Optimal Linear LQG Control Over Lossy Networks Without Packet Acknowledgment

Bruno Sinopoli[1], Luca Schenato[2], Massimo Franceschetti[3], Kameshwar Poolla[2], and Shankar Sastry[2],
[1]Department of Electrical Engineering, UC Berkeley, Berkeley, CA, USA
[2]Department of Information Engineering, University of Padova, Italy
[3]Department of Electrical and Computer Engineering UC San Diego, LA Jolla, CA, USA
{sinopoli,poolla,sastry}@eecs.berkeley.edu
schenato@dei.unipd.it, massimo@ece.ucsd.edu

***Abstract*— This paper is concerned with control applications over lossy data networks. Sensor data is transmitted to an estimation-control unit over a network, and control commands are issued to subsystems over the same network. Sensor and control packets may be randomly lost according to a Bernoulli process. In this context, the discrete-time Linear Quadratic Gaussian (LQG) optimal control problem is considered.**

It is known that in the scenario described above, and for protocols for which there is no acknowledgement of successful delivery of control packets (e.g. UDP-like protocols), the LQG optimal controller is in general nonlinear. However, the simplicity of a linear sub-optimal solution is attractive for a variety of applications. Accordingly, this paper characterizes the optimal linear static controller and compares its performance to the case when there is acknowledgement of delivery of packets.

I. Introduction

Today, an increasing number of applications demands remote control of plants over unreliable networks. The recent development of sensor web technology [1] enables the development of wireless sensor networks that can be immediately used for estimation and control. In these systems issues of communication delay, data loss, and time-synchronization play critical roles. Communication and control become tightly coupled and these two issues cannot be addressed independently. The goal of this paper is to provide some partial answers to the question of how control loop performance is affected by communication constraints and what are the basic system-theoretic implications of using unreliable networks for control. This requires a generalization of classical control techniques that explicitly takes into account the stochastic nature of the communication channel.

We consider a generalized formulation of the Linear Quadratic Gaussian (LQG) optimal control problem by modeling the arrival of both observations and control packets as random processes whose parameters are related to the characteristics of the communication channel. Accordingly, two independent Bernoulli processes are considered, with parameters $\overline{\gamma}$ and $\overline{\nu}$, that govern packet losses between the sensors and the estimation-control unit, and between the latter and the actuation points (see Figure 1).

In our analysis, we distinguish between two classes of protocols. The distinction resides simply in the availability of packet acknowledgements. Adopting the framework proposed by Imer *et al.* [2], we will refer therefore to TCP-like protocols if packet acknowledgements are available and to UDP-like protocols otherwise.

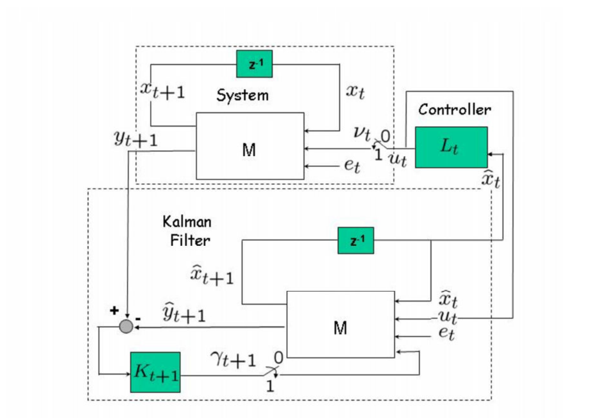

Fig. 1. **Overview of the system.** We study the statistical convergence properties of the expected state covariance of the discrete time LQG control system, when both the observation and the control signal, transmitted over an unreliable communication channel, can be lost at each time step with probability $1-\bar{\gamma}$ and $1-\bar{\nu}$ respectively.

Our previous results on this topic [3], [4], [5] are summarized in Figure 2. We have shown the existence of a critical domain of values for the parameters of the Bernoulli arrival processes, $\overline{\nu}$ and $\overline{\gamma}$, outside which a transition to instability occurs and the optimal controller fails to stabilize the system. In particular, we have shown that under TCP-like protocols the critical arrival probabilities for the control and observation channel are independent of each other. This is another consequence of the fact that the separation principle holds for these protocols. A more involved situation regards UDP-like protocols. In this case the critical arrival probabilities for the control and observation channels are coupled. The stability domain and the performance of the optimal controller degrade considerably as compared with TCP-like protocols as shown in Figure 2.

We have also shown that for the TCP-like case the classic separation principle holds, and consequently the controller and estimator can be designed independently. Moreover, the optimal controller is a linear function of the state. In sharp contrast, for the UDP-like case, the optimal controller is in general non-linear. In this case, a natural sub-optimal solution is to use the optimal static linear gain. This is particularly attractive for sensor networks, where simplicity of implementation and complexity issues are a primary concern. Accordingly, in this paper we focus on the performance of this UDP controller and compare it with the optimal one in the TCP case.

First, we formulate the problem of finding the optimal linear controller as a non-convex optimization problem. Then, we write, using Lagrange multipliers, a solution to a necessary condition for the optimum. Using a result of De Koning [6], we determine when such condition is also sufficient. We provide some numerical convergence results for the scalar case, and finally we show that the performance of the obtained solution is comparable to the one of the optimal controller in the TCP case.

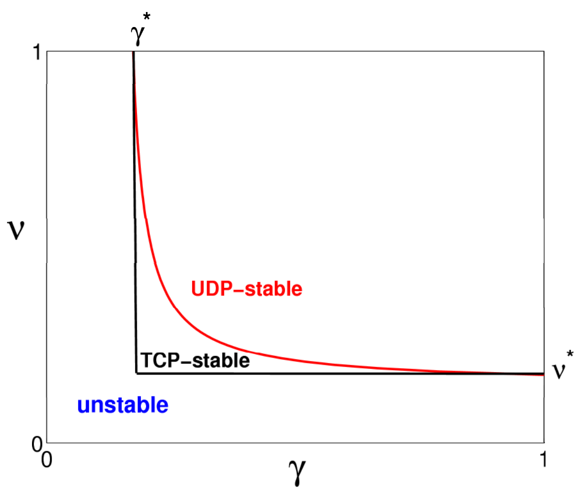

Fig. 2. Region of stability for UDP-like and TCP-like optimal control relative to measurement packet arrival probability γ, and the control packet arrival probability ν.

We now wish to mention some closely related research. Study of stability of dynamical systems where components are connected asynchronously via communication channels has received considerable attention in the past few years and our contribution can be put in the context of the previous literature. In [7] and [8], the authors proposed to place an estimator, i.e. a Kalman filter, at the sensor side of the link without assuming any statistical model for the data loss process. In [9], Smith *et al.* considered a suboptimal but computationally efficient estimator that can be applied when the arrival process is modeled as a Markov chain, which is more general than a Bernoulli process. Drew et al [10] analyze the problem of designing a controller over a wireless LAN. Control design has been investigated in the context of Cross Layer Design by Liu et al [11]. Finally, Elia [12][13] proposed to model the plant and the controller as deterministic time invariant discrete-time systems connected to zero-mean stochastic structured uncertainty. The variance of the stochastic perturbation is a function of the Bernoulli parameters, and the controller design is posed an an optimization problem to maximize mean-square stability of the closed loop system. This approach allows analysis of Multiple Input Multiple Output (MIMO) systems with many different controller and receiver compensation schemes [12], however, it does not include process and observation noise and the controller is restricted to be time-invariant, hence sub-optimal. The remainder of this paper is organized as follows. Section 2 provides the problem formulation. In Section 3 we summarize our previous results that are needed to understand the new contribution. In Section 4 we consider the optimization problem leading to the optimal linear UDP controller and discusses a solution to a weaker, necessary solution for optimality. Section 5 shows the results and compares them to the optimal TCP controller (which is always linear). Finally, Section 6 draws conclusions and outlines the agenda for future work.

II. Problem formulation

Consider the following linear stochastic system with intermittent observation and control packets:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k && (1)\\ u_k^a &= \nu_k u_k^c && (2)\\ y_k &= \gamma_k C x_k + v_k, && (3) \end{aligned}$$

where u_k^a is the control input to the actuator, u_k^c is the desired control input computed by the controller, (x_0, w_k, v_k) are Gaussian, uncorrelated, white, with mean $(\bar{x}_0, 0, 0)$ and covariance (P_0, Q, R) respectively, and (γ_k, ν_k) are i.i.d. Bernoulli random variables with $P(\gamma_k = 1) = \bar{\gamma}$ and $P(\nu_k = 1) = \bar{\nu}$. The stochastic variable ν_k models the loss of packets between the controller and the actuator: if the packet is correctly delivered then $u_k^a = u_k^c$, otherwise if it is lost then the actuator does nothing, i.e. $u_k^a = 0$. This compensation scheme is summarized by Equation (2). The stochastic variable γ_k models the packet loss between the sensor and the controller: if the packet is delivered then $y_k = Cx_k + v_k$, otherwise if it is lost then the controller reads pure noise, i.e. $y_k = v_k$. This observation model is summarized by Equation (3). A different observation formalism was proposed in [14], where the missing observation was modeled as an observation for which the measurement noise had infinite covariance. It is possible to show that both models are equivalent, but the one considered in this paper has the advantage to give rise to simpler analysis. This arises from the fact that when no packet is delivered, then the optimal estimator does not use the observation y_k at all, therefore its value is irrelevant. Let us define the following information sets:

$$\mathcal{I}_k = \begin{cases} \mathcal{F}_k \triangleq \{\mathbf{y}^k, \boldsymbol{\gamma}^k, \boldsymbol{\nu}^{k-1}\}, & \text{TCP-like} \\ \mathcal{G}_k \triangleq \{\mathbf{y}^k, \boldsymbol{\gamma}^k\}, & \text{UDP-like} \end{cases} \quad (4)$$

where $\mathbf{y}^k = (y_k, y_{k-1}, \ldots, y_1)$, $\boldsymbol{\gamma}^k = (\gamma_k, \gamma_{k-1}, \ldots, \gamma_1)$, and $\boldsymbol{\nu}^k = (\nu_k, \nu_{k-1}, \ldots, \nu_1)$. Consider also the following cost function:

$$\begin{aligned} &J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0) = \\ &= \mathbb{E}\Big[x_N' W_N x_N + \textstyle\sum_{k=0}^{N-1}(x_k' W_k x_k + \nu_k u_k' U_k u_k)\Big| \mathbf{u}^{N-1}, \bar{x}_0, P_0\Big] \end{aligned} \quad (5)$$

where $\mathbf{u}^{N-1} = (u_{N-1}, u_{N-2}, \ldots, u_1)$. Note that we are weighting the input only if it is successfully received at the plant. In fact, if it is not received, the plant applies zero input and therefore there is no energy expenditure.

We now look for a control input sequence $\mathbf{u}^{*N-1}$ as a function of the admissible information set $\mathcal{I}_k$, i.e. $u_k = g_k(\mathcal{I}_k)$, that minimizes the functional defined in Equation (5), i.e.

$$J_N^*(\bar{x}_0, P_0) \triangleq \min_{\mathbf{u_k}=\mathbf{g}_k(\mathcal{I}_k)} J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0), \qquad (6)$$

where $\mathcal{I}_k = \{\mathcal{F}_k, \mathcal{G}_k\}$ is one of the sets defined in Equation (4). The set $\mathcal{F}$ corresponds to the information provided under an acknowledgement-based communication protocols (TCP-like) in which successful or unsuccessful packet delivery at the receiver is acknowledged to the sender within the same sampling time period. The set $\mathcal{G}$ corresponds to the information available at the controller under communication protocols in which the sender receives no feedback about the delivery of the transmitted packet to the receiver (UDP-like). The UDP-like schemes are simpler to implement than the TCP-like schemes from a communication standpoint. However the price to pay is a less rich set of information.

III. Previous Work

Before introducing new results, it is necessary to review recently published results [3], [4], [5], for both the TCP-like and the UDP-like case.

A. TCP-like case: estimator and controller design

The LQG control problem for the TCP-like case has been solved in full generality in [3].

Finite Horizon LQG. The main results are summarized below:

- The separation Principle holds under TCP-like communication, since the optimal estimator is independent of the control input u_k.
- The optimal estimator gain K_k is time-varying and stochastic since it depends on the past observation arrival sequence $\{\gamma_j\}_{j=1}^k$.
- The Optimal LQG controller is a linear function of estimated state $\hat{x}_{k|k}$, i.e. $u_k = L_k \hat{x}_{k|k}$.
- The final cost cannot be computed explicitly, since it depends on the realization of ν_t and γ_t, but can be analytically bounded.

Infinite Horizon LQG. Consider the system (1)-(3) with the following additional hypothesis: $W_N = W_k = W$ and $U_k = U$. Moreover, let (A, B) and $(A, Q^{\frac{1}{2}})$ be controllable, and let (A, C) and $(A, W^{\frac{1}{2}})$ be observable. There exist critical arrival probabilities ν_c and γ_c , such that, for $\bar{\nu} > \nu_c$ and $\bar{\gamma} > \gamma_c$:

(a) The infinite horizon optimal controller gain is constant:

$$\lim_{k\to\infty} L_k = L_\infty = -(B'S_\infty B + U)^{-1} B' S_\infty A \qquad (7)$$

(b) The infinite horizon optimal estimator gain K_k is stochastic and time-varying since it depends on the past observation arrival sequence $\{\gamma_j\}_{j=1}^k$.

(c) The expected minimum cost can be bounded by two deterministic sequences:

$$\frac{1}{N} J_N^{min} \leq \frac{1}{N} J_N^* \leq \frac{1}{N} J_N^{max} \qquad (8)$$

where J_N^{min}, J_N^{max} converge to the following values:

$$\begin{aligned} J_\infty^{max} &\triangleq \lim_{N\to+\infty} \tfrac{1}{N} J_N^{max} \\ &= \text{trace}((A'S_\infty A + W - S_\infty)(\widehat{P}_\infty - \\ &\quad +\bar{\gamma}\widehat{P}_\infty C'(C\widehat{P}_\infty C' + R)^{-1} C\widehat{P}_\infty)) + \text{trace}(S_\infty Q) \\ J_\infty^{min} &\triangleq \lim_{N\to+\infty} \tfrac{1}{N} J_N^{min} \\ &= (1-\bar{\gamma})\text{trace}\left((A'S_\infty A + W - S_\infty)\widetilde{P}_\infty\right) + \\ &\quad +\text{trace}(S_\infty Q), \end{aligned}$$

and the matrices $S_\infty, \overline{P}_\infty, \underline{P}_\infty$ are the positive definite solutions of the following equations:

$$S_\infty = A'S_\infty A + W - \bar{\nu}\, A'S_\infty B(B'S_\infty B + U)^{-1} B'S_\infty A$$

$$\overline{P}_\infty = A\overline{P}_\infty A' + Q - \bar{\gamma}\, A\overline{P}_\infty C'(C\overline{P}_\infty C' + R)^{-1} C\overline{P}_\infty A'$$

$$\underline{P}_\infty = (1-\bar{\gamma})A\underline{P}_\infty A' + Q$$

The critical probability ν_c can be numerically computed via the solution of a quasi-convex LMIs optimization problem, as shown in [3]. Also the following analytical bounds are provided:

$$\begin{aligned} p_{min} &\leq \nu_c, \gamma_c \leq p_{max} \\ p_{min} &\triangleq 1 - \tfrac{1}{\max_i |\lambda_i^u(A)|^2} \\ p_{max} &\triangleq 1 - \tfrac{1}{\prod_i |\lambda_i^u(A)|^2}, \end{aligned}$$

where $\lambda_i^u(A)$ are the unstable eigenvalues of A. Moreover, $\nu_c = p_{min}$ when B is square and invertible [15] , and $\nu_c = p_{max}$ when B is rank one [13]. Dually, $\gamma_c = p_{min}$ when C is square and invertible, and $\gamma_c = p_{max}$ when C is rank one.

B. UDP-like case: estimator and controller design

As stated above, the LQG optimal control problem for the UDP-like case presents analytical complications. The lack of acknowledgement of the arrival of a control packet has dramatic effects on the controller design. Complete derivations for this case are presented in [4]. Here is a summary of them:

- The innovation step in the design of the estimator now explicitly depends on the input u_k ;
- the **separation principle** is not valid anymore in this setting.
- the LQG optimal control feedback $u_k = g_k^*(\mathcal{G}_k)$ with horizon $N \geq 2$ that minimizes the functional (5) under UDP-like communication is, in general, a **nonlinear** function of information set $\mathcal{G}_k$.
- In the particular case where the full state can be observed whenever the observation packet arrives, i.e. C is invertible and $R = 0$, the LQG controller is linear in the state, although the separation principle does not hold.

Our experience in the design of control systems over wireless sensor networks has taught us that it may be extremely difficult to design and implement a TCP-like protocol on such infrastructure. Therefore, there arises the need to design an easily computable controller that, although suboptimal, can guarantee “acceptable” performance in UDP-like scenarios. The rest of paper will deal with finding such regulator in the class of linear static controllers.

IV. A Linear Static Controller for UDP-like Networked Systems

We want to find optimal static gains L, K for the LQG controller and estimator respectively. The estimator equations are:

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + \bar{\nu}Bu_k + \gamma_k K(y_k - \hat{y}_k) \\ u_k &= -L\hat{x}_k \quad (9) \\ \hat{y}_k &= C\hat{x}_k, \quad (10) \end{aligned}$$

After some simple algebra the close loop dynamics can be written as,

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A & -\nu_k BL \\ \gamma_k KC & A - \bar{\nu}BL - \gamma_k KC \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \\ + \begin{bmatrix} w_k \\ \gamma_k K v_k \end{bmatrix}$$

If we define the vector $z_k = [x_k \;\; \hat{x}_k]^T \in \mathbb{R}^{2n}$, the the previous equation can be written in a more compact form as:

$$z_{k+1} = G_{\gamma_k,\nu_k}(K, L)z_k + d_k \quad (11)$$

Now let

$$P_k \triangleq \mathbb{E}\left[\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}\right] = \begin{bmatrix} P_k^{11} & P_k^{12} \\ P_k^{12\,T} & P_k^{22} \end{bmatrix}.$$

where P_k is the covariance of the vector z_k. Its evolution is given by:

$$\begin{aligned} P_{k+1} &= \mathbb{E}[G_{\gamma_k,\nu_k}(K,L)z_k z_k^T G^T_{\gamma_k,\nu_k}(K,L)] + \mathbb{E}[d_k d_k^T] \\ &= \mathbb{E}_{\nu,\gamma}[G_{\gamma_k,\nu_k}(K,L)P_k G^T_{\gamma_k,\nu_k}(K,L)] + D(K) \\ &= \overline{G}(K, L, P_k) + D(K) \quad (12) \end{aligned}$$

where:

$$\begin{aligned} D(K) &= \begin{bmatrix} Q & 0 \\ 0 & \bar{\gamma}KRK^T \end{bmatrix} \quad (13) \\ \overline{G}(K,L,P) &= \bar{\gamma}\bar{\nu}G_{11}PG_{11}^T + \bar{\gamma}(1-\bar{\nu})G_{10}PG_{10}^T + \\ &+ (1-\bar{\gamma})\bar{\nu}G_{01}PG_{01}{}^T + (1-\bar{\gamma})(1-\bar{\nu})G_{00}PG_{00}^T \quad (14) \\ G_{11} &= \begin{bmatrix} A & -BL \\ KC & A - \bar{\nu}BL - KC \end{bmatrix} \\ G_{10} &= \begin{bmatrix} A & 0 \\ \bar{\gamma}KC & A - \bar{\nu}BL - \bar{\gamma}KC \end{bmatrix} \\ G_{01} &= \begin{bmatrix} A & -\bar{\nu}BL \\ 0 & A - \bar{\nu}BL \end{bmatrix} \\ G_{00} &= \begin{bmatrix} A & 0 \\ 0 & A - \bar{\nu}BL \end{bmatrix} \end{aligned}$$

We next define the following cost:

$$\begin{aligned} c_k &= \mathbb{E}\left[x_k^T W x_k + \bar{\nu}u_k^T U u_k\right] \\ &= \text{Trace}\left(\begin{bmatrix} W & 0 \\ 0 & \bar{\nu}L^T UL \end{bmatrix} P_k\right) \\ &= \text{Trace}\big(N(L)P_k\big), \quad (15) \end{aligned}$$

where:

$$N(L) = \begin{bmatrix} W & 0 \\ 0 & \bar{\nu}L^T UL \end{bmatrix} \quad (16)$$

Clearly, if P_k converges to a finite value P_∞, then does the cost, i.e. c_k converges to c_∞. . Therefore, our objective to minimize this cost function with respect to K, L. The optimization problem can be written as follows:

$$\begin{array}{ll} \text{Min}_{K,L} & Tr\big(PN(L)\big) \\ s.t. & P = \overline{G}(K,L,P) + D(K), \;\; P \geq 0 \end{array} \quad (17)$$

This is a non convex optimization problem, and in the next section we will find necessary conditions for the existence of an optimum.

A. Necessary conditions

Using Lagrange multipliers the optimization problem can be rewritten as:

$$\begin{array}{ll} \text{Min}_{K,L,P,\Lambda} & J = Tr\big(PN(L)\big) + Tr\Big(\Lambda\big(\bar{G}(K,L,P) + D(K)\big) - P\Big) \\ s.t. & P \geq 0, \Lambda \geq 0. \end{array} \quad (18)$$

According to the minimum matrix principle [16], necessary conditions for the optimum are:

$$\frac{\partial J}{\partial \Lambda} = 0, \quad \frac{\partial J}{\partial P} = 0, \quad \frac{\partial J}{\partial K} = 0, \quad \frac{\partial J}{\partial L} = 0. \quad (19)$$

The first two conditions above can be written respectively as:

$$\begin{aligned} P &= \overline{G}(K,L,P) + D(K), \;\; P \geq 0 \quad (20) \\ \Lambda &= \underline{G}(K,L,\Lambda) + N(L), \;\; \Lambda \geq 0 \quad (21) \end{aligned}$$

where

$$\begin{aligned} \underline{G}(K,L,P) &= \bar{\gamma}\bar{\nu}G_{11}^T PG_{11} + \bar{\gamma}(1-\bar{\nu})G_{10}^T PG_{10} + \\ &+ (1-\bar{\gamma})\bar{\nu}G_{01}^T PG_{01} + (1-\bar{\gamma})(1-\bar{\nu})G_{00}^T PG_{00} \quad (22) \end{aligned}$$

Note that the operator $\underline{G}(K,L,P)$ is simply the dual of $\overline{G}(K,L,P)$. Let use consider the following partition of P and Λ and new matrices:

$$P = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_2 \end{bmatrix}$$

$$\overline{\Lambda} = \Lambda_1 - \Lambda_2, \;\; \underline{\Lambda} = \Lambda_2, \;\; \overline{P} = P_1 - P_2, \;\; \underline{P} = P_2$$

As shown in [17], the minimality assumption implies that:

$$\Lambda_{12} = -\underline{\Lambda} < 0, \;\; P_{12} = \underline{P} > 0 \quad (23)$$

An immediate result is that $\lim_{k\to\infty} \mathbb{E}[(x_k - \hat{x}_k)\hat{x}_k^T] = P_{12} - P_2 = 0$, i.e. the estimate is asymptotically uncorrelated with the error estimate, similarly to the standard Kalman filtering. If we substitute Eqn. (23) back into Eqn. (20) and (21), and after performing some straightforward algebraic manipulations we get:

$$\begin{aligned}
\overline{P} &= \bar{\gamma}(A-KC)\overline{P}(A-KC)^T + (1-\bar{\gamma})A\overline{P}A^T + \\
&\quad + \bar{\nu}(1-\bar{\nu})\underline{P} + Q + \bar{\gamma}KRK^T \qquad (24)\\
&= \Phi_1(\overline{P},\underline{P},K)\\
\underline{P} &= (A-\bar{\nu}BL)\underline{P}(A-\bar{\nu}BL)^T + \bar{\gamma}K(C\overline{P}C^T+R)K^T \qquad (25)\\
&= \Phi_2(\overline{P},\underline{P},K)\\
\overline{\Lambda} &= \bar{\nu}(A-BL)^T\overline{\Lambda}(A-BL) + (1-\bar{\nu})A^T\overline{\Lambda}A + \\
&\quad + W + \bar{\nu}(L^T\big(U+(1-\bar{\nu})B^T\underline{\Lambda}B\big)L \qquad (26)\\
&= \Phi_3(\overline{\Lambda},\underline{\Lambda},K)\\
\underline{\Lambda} &= \bar{\nu}(A-BL)^T\underline{\Lambda}(A-BL) + (1-\bar{\nu})A^T\underline{\Lambda}A + \\
&\quad + \bar{\nu}L^T\big(B^T\overline{\Lambda}B + (1-\bar{\nu})B^T\underline{\Lambda}B + U\big) \qquad (27)\\
&= \Phi_4(\overline{\Lambda},\underline{\Lambda},K)
\end{aligned}$$

Similarly, if we use Eqn. (23) into the last two partial derivatives of Eqn. (19), and after after performing some straightforward algebraic manipulations, we get:

$$\begin{aligned}
K &= A\overline{P}C^T\big(C\overline{P}C^T+R\big)^{\dagger} \qquad (28)\\
&= \Phi_5(\overline{P})\\
L &= \big(B^T\overline{\Lambda}B + (1-\bar{\nu})B^T\underline{\Lambda}B + U\big)^{\dagger}B^T\overline{\Lambda}A \qquad (29)\\
&= \Phi_6(\overline{\Lambda},\underline{\Lambda})
\end{aligned}$$

where the symbol † represents the Moore-Penrose pseudoinverse. An iterative solution to the set of Equations (24)-(29) shown above will provide necessary conditions for optimality. Clearly, if there exists only one minimum, the condition becomes also sufficient. Note that if $\bar{\nu} = \bar{\gamma} = 1$ and we substitute Eqn. (28) into Eqn. (24), and Eqn. (29) into Eqn. (26), we obtain the standard Algebraic Riccati equations for the Kalman filter and LQ optimal controller, respectively. Next section provide an iterative algorithm that converges to solution of the optimization problem if such a solution is finite.

B. Iterative solution and sufficient conditions

As described above, the six coupled nonlinear Equations (24)-(29), define a set of necessary conditions. A natural choice to try to find a fixed point is to use an iterative solution as the following:

$$\begin{aligned}
\overline{P}_{k+1} &= \Phi_1(\overline{P}_k,\underline{P}_k,K_k) && (30)\\
\underline{P}_{k+1} &= \Phi_2(\overline{P}_k,\underline{P}_k,K_k) && (31)\\
\overline{\Lambda}_{k+1} &= \Phi_3(\overline{\Lambda}_k,\underline{\Lambda}_k,L_k) && (32)\\
\underline{\Lambda}_{k+1} &= \Phi_4(\overline{\Lambda}_k,\underline{\Lambda}_k,L_k) && (33)\\
K_k &= \Phi_5(\overline{P}_k) && (34)\\
L_k &= \Phi_6(\overline{\Lambda}_k,\underline{\Lambda}_k) && (35)
\end{aligned}$$

For ease of notation, if we substitute the last two equations for the gains K_k, L_k into the previous four, the iterative update can be written in a more compact for as follows:

$$(\overline{P}_{k+1},\underline{P}_{k+1},\overline{\Lambda}_{k+1},\underline{\Lambda}_{k+1}) = \Phi(\overline{P}_k,\underline{P}_k,\overline{\Lambda}_k,\underline{\Lambda}_k) \qquad (36)$$

It was shown by De Koning in [6] that under some standard hypotheses, the necessary conditions given by Equations (24)-(29) are also sufficient and that the iterative solution given by Equations (30)-(35) converges to the fixed point solution. We adapt his results to our scenario in the following theorem:

Theorem 1: Let us consider the close loop control systems defined by Equations (1)-(2) and (9)-(10), where ν_k and γ_k are Bernoulli random variables with mean $\bar{\nu}$ and $\bar{\gamma}$, respectively. Assume that (A,B), (A^T,C^T), $(A,W^{\frac{1}{2}})$ and $(A^T,Q^{\frac{1}{2}})$ are all stabilizzable, and $U>0, R>0$. Then, the sequence defined by Equations (30)-(35) starting from initial conditions $\overline{P}_0 = \underline{P}_0 = \overline{\Lambda}_0 = \underline{\Lambda}_0 = 0$ converges to the unique solution of the optimization problem defined by Eqn. (17), i.e.

$$\lim_{k\to\infty}\Phi^k(0,0,0,0) = (\overline{P}^*,\underline{P}^*,\overline{\Lambda}^*,\underline{\Lambda}^*),$$

if and only if the sequence defined by Equations (30)-(35) where $W=Q=0$, $V=R=0$ and initial conditions $\overline{P}_0 = \overline{\Lambda}_0 = I$ and $\underline{P}_0 = \underline{\Lambda}_0 = 0$ converge to zero, i.e.

$$\lim_{k\to\infty}\Phi^k(I,0,I,0) = (0,0,0,0)$$

The proof of the previous theorem is rather involved and requires the use of the homotopic continuation method to prove convergence, therefore it is omitted. We refer the interested reader to [6] and [18] for details.

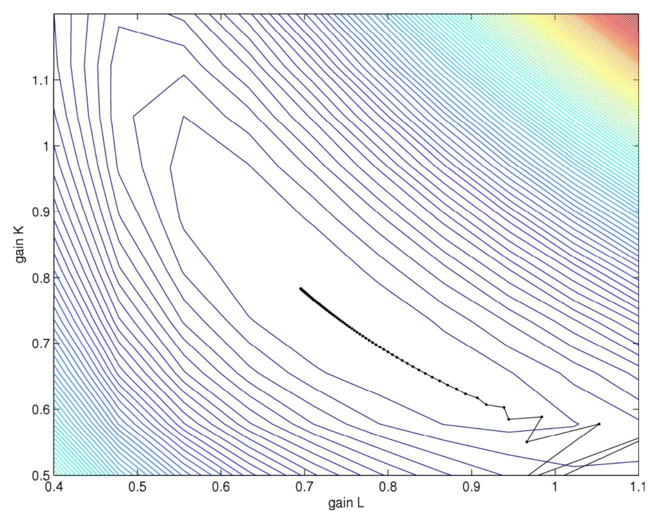

Fig. 3. **Convergence of the optimization problem.** The iterative method converges to a unique minimum

V. Discussion

In the previous section we provided necessary and sufficient conditions for the existence of an optimum, along with an iterative method to compute it. This section shows some numerical example and applications of the proposed iterative algorithm.

For the sake of simplicity, consider a scalar version of the system of Equations (1)-(3), with $B=C=Q=R=W=U=1, A=1.1, \nu=\gamma=0.8$. Figure 3 shows a contour plot of the infinite horizon cost as a function of the controller an estimator gains. Note that the cost function is non-convex, but that there is a unique minimum. The same

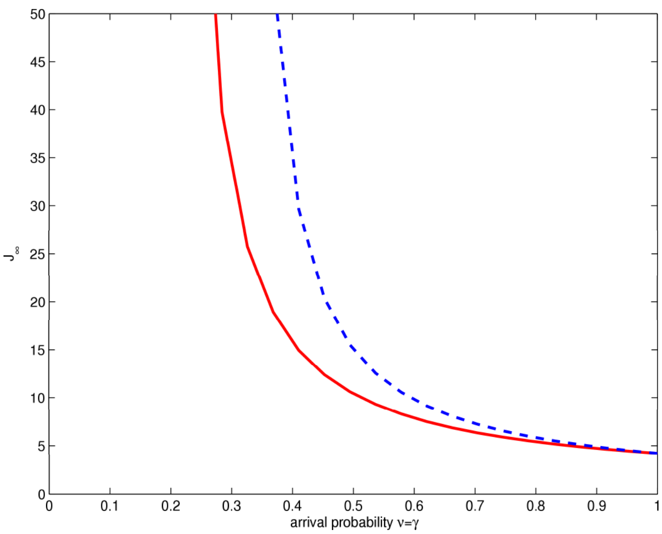

Fig. 4. Comparison between the optimal TCP-like LQG controller and the suboptimal UDP-like controller. There is little loss of performance in choosing a suboptimal UDP controller, while saving much complexity in protocol design.

figure shows how the cost converges after just a few iteration steps, suggesting that the algorithm is also computationally efficient. Figure 4, instead, shows a comparison between the optimal TCP-like LQG controller and the suboptimal UDP-like controller derived above, for different values of ν, γ. The figure suggests that for sufficiently high arrival rate, implementing an optimal controller over a TCP-like network does not provide a significant advantage. This is particularly useful to the designer, who can trade off high complexity in the network design with a little performance loss.

VI. Conclusion and Future Work

In this paper we analyzed a generalized version of the LQG control problem in the case where both observation and control packets may be lost during transmission over a communication channel. This situation arises frequently in distributed systems where sensors, controllers and actuators reside in different physical locations and have to rely on data networks to exchange information. In this context controller design heavily depends on the communication protocol used. In fact, in TCP protocols, acknowledgements of successful transmissions of control packets are provided to the controller, while in UDP protocols, no such feedback is provided. In the first case, the separation principle holds and the optimal control is a linear function of the state. As a consequence, controller and estimator design problems are decoupled. UDP-like protocols present a much more complex problem. We have shown that the lack of acknowledgement of control packets results in the failure of the separation principle. Estimation and control are now intimately coupled. We have shown that the LQG optimal control is, in general, nonlinear in the estimated state. In the particular case, where we have access to full state information, the optimal controller is linear in the state. To fully exploit UDP-like protocols it is necessary to have a controller/estimator design methodology for the general case when there is measurement noise and under partial state observation. As UDP protocols are the only practical solution in many cases where the channel is too unreliable to guarantee successful delivery of acknowledgement, it would prove extremely valuable to determine the optimal time-invariant LQG controller. Among all possible choices we focused on the class of linear controllers, for their simplicity in implementation. After describing the optimization problem, we derived necessary and sufficient conditions for the existence of a unique solution. Probably our most interesting finding, for practical purposes, is that control performance is not greatly affected by lack of optimality of the linear controller.

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