Time Varying Optimal Control with Packet Losses.

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***Abstract*— The problem of using wireless sensor networks technology for estimation and control of dynamical systems has recently received widespread attention within the scientific community. Classical control theory is in general insufficient to model distributed control problems where issues of communication delay, jitter, and time synchronization between components cannot be ignored.**

The purpose of this paper is to extend our work on discrete time Kalman filtering with intermittent observations [1] that was motivated by data losses in a communication channel. Accordingly, we consider the Linear Gaussian Quadratic (LQG) optimal control problem in the discrete time setting, formally showing that the separation principle holds in the presence of data losses. Then, using our previous results, we show the existence of a critical arrival probability below which the resulting optimal controller fails to stabilize the system. This is done by providing analytic upper and and lower bounds on the cost functional, and stochastically characterizing their convergence properties as $t \to \infty$.

I. Introduction

Advances in VLSI and MEMS technology have boosted the development of micro sensor integrated systems. Such systems combine computing, storage, radio technology, and energy source on a single chip [2] [3]. When distributed over a wide area, networks of sensors can perform a variety of tasks that range from environmental monitoring and military surveillance, to navigation and control of a moving vehicle [4] [5] [6]. A common feature of these systems is the presence of significant communication delays and data loss across the network. From the point of view of control theory, significant delay is equivalent to loss, as data needs to arrive to its destination in time to be used for control. In short, communication and control become tightly coupled such that the two issues cannot be addressed independently.

Consider, for example, the problem of navigating a vehicle based on the estimate from a sensor web of its current position and velocity. The measurements underlying this estimate can be lost or delayed due to the unreliability of the wireless links. What is the amount of data loss that the control loop can tolerate to reliably perform the navigation task? Can communication protocols be designed to satisfy this constraint? Practical advances in the design of these systems are described in [7]. The goal of this paper is to examine some control-theoretic implications of using sensor networks for control. These require a generalization of classical control techniques that explicitly take into account the stochastic nature of the communication channel.

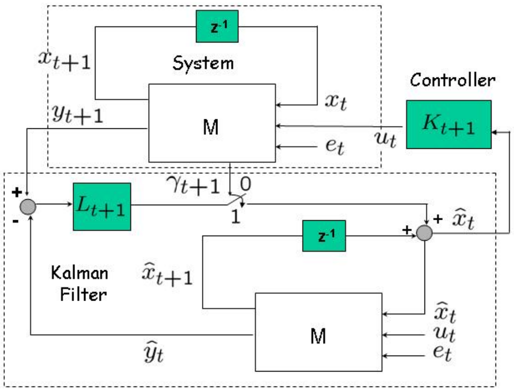

Fig. 1. **Overview of the system.** We study the statistical convergence of the expected state covariance of the discrete time LQG, where the observation, travelling over an unreliable communication channel, can be lost at each time step with probability $1 - \lambda$.

In our setting, the sensor network provides observed data that are used to estimate the state of a controlled system, and this estimate is then used for control. We study the effect of data losses due to the unreliability of the network links. We generalize the Linear Quadratic Gaussian (LQG) optimal control problem —modeling the arrival of an observation as a random process whose parameters are related to the characteristics of the communication channel, as shown in Figure 1. The separation principle states that observer and plant of a linear system can be designed independently. We first show that this principle continues to hold in the case of data loss between the sensor and the estimator. This allows us to use our result in [1], [8] to show the existence of a critical loss probability below which the resulting optimal controller fails to stabilize the system.

Consider the following discrete time linear dynamical system:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t, \end{aligned} \tag{1}$$

where $x_t \in \Re^n$ is the state vector, $y_t \in \Re^m$ the output vector, $u_t \in \Re^q$ is the input vector, $w_t \in \Re^n$ and $v_t \in \Re^m$ are Gaussian random vectors with zero mean and covariance matrices $Q \geq 0$ and $R > 0$, respectively. w_t is independent of w_s for $s < t$. Assume that the initial state, x_0, is also a Gaussian vector of zero mean and covariance Σ_0. LQG theory provides optimal solution to the control problem by minimizing the functionals

$$\begin{aligned} J_N &= \mathbb{E}[x_N' W_N x_N + \sum_{k=0}^{N-1}(x_k' W_k x_k + u_k' V_k u_k)|\mathbf{y}_{N-1}] \\ J_\infty &= \lim_{n\to\infty} \frac{1}{N}\mathbb{E}[\sum_{k=0}^{N-1}(x_k' W x_k + u_k' V u_k)|\mathbf{y}_{N-1}], \end{aligned} \tag{2}$$

for the finite and infinite horizon cases respectively, where $\mathbf{y}_N = (y_1, \ldots, y_N)$ is the observation history vector. In our previous work on Kalman Filtering with intermittent observations [1], [8] we proved the existence of a critical loss probability under which the expected error covariance of the filter diverges. The aim of this work is to extend this result to the optimal control problem showing the existence of a transition from bounded to unbounded states in the closed loop system as well, when the rate of observation loss exceeds a given threshold λ_c.

In some related work [9] Nilsson presents the LQG optimal regulator with bounded delays between sensors and controller, and between the controller and the actuator, but he does not address the packet-loss case. This is considered by Hadijcostis and Touri [10]. Their analysis is restricted to the static scalar case. Other approaches include using the last received sample for control, or designing a dropout compensator [11], [12]. We consider the alternative approach where the external compensator feeding the controller is the optimal time varying Kalman gain. Moreover, we analyze the proposed solution in state space domain rather than in frequency domain as it was presented in [12], and we consider the more general Multiple Input Multiple Output (MIMO) case.

Following the procedure and using the result in [1], [8] we are able to prove the existence of a critical value for the arrival rate above which the optimization problem is bounded, and below which the cost J goes unbounded. This is accomplished by finding deterministic upper and lower bounds for the expected optimal cost and their convergence conditions.

The LQG optimal control problem with missing observations can also be modelled using the well known Jump Linear System (JLS) theory [13], where the observer switches between open loop and closed loop configuration, depending on whether the packet containing the observation is lost, or arrives at the estimator in time. However, convergence results in this case can be obtained only when each jump sub-system is stabilizable and detectable. The detectability assumption fails in our case, producing a non-stationary state random process.

Finally, we mention that philosophically our result can be seen as another manifestation of the well known *uncertainty threshold principle* [14], [15]. This principle states that optimum long-range control of a dynamical system with uncertainty parameters is possible if and only if the uncertainty does not exceed a given threshold. The uncertainty is modelled as white noise scalar sequences acting on the system and control matrices. In our case the uncertainty is due to the random arrival of the observation, with the randomness arising from losses in the network.

The paper is organized as follows. In section II we formalize the LQG optimal control problem with intermittent observations. We provide upper and lower bounds on the cost functional of the LQG problem, and find the conditions on the observation arrival probability λ for which the upper bound converges to a fixed point, and for which the lower bound diverges. Finally, in section III, we state our conclusions and give directions for future work.

II. Problem Formulation

We define the arrival of the observation at time t as a binary random variable γ_t, with probability distribution $p_{\gamma_t}(1) = \lambda$, and with γ_t independent of γ_s if $t \neq s$. The output noise v_t is defined in the following way:

$$p(v_t|\gamma_t) = \begin{cases} \mathcal{N}(0, R) & : \quad \gamma_t = 1 \\ \mathcal{N}(0, \sigma^2 I) & : \quad \gamma_t = 0, \end{cases}$$

for some σ^2 . That is, the variance of the observation at time t is R if γ_t is 1, and $\sigma^2 I$ otherwise. Note that the absence of observation corresponds to the limiting case of $\sigma \to \infty$. Our approach is to derive the LQG equations using a "dummy" observation with a given variance when the real observation does not arrive, and then take the limit as $\sigma \to \infty$. Let us now consider the modified objective functionals:

$$\begin{aligned} & J_N(\boldsymbol{\gamma}_{N-1}, \mathbf{u}_{N-1}) = \\ & = \mathbb{E}[x_N' W_N x_N + \textstyle\sum_{k=0}^{N-1}(x_k' W_k x_k + u_k' V_k u_k)|\mathbf{y}_{N-1}, \boldsymbol{\gamma}_{N-1}] \end{aligned} \tag{3}$$

where $\boldsymbol{\gamma}_N = (\gamma_1, \ldots, \gamma_N)$ is the history vector of the observation arrival process. Since in the modified functional the arrival sequence is supposed to be known, then its minimization correspond to the minimization of a time varying system given by Equations (1) where $\mathbb{E}[x_0 x_0'] = P_0$, $\mathbb{E}[w_t w_t'] = Q$, $\mathbb{E}[v_t v_t'] = R_t$, and $R_t = \gamma_t R + (1 - \gamma_t)\sigma^2 I$. The only time-varying part of the system is the output noise, which depends on the arrival sequence. The minimization of the functional given in Equation (3) is given by a time-varying LQG, which is summarized in the following theorem:

Theorem 1 (finite horizon LQG). *Consider the following linear stochastic system with intermittent observations:*

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t, \end{aligned} \tag{4}$$

where (x_0, w_t, v_t) are Gaussian, uncorrelated, white, with zero mean and covariance (P_0, Q, R_t) respectively, and $R_t = \gamma_t R + (1-\gamma_t)\sigma^2 I$. The control inputs that minimize the quadratic functional given by Equation (3) are given by the following linear feedback:

$$u_t = -L_t \hat{x}_t \tag{5}$$

where $\hat{x}_t = \mathbb{E}[x_t|\mathbf{y}_{t-1}, \boldsymbol{\gamma}_{t-1}]$ is the optimal estimator of the unknown state x_t obtained by the time-varying Kalman filter, and the controller gain L_t is obtained by the following recursive algorithm:

$$S_N = W_N \tag{6}$$

$$L_t = (V_t + B'S_{t+1}B)^{-1}B'S_{t+1}A, \quad t = N-1, \ldots, 1 \tag{7}$$

$$S_t = W_t + A'S_{t+1}A - L_t'(V_t + B'S_{t+1}B)L_t \tag{8}$$

The optimal Kalman filter estimator with missing observations, i.e. $\sigma \to \infty$, is given by:

$$\hat{x}_{t+1} = (A - BL_t)\hat{x}_t + \gamma_t AK_t(y_t - C\hat{x}_t) \tag{9}$$

where the estimator gain K_t is given by:

$$P_0 = \mathbb{E}[x_0 x_0'] \tag{10}$$

$$K_{t+1} = P_t C'(CP_tC' + R)^{-1}, \qquad t = 0, \ldots, N-1 \tag{11}$$

$$\begin{aligned} P_{t+1} &= AP_tA' + Q + \\ &\quad -\gamma_t AP_tC'(CP_tC' + R)^{-1}CP_tA' \end{aligned} \tag{12}$$

The minimum of the functional (3) using optimal LQG is given by:

$$J_{min}(\boldsymbol{\gamma}_{N-1}) = \min_{\mathbf{u}_{N-1}} J_N(\boldsymbol{\gamma}_{N-1}, \mathbf{u}_{N-1}) =$$

$$= tr(S_0P_0) + \sum_{t=0}^{N-1} tr(S_{t+1}Q) + \tag{13}$$

$$+ \sum_{t=1}^{N-1} tr\left(P_t L_t'(B'S_{t+1}B + V_t)L_t\right) \tag{14}$$

Proof: For a finite value of σ, the proof of the theorem follows directly from standard time-varying finite horizon LQG, since the sequence of the observation arrivals is fixed and R_t is thus known (see [16] for example). The optimal controller gain L_t is independent of the arrival process $\{\gamma_t\}$ and the noise. This is a consequence of the separation principle. Therefore, the arrival process $\{\gamma_t\}$ affects only the Kalman estimator $\hat{x}_t = \mathbb{E}[x_t|\mathbf{y}_{t-1}, \boldsymbol{\gamma}_{t-1}]$. The optimal Kalman estimator for the limiting case corresponding to $\sigma \to +\infty$ is given by Equations (9)-(12) that were derived in [8]. ∎

The previous theorem shows that the separation principle holds also for the case of missing observations, therefore the optimal controller design and the optimal estimator design can be computed separately. It is important to see that the optimal estimator given by Equation (9) is causal, i.e. requires only the knowledge of arrival process $\{\gamma_t\}$ up to time t and can then be implemented on-line. However, the minimal functional $J_{min}(\boldsymbol{\gamma}_{N-1})$ depends on the exact arrival sequence and it is therefore a stochastic variable.

It is therefore interesting to study the expected value of the stochastic finite horizon LQG, i.e. computing $\bar{J}_{min} = \mathbb{E}[J_{min}(\boldsymbol{\gamma}_{N-1})]$. Following the same analysis developed in [8], although it is not possible to compute exactly the the estimate $\bar{J}_{min}$, some bounds can be computed as follows:

Theorem 2. *Assume the arrival process γ_t is a bernoulli process where $P[\gamma_t = 1] = \lambda$. Then the expected value of the functional satisfies the following inequalities:*

$$\underline{J}_N \leq \mathbb{E}[J_{min}(\boldsymbol{\gamma}_{N-1})] \leq \overline{J}_N \tag{15}$$

where

$$\underline{J}_N = tr(S_0P_0) + \sum_{t=0}^{N-1} tr(S_{t+1}Q) + \tag{16}$$

$$+ \sum_{t=1}^{N-1} tr\left(\underline{F}_t L_t'(B'S_{t+1}B + V_t)L_t\right),$$

$$\overline{J}_N = tr(S_0P_0) + \sum_{t=0}^{N-1} tr(S_{t+1}Q) + \tag{17}$$

$$+ \sum_{t=1}^{N-1} tr\left(\overline{F}_t L_t'(B'S_{t+1}B + V_t)L_t\right)$$

and

$$\begin{aligned} \underline{F}_0 &= \overline{F}_0 = P_0 \\ \underline{F}_{t+1} &= (1-\lambda)A\underline{F}_tA' + Q \\ \overline{F}_{t+1} &= A\overline{F}_tA' + Q - \lambda A\overline{F}_tC'(C\overline{F}_tC' + R)^{-1}C\overline{F}_tA' \end{aligned} \tag{18}$$

Proof: The expectation of Equation (13) is given by:

$$\mathbb{E}[J_{min}(\boldsymbol{\gamma}_{N-1})] = tr(S_0P_0) + \sum_{t=0}^{N-1} tr(S_{t+1}Q) + \tag{19}$$

$$+ \sum_{t=1}^{N-1} tr\left(\mathbb{E}[P_t] L_t'(B'S_{t+1}B + V_t)L_t\right)$$

where we used the facts that the trace is a linear operator and that only the matrices $\{P_t\}$ depends on the arrival process $\{\gamma_t\}$. It was shown in [8] that, although $\mathbb{E}[P_t]$ cannot be computed exactly, it is possible to find lower and upper bounds that can be computed exactly, i.e.

$$\underline{F}_t \leq \mathbb{E}[P_t] \leq \overline{F}_t$$

where the matrices $\underline{F}_t, \overline{F}_t$ are given by Equations (18). Therefore the bounds of Equation (15) follow directly from the bounds on $\mathbb{E}[P_t]$ and the fact that last term in Equation (13) is monotonic in P. In fact, $P_1 \geq P_2 \geq$

$0,\ T \geq 0 \Rightarrow tr(P_1T) = tr(P_1T^{\frac{1}{2}}T^{\frac{1}{2}}) = tr(T^{\frac{1}{2}}P_1T^{\frac{1}{2}}) \geq tr(T^{\frac{1}{2}}P_2T^{\frac{1}{2}}) = tr(P_2T^{\frac{1}{2}}T^{\frac{1}{2}}) = tr(P_2T)$, and by letting $Q = L_t'(B'S_{t+1}B + V_t)L_t \geq 0$ this concludes the proof. ■

We can now extend the results of the finite horizon LQG to the infinite horizon case:

Theorem 3 (infinite horizon LQG). *Consider the linear stochastic system with intermittent observations of theorem 1. The control inputs that minimize the quadratic functional given by:*

$$\frac{1}{N}J_N(\boldsymbol{\gamma}_{N-1}, \mathbf{u}_{N-1}) \tag{20}$$

converges to the following linear feedback:

$$u_t = -L_\infty \hat{x}_t, \qquad as \qquad N \to +\infty \tag{21}$$

where $\hat{x}_t = E[x_t|\mathbf{y}_{t-1}, \boldsymbol{\gamma}_{t-1}]$ is the optimal estimator of the unknown state x_t obtained by the standard time-varying Kalman filter, and the controller gain L_∞ is the solution of the following Riccati Equation:

$$\begin{aligned} S_\infty &= W + A'S_\infty A + \\ &\quad -A'S_\infty B(V + B'S_\infty B)B'S_\infty A \qquad (22) \\ L_\infty &= (V + B'S_\infty B)^{-1}B'S_\infty A \qquad (23) \end{aligned}$$

The optimal Kalman filter estimator with missing observations, i.e. $\sigma \to \infty$, is given by:

$$\hat{x}_{t+1} = (A - BL_\infty)\hat{x}_t + \gamma_t AK_t(y_t - C\hat{x}_t) \tag{24}$$

where the estimator gain K_t is given by:

$$\begin{aligned} P_0 &= E[x_0x_0'] \qquad (25) \\ K_{t+1} &= P_tC'(CP_tC' + R)^{-1} \qquad t = 0, 1, \ldots \qquad (26) \\ P_{t+1} &= AP_tA' + \\ &\quad +Q - \gamma_t AP_tC'(CP_tC' + R)^{-1}CP_tA' \qquad (27) \end{aligned}$$

The expected value of the minimum of the functional (20) using optimal LQG is bounded by:

$$\begin{aligned} \underline{J}_\infty &\leq \mathbb{E}[\min_{\mathbf{u}} J_\infty(\boldsymbol{\gamma}, \mathbf{u})] \leq \overline{J}_\infty \qquad (28) \\ \underline{J}_\infty &= tr(S_\infty Q) + tr\left(\underline{F}_\infty L_\infty'(B'S_\infty B + V)L_\infty\right) \qquad (29) \\ \overline{J}_\infty &= tr(S_\infty Q) + tr\left(\overline{F}_\infty L_\infty'(B'S_\infty B + V)L_\infty\right) \qquad (30) \end{aligned}$$

where the matrices $\underline{F}_\infty, \overline{F}_\infty$ are the solutions of the following equations:

$$\begin{aligned} &\underline{F}_\infty = (1 - \lambda)A\underline{F}_\infty A' + Q \qquad (31) \\ &\overline{F}_\infty = A\overline{F}_\infty A' + \\ &+Q - \lambda A\overline{F}_\infty C'(C\overline{F}_\infty C' + R)^{-1}C\overline{F}_\infty A' \qquad (32) \end{aligned}$$

Proof: The proof for the infinite horizon LQG with missing observations can be derived by taking the limit for $N \to +\infty$ of the finite horizon LQG. The separation principle still holds, therefore the sequence S_t converges to a finite limit S_∞ if and only if there exist a solution to the standard algebraic Riccati Equation (22), otherwise the sequence is unbounded. The Riccati equation (22) has a unique semi-definite solution if and only if $(A, W^{\frac{1}{2}})$ is observable and (A, B) is stabilizable. These are standard results that can be found in any optimal control textbook as in Chen et al.[16]. If the sequence S_t converges, then also the controller gain L_t converges to a finite gain L_∞ given by Equation (23). The equations for the estimator remain the same as for the finite horizon case, and once again they depend on the sequence of the observation arrivals. The minimum of the functional given by Equation (20) is a stochastic variable. Although the expected value of the minimum of this functional cannot be computed exactly, it is possible to give a lower and an upper bounds. In fact:

$$\frac{1}{N}\underline{J}_N \leq \mathbb{E}[\min_{\mathbf{u}} \frac{1}{N}J_N(\boldsymbol{\gamma}, \mathbf{u})] \leq \frac{1}{N}\overline{J}_N \quad \forall N$$

It was shown in [8] that if the Equations (31)-(32) have a solution, then $\underline{F}_t \to \underline{F}_\infty, \overline{F}_t \to \overline{F}_\infty$ for $t \to +\infty$, otherwise the sequence is unbounded. Moreover, since $S_t \to S_\infty, L_t \to L_\infty$ as shown above, then we have:

$$\begin{aligned} &\lim_{N\to+\infty} \frac{1}{N}tr(S_0P_0) = 0 \\ &\lim_{N\to+\infty} \frac{1}{N}\sum_{t=0}^{N-1} tr(S_{t+1}Q) = \\ &= \lim_{N\to+\infty} \frac{1}{N}\sum_{t=0}^{N-1} tr(S_\infty Q) = tr(S_\infty Q) \\ &\lim_{N\to+\infty} \frac{1}{N}\sum_{t=0}^{N-1} tr(F_tL_t'(B'S_{t+1}B + V)L_t) = \\ &= \lim_{N\to+\infty} \frac{1}{N}\sum_{t=0}^{N-1} tr(F_\infty L_\infty'(B'S_\infty B + V)L_\infty) = \\ &= tr((F_\infty L_\infty'(B'S_\infty B + V)L_\infty) \end{aligned}$$

Substituting the above limits into Equations (17)-(18) we obtain the desired bounds on the expected value of the minimum cost functional, which concludes the theorem ■

The theorem above states that the separation principle holds also for the infinite horizon LQG with missing observation. With this in mind, convergence conditions for the functionals are equivalent to the ones derived for the estimator alone [1], [8]. Therefore there exists a critical probability λ_c below which the closed loop systems is unbounded and above which it is mean square stable.

III. Conclusion

Motivated by applications where control is performed over a communication network, in this paper we extend our previous results on estimation with intermittent observations to the optimal control problem. First, we show that the separation principle holds also in the case when the observed state can be lost at each time step with some probability λ. Then, we show how the optimal control problem formally reduces to the solution of a standard Riccati equation for the controller and the same modified Riccati equation that was studied in [1], [8] for the estimator. Accordingly, we provide upper and lower bounds on the expected optimal cost functional and characterize its convergence conditions, showing a transition to an unbounded cost beyond a critical arrival probability. We also provide upper and lower bounds for the cost in the finite horizon case.

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