \color{red}{+} & & & & & & & & \\
\hline
\color{red}{\Rightarrow \min \geq} & & 2x_{A,1} & +3x_{A,2} & +5x_{A,3} & & +1x_{B,2} & +3x_{B,3} & = & \mathbf{53}
\end{array}
$$

Solution is optimal!

Primal LP

$\min \left\{ c^T x \mid Ax = b, x \geq 0 \right\}$

Dual LP

$\max \left\{ b^T y \mid y^T A \leq c^T, y \in \mathbb{R}^m \right\}$

Simple observation: For any x, y feasible,

$$c^T x \geq y^T A x = b^T y.$$

Primal LP

$\min \{ c^T x \mid Ax = b, x \geq 0 \}$

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- **Weak Duality:**

$$\min \{ c^T x \mid Ax = b, x \geq 0 \} \geq \max \{ b^T y \mid y^T A \leq c^T, y \in \mathbb{R}^m \}$$

Primal LP

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- **Weak Duality:**

$$\min \left\{ c^T x \mid Ax = b, x \geq 0 \right\} \geq \max \left\{ b^T y \mid y^T A \leq c^T, y \in \mathbb{R}^m \right\}$$

- **Strong Duality:**

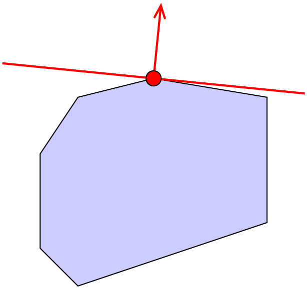
$$\min \left\{ c^T x \mid Ax = b, x \geq 0 \right\} = \max \left\{ b^T y \mid y^T A \leq c^T, y \in \mathbb{R}^m \right\}$$

Consider n variables, m constraints:

$$\min \left\{ c^T x \mid Ax = b, x \geq 0 \right\}$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

- If optimal: there always exists an optimal vertex solution.



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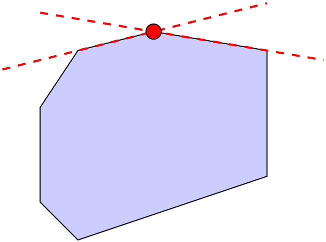
- Vertices uniquely determined by n tight inequalities:



- In standard from that means
  - m equality constraints $Ax = b$
  - $n - m$ tight bounds $x_i = 0$

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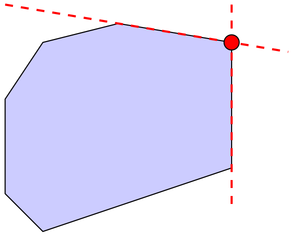

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Primal solution

- Fix $n - m$ variables: $x_i = 0$ for $i \in \mathcal{N} \subseteq \{1, \ldots, n\}$
- m variables remain: x_i for $i \in \mathcal{B} = \{1, \ldots, n\} \setminus \mathcal{N}$
- Solve linear system with m equations, m variables:

$$Ax = b \ \rightsquigarrow \ A_{\mathcal{B}} x_{\mathcal{B}} = b \ \rightsquigarrow \ x_{\mathcal{B}} = A_{\mathcal{B}}^{-1} b$$

Primal solution

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Dual multipliers

- Globally: find y such that $y^T A \leq c^T$
- Locally: ignore fixed variables and solve

$$y^T A_{\mathcal{B}} = c_{\mathcal{B}}^T \rightsquigarrow y^T = c_{\mathcal{B}}^T A_{\mathcal{B}}^{-1}$$

Primal solution

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- Globally: find y such that $y^T A \leq c^T$
- Locally: ignore fixed variables and solve

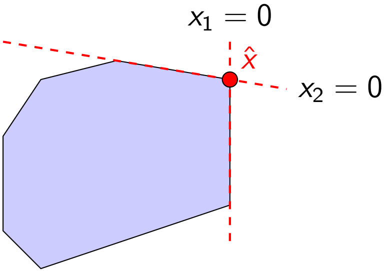
$$y^T A_{\mathcal{B}} = c_{\mathcal{B}}^T \rightsquigarrow y^T = c_{\mathcal{B}}^T A_{\mathcal{B}}^{-1}$$

Basic solution = discrete basis $\mathcal{B}$ + primal sol. x + dual mult. y

- In theory: could enumerate $\binom{n}{m}$ basic solutions.

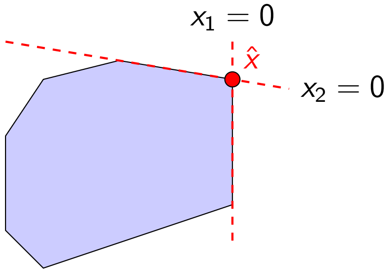
- $x_\mathcal{B}$ is a function of $x_\mathcal{N}$:

$$A_\mathcal{B} x_\mathcal{B} + A_\mathcal{N} x_\mathcal{N} = b$$



- $x_{\mathcal{B}}$ is a function of $x_{\mathcal{N}}$:

$$A_{\mathcal{B}} x_{\mathcal{B}} = b - A_{\mathcal{N}} x_{\mathcal{N}}$$



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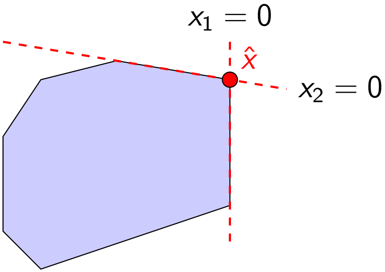
$$x_{\mathcal{B}} = A_{\mathcal{B}}^{-1}(b - A_{\mathcal{N}} x_{\mathcal{N}})$$



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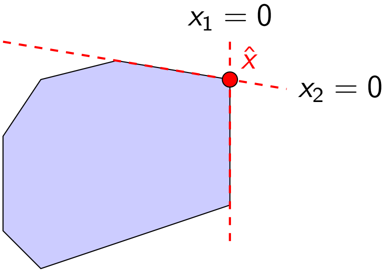

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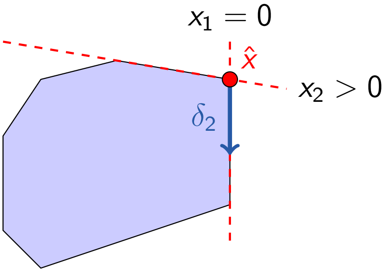

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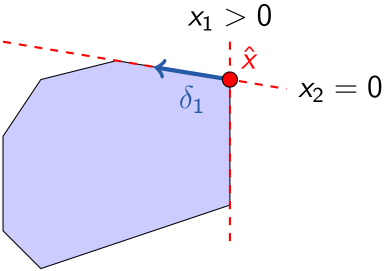

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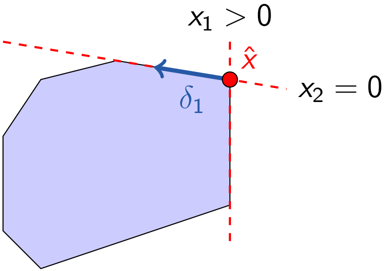

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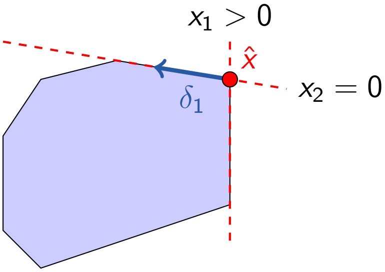

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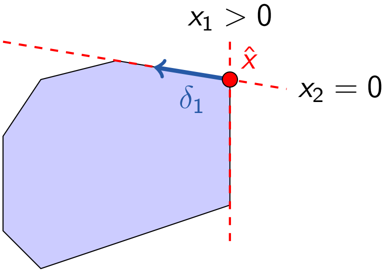

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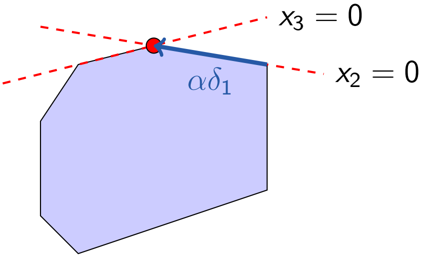

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Dual simplex

- basic procedures as in primal simplex
- maintains dual feasibility and moves towards primal feasibility
- objective value increases towards optimum
- typically **very** few iterations to re-optimize

Dual simplex algorithm

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Repeat while $c^T x < z^*$ obj. limit

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Summary

- Discrete and continuous:
  - vertices are uniquely determined by n equalities
  - if optimal: there always exists an optimal vertex solution
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Further aspects:

- general bounds: $lb_i \leq x_i \leq ub_i$
- feasible starting basis for simplex ("phase 1"), pricing strategies, linear algebra tricks, ...
- exponential worst-case complexity of simplex vs. performance in practice
- interior point algorithms
- algorithms for specially structured LPs: networks, ...

Integer Programming for Constraint Programmers

Linear program

Objective function:

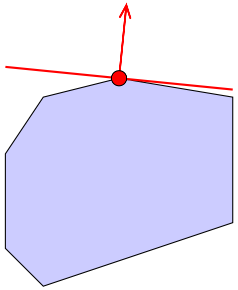
▷ linear function

Feasible set:

▷ described by linear constraints

Variable domains:

▷ real values



$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \in \mathbb{R}^n_{\geq 0} \end{aligned}$$

▷ convex set

▷ “basic” solutions

Integer Program

Objective function:

- ▷ linear function

Feasible set:

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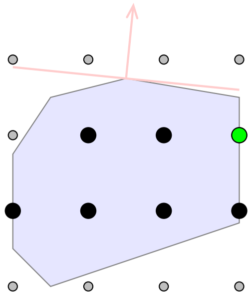
Variable domains:

- ▷ integer values



$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \leq b \\ & x \in \mathbb{Z}_{\geq 0} \end{aligned}$$

- ▷ not even connected
- ▷ $\mathcal{NP}$-hard problem

An incomplete history on integer programming

Cutting plane algorithm

- R. E. Gomory, “Outline of an algorithm for integer solutions to linear programs”. Bull. AMS 64, 1958, pp. 275–278.

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Branch-and-bound

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- R. J. Dakin, “A tree-search algorithm for mixed integer programming problems”. The Computer Journal, Volume 8, 1965, pp. 250–255
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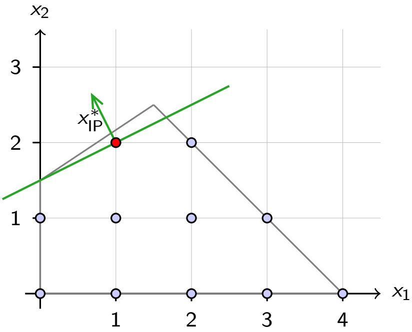
Branch-and-cut

- Grötschel, Jünger, Reinelt (1984, 1985, 1987)
- Padberg, Rinaldi (1991)

General cutting plane method

$$\mathcal{F}_{\text{IP}} := \{x \in \mathbb{Z}^n_+ : Ax \leq b\}$$

$$\mathcal{F}_{\text{LP}} := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$$



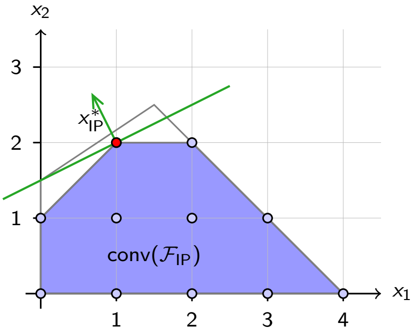
$\min\{c^T x : x \in \mathcal{F}_{\text{IP}}\}$

Observation

- $\text{conv}(\mathcal{F}_{\text{IP}})$ is a polyhedron
- IP could be formulated as LP

Problems with $\text{conv}(\mathcal{F}_{\text{IP}})$:

- linear description not known
- large nr. of constraints needed



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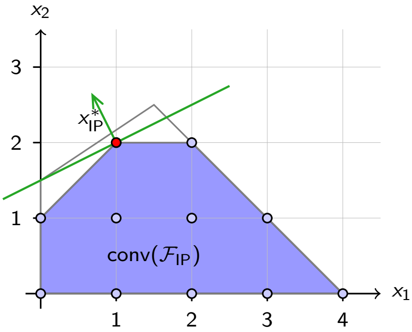

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$$\mathcal{F}_{\mathrm{LP}} \quad \supseteq \qquad \mathcal{F} \quad \supseteq \qquad \mathrm{conv}(\mathcal{F}_{\mathrm{IP}})$$

$$\min\{c^T x : x \in \mathcal{F}_{\mathrm{LP}}\} \leq \min\{c^T x : x \in \mathcal{F}\} = \min\{c^T x : x \in \mathrm{conv}(\mathcal{F}_{\mathrm{IP}})\}$$

General cutting plane method

Algorithm

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3. If $x^* \in \mathcal{F}_{\text{IP}}$: Stop
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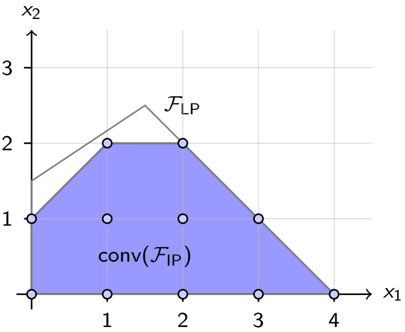

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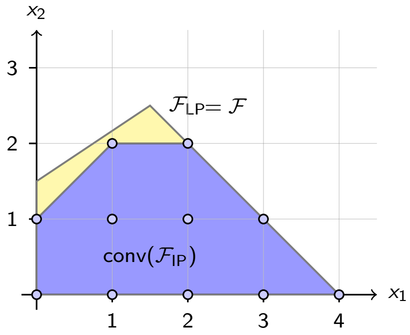

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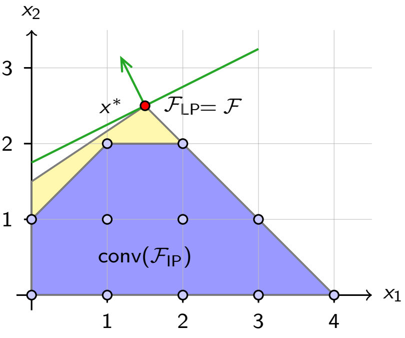

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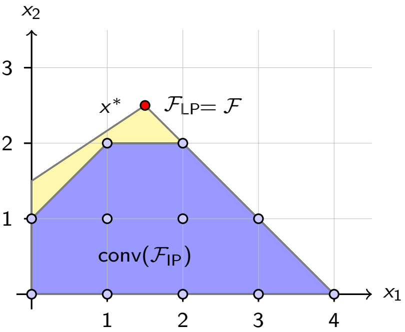

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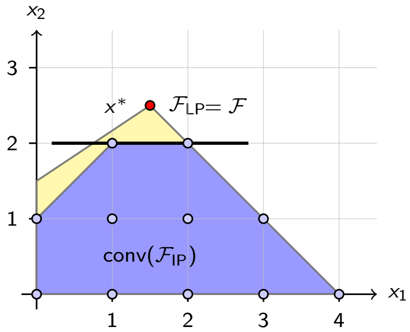

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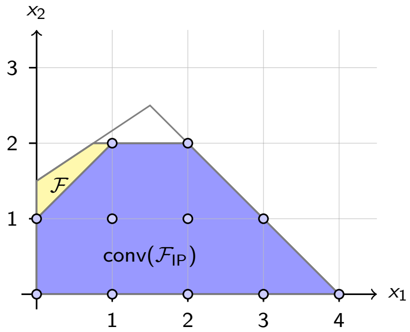

$\min\{c^T x : x \in \mathrm{conv}(\mathcal{F}_{\mathrm{IP}})\}$

General cutting plane method

Algorithm

1. $\mathcal{F} \leftarrow \mathcal{F}_{\text{LP}}$
2. Solve

$$\begin{array}{rr} \min & c^T x \\ \text{s.t.} & x \in \mathcal{F} \end{array}$$

3. If $x^* \in \mathcal{F}_{\text{IP}}$: Stop
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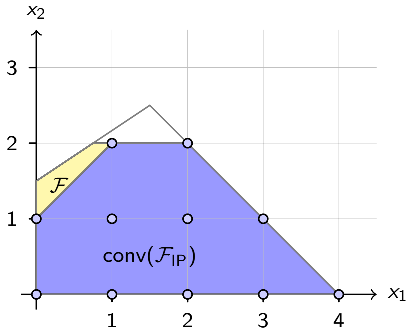

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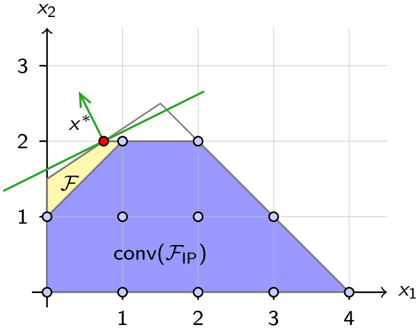

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- Resolving is cheap since dual feasible (hot start)

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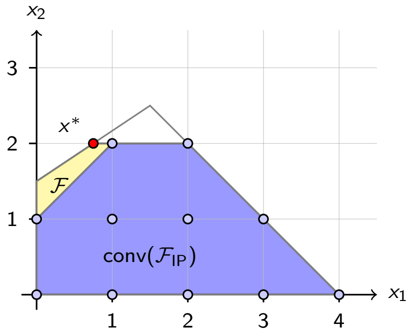

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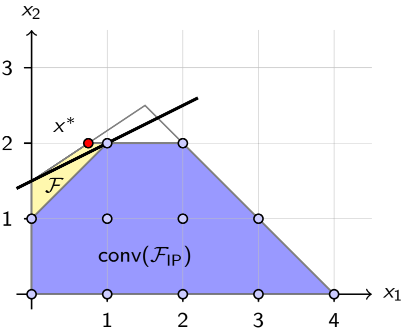

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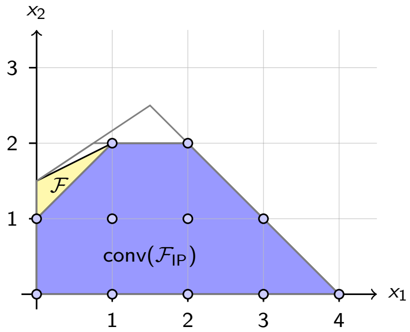

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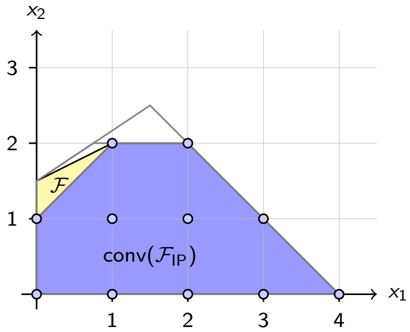

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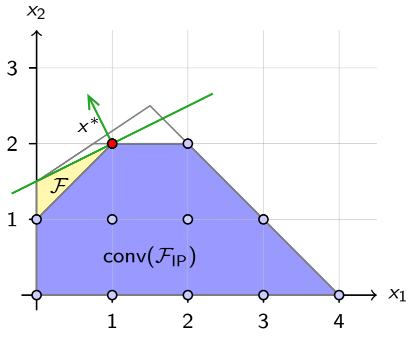

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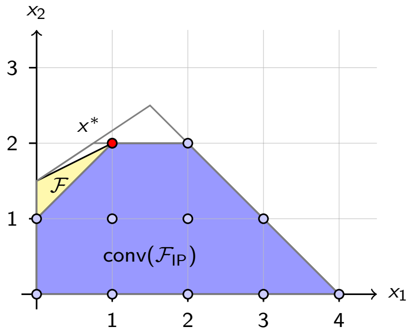

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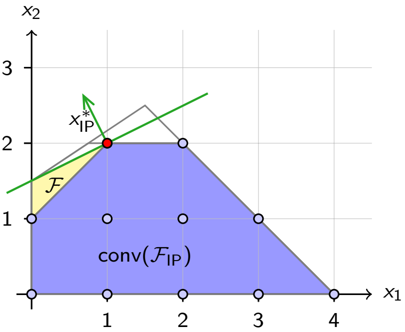

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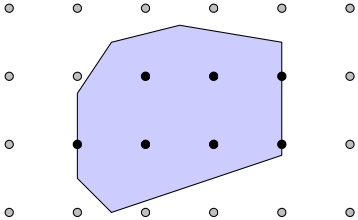

LP-based branch-and-bound (colorful picture)

Steps

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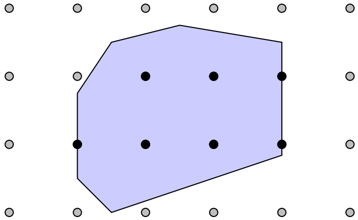

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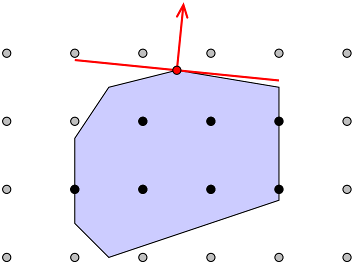

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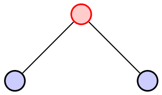

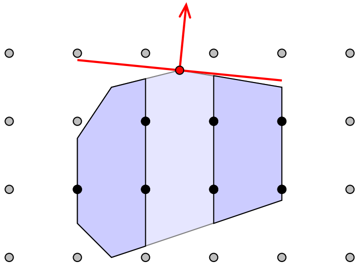

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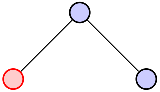

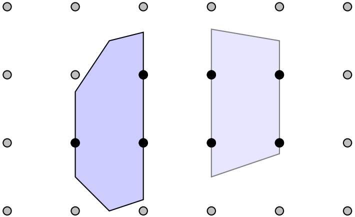

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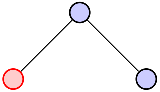

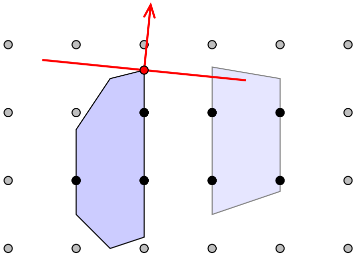

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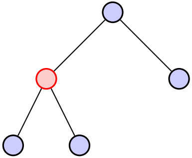

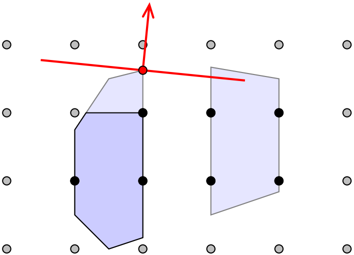

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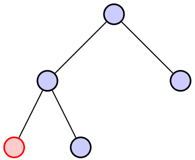

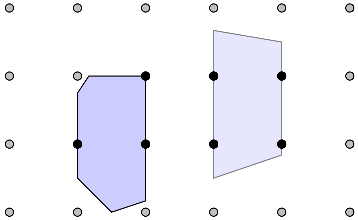

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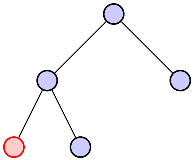

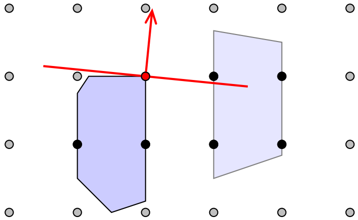

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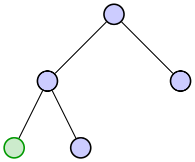

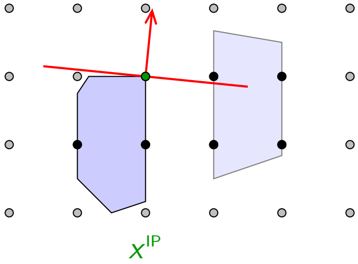

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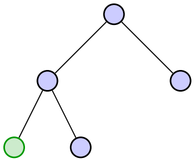

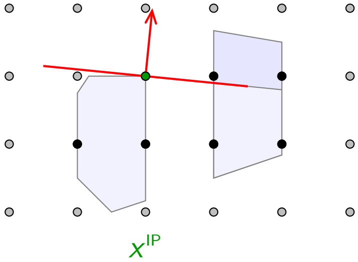

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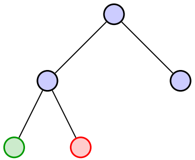

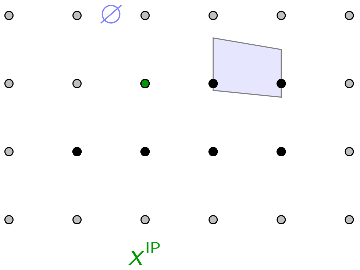

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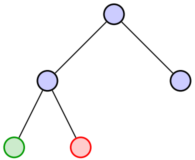

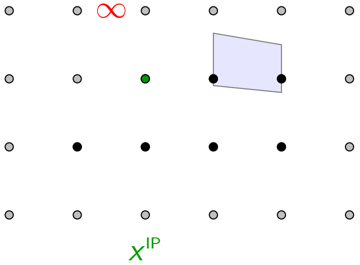

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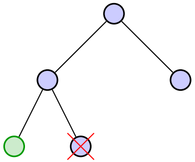

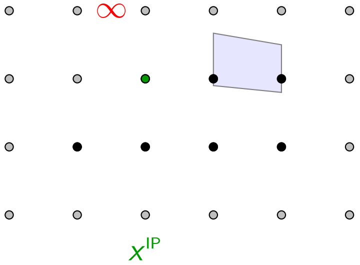

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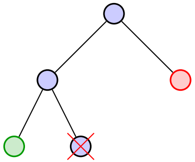

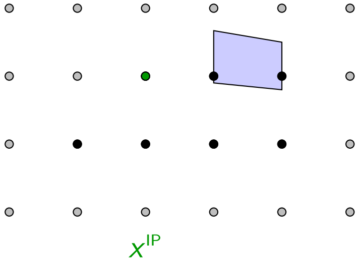

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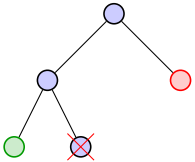

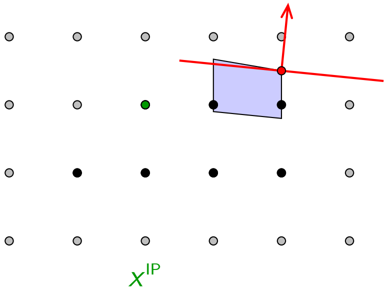

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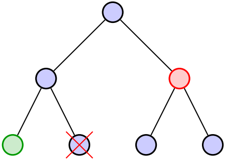

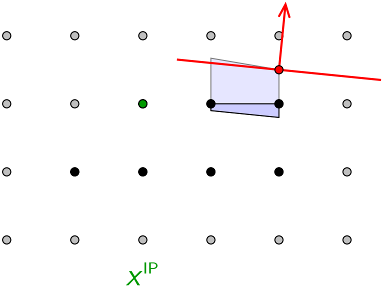

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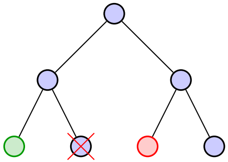

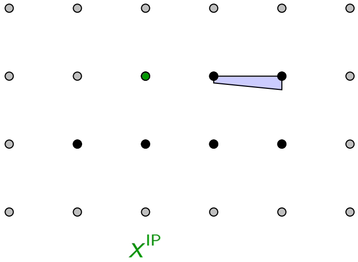

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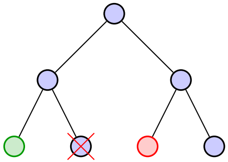

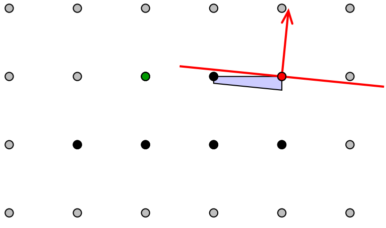

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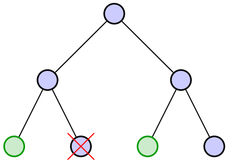

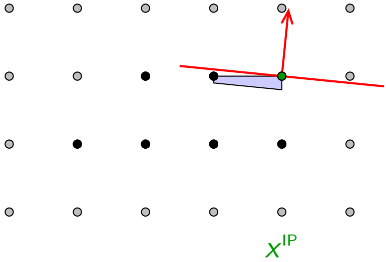

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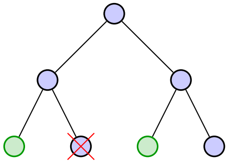

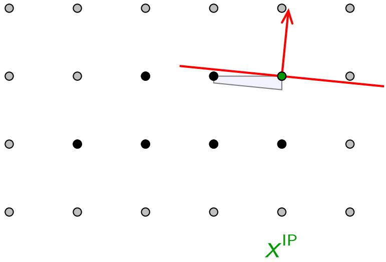

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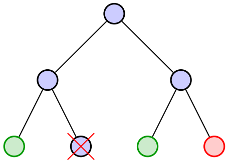
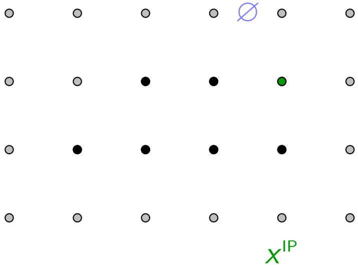
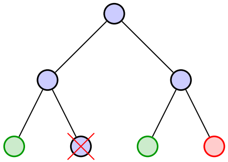


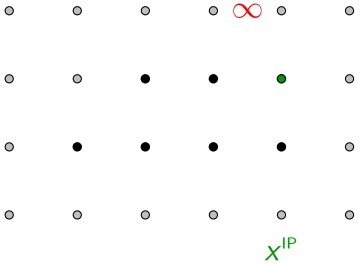


LP-based branch-and-bound (colorful picture)

Steps

1. Abort criterion
2. Node selection
3. Solve relaxation
4. Bounding
5. Feasibility check
6. Branching





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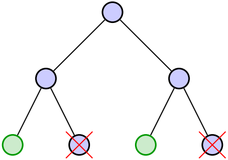

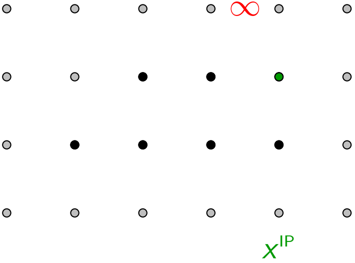

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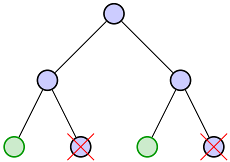
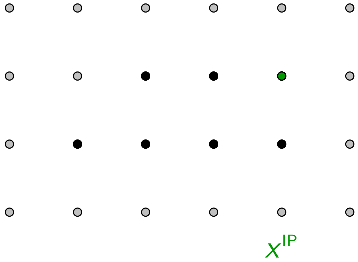
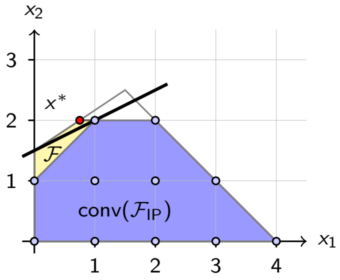
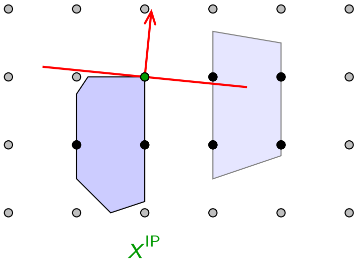






$\min\{c^T x : x \in \text{conv}(\mathcal{F}_{\text{IP}})\}$

cutting planes



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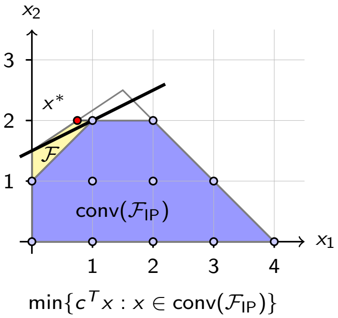

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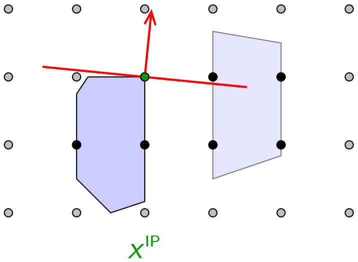
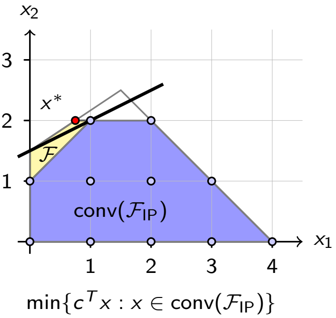


branch-and-bound

- Both approaches solve an initial linear program.

Solving an integer program



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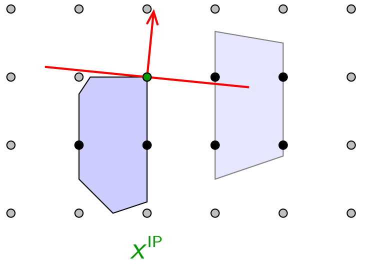

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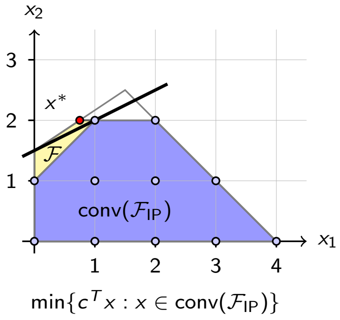
- Classically using simplex algorithm for efficent hot start



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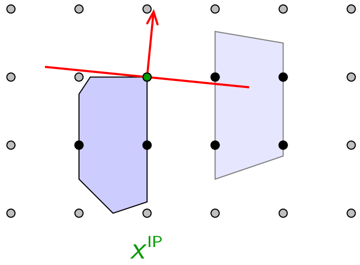

branch-and-bound

- Both approaches solve an initial linear program.
- Classically using simplex algorithm for efficent hot start
  - A cutting plane or a bound change is an additional row (linear constraint).

Branch-and-cut $=$ Branch-and-bound $+$ Cutting planes

Branch-and-cut = Branch-and-bound + Cutting planes

- Use branch-and-bound as global solver paradigm.
- Perform limited cutting plane generation within each search node.
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- Pure branch-and-bound fails in general
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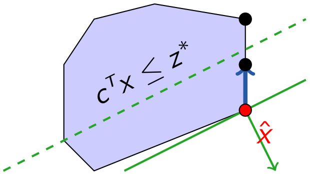
- Branch-and-cut fails later
  - still exponential search tree
  - but shifts the exponential grow significantly

- How can a linear program be solved?
- How can an integer program be solved?
- For what is the linear programming relaxation used within an integer programming solver?

Reduced cost propagation

- z^*: best objective value $\rightarrow$ $c^T x \leq z^*$
- $\hat{x}$: LP optimum
- For variables x_i with reduced cost $r_i \neq 0$
  - variables are not in the basis
  - variables sitting on one of their bounds
  - $r_i > 0 \rightarrow \hat{x}_i = \mathrm{lb}_i$ (lower bound)
  - $r_i < 0 \rightarrow \hat{x}_i = \mathrm{ub}_i$ (upper bound)



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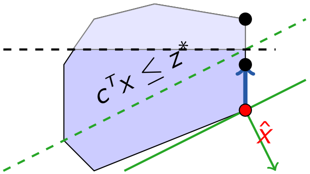

Case 1 $r_i > 0$:

$$c^T\hat{x} + r_i\,(x_i - \mathrm{lb}_i) \leq z^* \Leftrightarrow x_i \leq \frac{z^* - c^T\hat{x}}{r_i} + \mathrm{lb}_i$$

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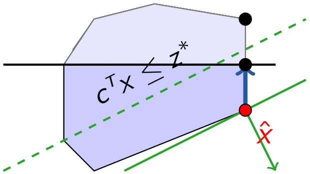

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$$c^T \hat{x} + r_i (x_i - \text{lb}_i) \leq z^* \Leftrightarrow x_i \leq \frac{z^* - c^T \hat{x}}{r_i} + \text{lb}_i \Rightarrow x_i \leq \left\lfloor \frac{z^* - c^T \hat{x}}{r_i} + \text{lb}_i \right\rfloor$$

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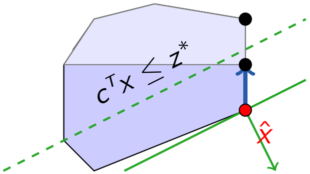

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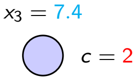
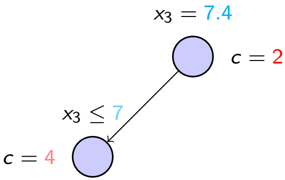
Estimating the objective



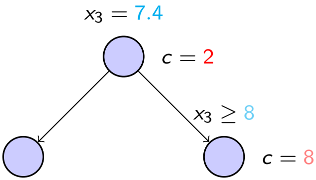
Estimating the objective

- objective gain per unit:
  - $\zeta^{-}(x_3) = \frac{4-2}{7.4-7} = \frac{2}{0.4} = 5$



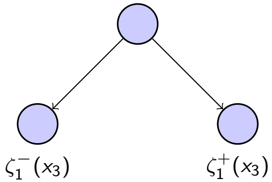
Estimating the objective

- objective gain per unit:
  - $\zeta^+(x_3) = \frac{8-2}{8-7.4} = \frac{6}{0.6} = 10$



Estimating the objective

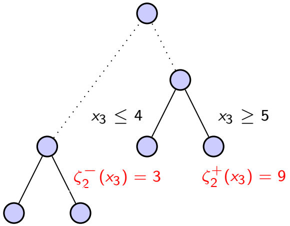
- objective gain per unit:
  - $\zeta_1^-(x_3) = 5$, $\zeta_1^+(x_3) = 10$



Estimating the objective

▷ objective gain per unit:

- $\zeta_1^-(x_3) = 5$, $\zeta_1^+(x_3) = 10$
- other values at other nodes

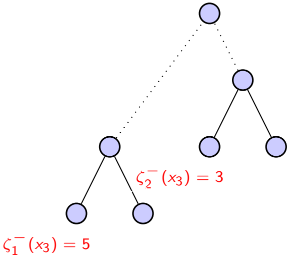


Estimating the objective

- objective gain per unit:
  - $\zeta_1^-(x_3) = 5$, $\zeta_1^+(x_3) = 10$
  - other values at other nodes
- pseudocosts:

  average objective gain

  $\psi^-(x_3) = \frac{\zeta_1^-(x_3) + \ldots + \zeta_n^-(x_3)}{n} = \frac{5+3}{2} = 4$



Estimating the objective

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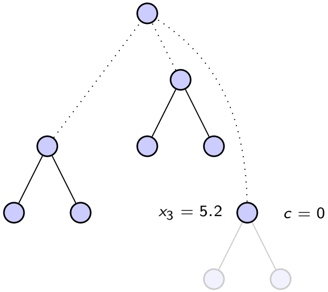
  $\psi^-(x_3) = 4$, $\psi^+(x_3) = 9.5$
- estimate increase of objective
  by pseudocosts and fractionality:



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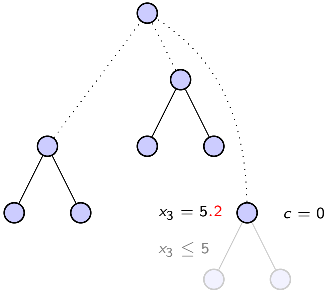

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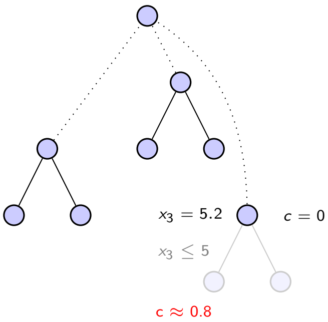
  $\psi^-(x_3) \cdot \mathrm{frac}(x_3) = 4 \cdot 0.2 = 0.8$,



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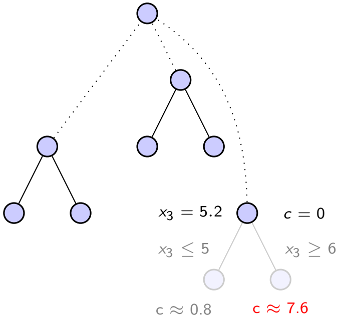
- estimate increase of objective
  by pseudocosts and fractionality:
  $\psi^-(x_3) \cdot \mathrm{frac}(x_3) = 4 \cdot 0.2 = 0.8$,
  and $\psi^+(x_3)(1 - \mathrm{frac}(x_3)) = 7.6$



RENS – Relaxation Enforced Neighborhood Search

Idea: Search the vicinity of a relaxation solution

Algorithm

1. $\bar{x} \leftarrow$ LP optimum;
2. Fix all integral variables:
   $x_i := \bar{x}_i \quad$ for all $i : \bar{x}_i \in \mathbb{Z}$;
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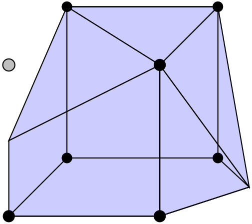

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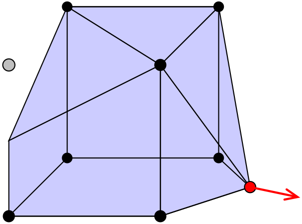

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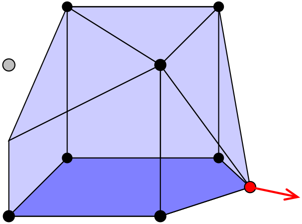

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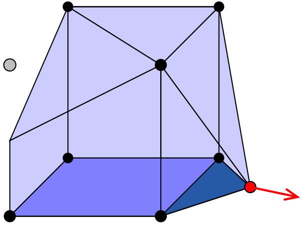

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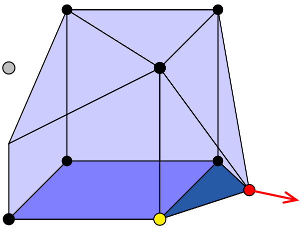

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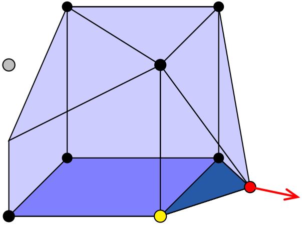

Crucial point: Does not need a feasible start solution

Integer Programming for Constraint Programmers

Linear relaxation

- gives a global view
- provides a proven dual bound for the original problem
  - quality guarantee
- can be used for more than getting a dual bound
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As in CP, the chosen model has a huge impact on the performance of a solver

An incomplete list of IP Solvers

Non-commercial solvers

- CBC (IBM) `https://projects.coin-or.org/Cbc`
- GLPK `http://www.gnu.org/s/glpk/`
- LPSOLVE `http://lpsolve.sourceforge.net/`
- SCIP `http://scip.zib.de`
- SYMPHONY `https://projects.coin-or.org/SYMPHONY`

Commercial solvers

- CPLEX (IBM) `http://www.cplex.com`
- GUROBI `http://www.gurobi.com`
- MOPS `http://www.mops-optimizer.com`
- MOSEK `http://www.mosek.com`
- XPRESS (Fico) `http://www.fico.com`

Integer Programming for Constraint Programmers

Questions

Tutorial

Integer Programming for Constraint Programmers

Ambros Gleixner and Stefan Heinz

Zuse Institute Berlin (ZIB)

Chris Beck, Timo Berthold, and Kati Wolter

DFG Research Center MATHEON
Mathematics for key technologies