Via Gradenigo 6/b
Department of Information Engineering
University of Padova
35121 Padova, Italy
Email: schenato@dei.unipd.it

November 9, 2007

Professor Roberto Tempo
Editor, Technical Notes and Correspondence
IEEE Transactions on Automatic Control
IEIIT-CNR, Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy

Dear Professor Tempo,

Enclosed please find a copy of the manuscript of the paper "To zero or to hold control inputs with lossy links ?" which I would like to submit to the *IEEE Transactions on Automatic Control* for possible publication as a *Technical Note*. The work being submitted has neither been published elsewhere nor it is currently under review by any other publication. A preliminary version of this paper titled "To zero or to hold control inputs in lossy networked control systems?" has been published in the Proceedings of the European Control Conference (ECC'07), July 2008. The author's contact information including mailing address, phone, fax, and email is attached to this letter. I have also included the abstract of the paper.

I will serve as the corresponding author in any future communication. Please do not hesitate to contact me if anything else if needed.

Sincerely,

Luca Schenato

Author list:

Luca Schenato
Associate Professor, Department of Information Engineering
University of Padova
Via Gradenigo 6/b
35121 Padova, Italy
Phone: +39 049 827 7925
Fax: +39 049 827 7699
Email: schenato@dei.unipd.it

To zero or to hold control inputs with lossy links ?

Luca Schenato

Abstract: This paper studies the LQ performance of networked control systems where control packets are subject to loss. In particular we explore the two simplest compensation strategies commonly found in the literature: the zero-input strategy, in which the input to the plant is set to zero if a packet is dropped, and the hold-input strategy, in which the previous control input is used if packet is lost. We derive expressions for computing the optimal static gain for both strategies and we compare their performance on some numerical examples. Interestingly, none of the two can be claimed superior to the other, even for simple scalar systems, since there are scenarios where one strategy performs better then the other and scenarios where the converse occurs.

To zero or to hold control inputs with lossy links ?

Luca Schenato

Abstract

This paper studies the LQ performance of networked control systems where control packets are subject to loss. In particular we explore the two simplest compensation strategies commonly found in the literature: the zero-input strategy, in which the input to the plant is set to zero if a packet is dropped, and the hold-input strategy, in which the previous control input is used if packet is lost. We derive expressions for computing the optimal static gain for both strategies and we compare their performance on some numerical examples. Interestingly, none of the two can be claimed superior to the other, even for simple scalar systems, since there are scenarios where one strategy performs better then the other and scenarios where the converse occurs.

I. Introduction

Today's technological advances in wireless communications and in the fabrication of inexpensive embedded electronic devices, are creating a new paradigm where a large number of systems are interconnected, thus providing an unprecedented opportunity for totally new applications. This is particularly true for real-time control systems where access to information from many sensors and distributed actuators can potentially lead to better performances. These systems are commonly referred as networked control systems. However, these advantages come at the price of unreliable or at least not-ideal communication links which lead to packet drops, random delay, quantization errors, thus leading to degradation from the ideal performance. Recently, a great effort has been given to understand and analyze these systems with respect to the interaction of communications and control, which has been recently surveyed in [1].

In particular, one of the most common problems in networked control systems, especially in wireless sensor networks, is packet drop, i.e. packets can be lost due to communication noise, interference, or congestion. If the controller is not co-located with the sensor and the actuator and it is placed in a remote location, then both sensor measurement packets and control packets can be lost. This would be the case, for example, in a pursuit-evasion-game scenario where locations of evaders are obtained through a wireless sensor network, then processed by a centralized controller, and finally optimal control inputs are dispatched to the mobile pursuers via wireless communication [2]. A large number of works in the literature have analyzed estimation and filter design under lossy communication between the sensors and the controller [3] [4][5][6][7][8][9]. However, there are also several works that studied the close loop performance when control packets can be dropped [10][11][12][13][14][15]. In general, in most of the literature two different strategies are considered for dealing with packet drops. In the first one, which we refer as *zero-input*, the actuator input to the plant is set to zero when the control packet from the controller to the actuator is lost [13][14][15], while in the second, which we refer as *hold-input*, the latest control input stored in the actuator buffer is used when a packet is lost [11][10][12]. These are not the only strategies that can be adopted. In fact, if smart actuators are available, i.e. if actuators are provided with computational resources, then the whole controller [15] or a compensation filter [16] can be placed on the actuator. Another strategy is to use a model predictive controller which sends not only the current input but also a finite window of future control inputs into a single packet so that if a packet is lost the actuator can pop up from its buffer the corresponding predicted input from the latest received packet [17] [18]. Nonetheless, even this strategy requires more computational resources and communication bandwidth than the zero-input or hold-input strategies.

This research is supported partially by CARIPARO foundation.

L. Schenato is with Faculty of Information Engineering, University of Padova, Via Gradenigo 6/b, 35131 Padova, Italy, schenato@dei.unipd.it

To the author's knowledge there is no study present in the literature which directly compares the hold-input and zero-input strategies, except for a simple empirical example in [15]. In particular, it seems that the zero-input strategy is mainly used for mathematical convenience as it gives simpler equations than the hold-input strategy, rather than being based on performance considerations. Indeed, intuitively one is led to think that the use the latest control input stored in the actuator buffer provides better performance than using a zero input in particular during the transient, since the true current optimal control input is likely to be close to the previous value. The zero-input strategy, however, it is not so unreasonable, since the optimal control input eventually converges to zero for a stable closed loop system in steady state. Motivated by these observations, the goal of this paper is to explicitly quantify the performance of these two strategies by adopting a LQ approach for discrete time linear system where the control input packets are dropped according to a Bernoulli stochastic process as described in details the Section II. In particular, we derive equations to compute the optimal static control gains for both strategies. While the equations for optimal control under the zero-input strategy in Section III have been previously derived [15], the equations for optimal gain design under the hold-input strategy presented in Section IV are novel. The equations are then used to compare the performance of the two strategies for scalar systems in Section V. In particular, we show that none of the two strategies is always superior to the other, but the performance depends on the packet loss probability and the cost weights. Finally, in Section VI we summarize the results and discuss future research directions.

II. Problem formulation

Consider the following linear stochastic system:

$$x_{k+1} = Ax_k + Bu_k^a \tag{1}$$

where u_k^a is the control input to the actuator. We assume that the full state x_k is available to a remote controller which adopts a simple linear feedback:

$$u_k^c = Lx_k$$

The link between the controller and the actuator is lossy, and the stochastic binary variable $\nu_k \in \{0,1\}$ models the packet loss between the controller and the actuator. We consider two control strategies. In the zero-input strategy, if the packet is correctly delivered then $u_k^a = u_k^c$, otherwise the actuator does nothing, i.e. $u_k^a = 0$, which gives the following closed loop system:

$$\begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k^a \\ u_k^a & = & \nu_k u_k^c \\ u_k^c & = & L_z x_k \end{array} \tag{2}$$

In the hold-input strategy, instead, when the packet is lost we use the previous control value stored in actuator, i.e. $u_k^a = u_{k-1}^a$, which leads to the following closed loop dynamics:

$$\begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k^a \\ u_k^a & = & \nu_k u_k^c + (1-\nu_k)u_{k-1}^a \\ u_k^c & = & L_h x_k \end{array} \tag{3}$$

These two control packet loss compensation strategies are graphically illustrated in Figure 1.

We compare the performance in terms of the infinite horizon expected total cost:

$$J_\infty(L) = \mathbb{E}[\sum_{k=0}^{\infty} x_k^T W x_k + (u_k^a)^T U u_k^a] \tag{4}$$

where $W_k = W \geq 0$ and $U_k = U \geq 0$. Note that only the inputs that actually enter the plant u_k^a, and not the desired control inputs u_k^c, are penalized. We also assume that the packet drops are i.i.d. Bernoulli random variables:

$$\mathbb{P}[\nu_k = 0] = \nu$$

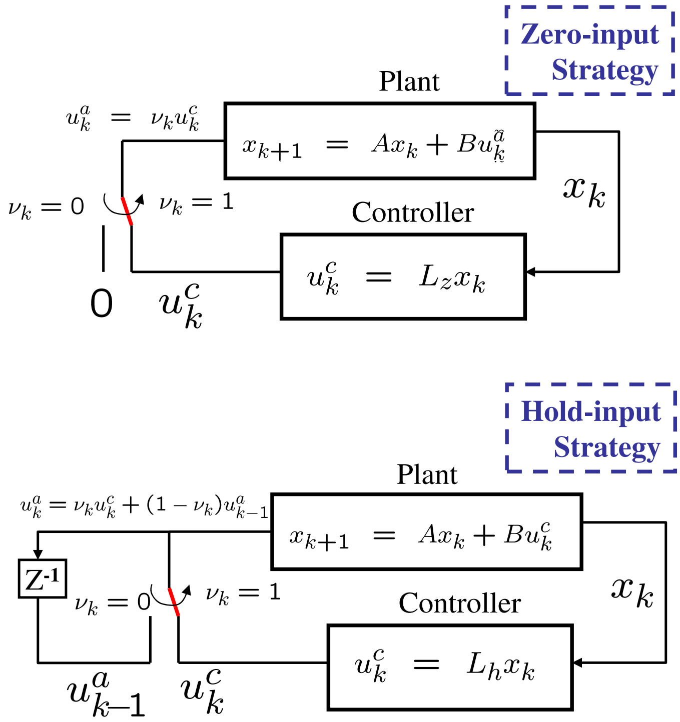

Fig. 1. Compensation approaches for actuators with no computational resources when a control packet is lost: zero-input approach $u_k^a = 0$ (*top*) and hold-input approach $u_k^a = u_{k-1}^a$ (*bottom*).

A useful result that links stability to infinite horizon cost is the following:

Proposition 1: Let the pair $(A, W^{1/2})$ be detectable, then the closed loop systems given by Equations (2) and (3) are mean square asymptotically stable, i.e. $\lim_{k\to\infty} \mathbb{E}[||x_k||^2] = 0$, if and only if the cost defined in Equation (4) is bounded, i.e. $J_\infty(L) < \infty$.

Proof: The proof is rather standard since it can be derived similarly to standard LQ-control results [19], therefore it is omitted. ■

This proposition states that minimizing the infinite horizon cost implicitly solves the problem of finding a stabilizing gain L. In the next two sections, we will derive the optimal infinite horizon cost and corresponding optimal gain for the two strategies.

III. LQ OPTIMAL CONTROL: ZERO-INPUT STRATEGY

The equations in this sections have been previously derived in [15] in a more general LQG optimal control setting, but are reported here in the context of LQ control to ease comparison with the hold input strategy developed in the next section.

The optimal control equations are obtained using the standard dynamic programming approach, i.e. we compute the cost-to-go function iteratively. First note that system (2) can be written as

$$\begin{aligned} x_{k+1} &= (A + \nu_k BL)x_k \\ u_k^a &= \nu_k L x_k \end{aligned}$$

Let us define the cost-to-go function C_k as follows

$$C_k^N(x_k) = \mathbb{E}[\sum_{h=k}^{N} x_k^T W_k x_k + u_k^{aT} U_k u_k^a | x_k] \tag{5}$$

where $W_k = W$ and $U_k = U$ except for the terminal cost $U_N = 0$. We claim that the cost-to-go function can be written as

$$C_k^N(x_k) = \mathbb{E}[x_k^T S_k x_k | x_k] \tag{6}$$

This is clearly true for $k = N$ with $S_N = W$. Then by induction, we show that this is true for all k. Suppose that it is true for $k+1$, then we have:

$$\begin{aligned} C_k^N(x_k) &= \mathbb{E}[\sum_{h=k}^{N} x_k^T W x_k + u_k^{aT} U u_k^a | x_k] \\ &= \mathbb{E}[x_k^T W x_k + u_k^{aT} U u_k^a + C_{k+1}^N | x_k] \\ &= \mathbb{E}[x_k^T W x_k + \nu_k x_k^T L^T U L x_k + x_k^T (A + \nu_k BL)^T S_{k+1} (A + \nu_k BL) x_k | x_k] \\ &= \mathbb{E}\big[x_k^T \big(W + (1-\nu) L^T U L + \nu A^T S_{k+1} A + (1-\nu)(A+BL)^T S_{k+1}(A+BL)\big) x_k | x_k\big] \end{aligned}$$

where we used the fact that ν_k is independent of x_k. Therefore the claim above is true and the matrix S_k is given by:

$$S_k = W + \nu A^T S_{k+1} A + (1-\nu)\big(L^T U L + (A+BL)^T S_{k+1}(A+BL)\big) = \mathcal{F}(S_{k+1}, L) \tag{7}$$

where the operator $\mathcal{F}(S, L)$ is affine in S for fixed L, and quadratic in L for fixed S. The infinite horizon cost can be obtained from the cost-to-go function as follows:

$$J_\infty(L) = \lim_{N\to\infty} C_0^N(x_0) = x_0^T S x_0$$

where S is the solution of the Lyapunov-like equation $S = \mathcal{F}(S, L)$, if such solution exists. The optimal gain L^* is defined as the minimizer of the infinite horizon cost, i.e. $L^* = \text{argmin}_L x_0^T S x_0$. It was shown in [15] that the optimal gain is independent of the initial condition x_0 and can be obtained by solving a Riccati-like equation. We summarize those results in the following theorem:

Theorem 1 ([15]): Consider the system defined by Equations (2) and the infinite horizon cost defined in Equation (4). Assume that the pair (A, B) is stabilizable and $(A, W^{1/2})$ is detectable. Then the optimal infinite horizon cost $J_\infty^* = \min_L J_\infty(L)$ is given by $J_\infty^* = x_0 S^* x_0$ where S^* is the unique positive semidefinite solution of the Riccati-like equation:

$$S^* = A^T S^* A + W - (1-\nu) A^T S^* B (B^T S^* B + U)^{-1} B^T S^* A = \Phi(S^*) \tag{8}$$

and the optimal gain is given by

$$L^* = -(B^T S^* B + U)^{-1} B^T S^* A \tag{9}$$

The Riccati-like equation $S_\infty^* = \Phi(S_\infty^*)$ has a positive semidefinite solution if and only if $\nu < \nu_c$, where ν_c is a critical packet loss probability, which depends on the pair (A, B). The critical loss probability ν_c satisfies the following bounds:

$$\begin{aligned} \nu_m &\le \nu_c \le \nu_M \\ \nu_m &= \tfrac{1}{\max_i |\lambda_i^u|^2}, \qquad \nu_M = \tfrac{1}{\prod_i |\lambda_i^u|^2} \end{aligned} \tag{10}$$

where λ_i^u are the unstable eigenvalues of the matrix A. In particular $\nu_c = \nu_m$ if B is invertible, and $\nu_c = \nu_M$ if B is rank one.

IV. LQ OPTIMAL CONTROL: HOLD-INPUT STRATEGY

We now derive the equations to compute the infinite horizon cost for the hold-input strategy. We proceed similarly to the previous section by computing the cost-to-go function. We first define the augmented state $z_k = \begin{bmatrix} x_k \\ u^a_{k-1} \end{bmatrix}$. Then the system defined by Equations (3) can be written as:

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ u^a_k \end{bmatrix} &= \begin{bmatrix} A+\nu_k BL & (1-\nu_k)B \\ \nu_k L & (1-\nu_k)I \end{bmatrix} \begin{bmatrix} x_k \\ u^a_{k-1} \end{bmatrix} && (11) \\ &= F(\nu_k) z_k && (12) \end{aligned}$$

where I is the identity matrix. The evolution of the systems can be modeled as a Jump Markov Linear System (JMLS) since the dynamics jumps between two systems $F(\nu_k = 1)$ and $F(\nu_k = 0)$ according to a Bernoulli distribution. Many mathematical tools exist to study JMLS [20], and in particular they reframe the LQ-problem as the solution of a set of coupled algebraic Riccati equations (CARE). However, since the system we are considering has only two jumping states with a simple bernoulli switching, we can explicitly reduce it to a single Riccati-like equation that can be compared with the zero-input Riccati-like equation of the previous section. We start defining the cost-to-go function as in the previous section:

$$C^N_k(z_k) = \mathbb{E}[\sum_{h=k}^{N} x_k^T W_k x_k + u^{aT}_k U_k u^a_k | z_k] \qquad (13)$$

where $W_k = W$ and $U_k = U$ except for the terminal cost $U_N = 0$. We claim that the cost-to-go function can be written as

$$C^N_k(z_k) = \mathbb{E}[z_k^T V_k z_k | z_k] \qquad (14)$$

This is clearly true for $k = N$ with $V_N = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}$. Then by induction, we show this is true for all k. Suppose it is true for $k+1$, then we have:

$$\begin{aligned} C^N_k(z_k) &= \mathbb{E}[\sum_{h=k}^{N} x_k^T W x_k + u^{aT}_k U u^a_k | z_k] \\ &= \mathbb{E}[x_k^T W x_k + u^{aT}_k U u^a_k + C^N_{k+1} | x_k] \\ &= \mathbb{E}[z_k^T \begin{bmatrix} W+\nu_k L^T U L & \nu_k(1-\nu_k)L^T U \\ \nu_k(1-\nu_k)UL & (1-\nu_k)^2 U \end{bmatrix} z_k + z_k^T F^T(\nu_k) V_{k+1} F(\nu_k) z_k | z_k] \\ &= \mathbb{E}[z_k^T \begin{bmatrix} W+(1-\nu)L^T U L & 0 \\ 0 & \nu U \end{bmatrix} z_k + \nu z_k^T F^T(0) V_{k+1} F(0) z_k + \\ &\quad +(1-\nu) z_k^T F^T(1) V_{k+1} F(1) z_k | z_k] \end{aligned}$$

where we used the fact that ν_k is independent of z_k. Therefore the claim above is true and the matrix V_k is given by:

$$\begin{aligned} V_k &= \begin{bmatrix} W+(1-\nu)L^T U L & 0 \\ 0 & \nu U \end{bmatrix} + \nu \begin{bmatrix} A^T & 0 \\ B^T & I \end{bmatrix} V_{k+1} \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} + \\ &\quad +(1-\nu) \begin{bmatrix} (A+BL)^T & L^T \\ 0 & 0 \end{bmatrix} V_{k+1} \begin{bmatrix} A+BL & 0 \\ L & 0 \end{bmatrix} \\ &= \mathcal{L}(V_{k+1}, L) \end{aligned} \qquad (15)$$

where the operator $\mathcal{L}(V, L)$ is affine in V for fixed L, and quadratic in L for fixed V. Let us partition the matrix V and the operator $\mathcal{L}(V, L)$ as follows

$$V = \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_{12} \\ \mathcal{L}_{12}^T & \mathcal{L}_2 \end{bmatrix}$$

then Equation (15) can be written as:

$$\begin{aligned}\mathcal{L}_1(L,V) &= W+\nu A^TV_1A+(1-\nu)\Big(L^TUL+L^TV_2L+(A+BL)^TV_1(A+BL)+ \\ &\quad +L^TV_{12}^T(A+BL)+(A+BL)^TV_{12}L\Big) \qquad (16)\\ \mathcal{L}_{12}(V) &= \nu(A^TV_1B+A^TV_{12}) \qquad (17)\\ \mathcal{L}_2(V) &= \nu(U+B^TV_1B+V_{12}^TB+BV_{12}+V_2) \qquad (18)\end{aligned}$$

Note that only the upper left block $\mathcal{L}_1(L,V)$ depends on the gain L. Moreover, since it is quadratic in the gain L it can be written as follows:

$$\begin{aligned}\mathcal{L}_1(L,V) &= \Phi_\nu(V)+(1-\nu)(L-L_V)^TP_V(L-L_V) \qquad (19)\\ \Phi_\nu(V) &= W+A^TV_1A-(1-\nu)A^T(V_1B+V_{12})P_V^{-1}(B^TV_1+V_{12}^T)A \qquad (20)\\ P_V &= U+V_2+B^TV_1B+V_{12}^TB+B^TV_{12} \qquad (21)\\ L_V &= -P_V^{-1}(B^TV_1+V_{12}^T)A \qquad (22)\end{aligned}$$

We define the nonlinear operator $\Psi_\nu(V)$ as follows:

$$\Psi_\nu(V)=\begin{bmatrix}\Phi_\nu(V) & \mathcal{L}_{12}(V)\\ \mathcal{L}_{12}^T(V) & \mathcal{L}_2(V)\end{bmatrix} \qquad (23)$$

which has few useful properties summarized in the following proposition:

Proposition 2: Consider the operators $\mathcal{L}(L,V)$ and $\Psi(V)$ defined in Equations (15) and (23), respectively. Also assume that $V\geq 0$. Then the following facts are true:

(a) $\mathcal{L}(L,V)\geq\Psi_\nu(V)=\mathcal{L}(L_V,V)\geq 0,\forall L,V$, where L_V is defined in Equation (22).
(b) If $V_1\geq V_2$ then $\mathcal{L}(L,V_1)\geq\mathcal{L}(L,V_1)$ and $\Psi_\nu(V_1)\geq\Psi_\nu(V_2)$.
(c) If $\nu_1\geq\nu_2$ then $\Psi_{\nu_1}(V)\geq\Psi_{\nu_2}(V),\forall V$.
(d) If $(A,W^{1/2})$ is detectable and $V=\Psi_\nu(V)$ and $S=\mathcal{L}(L,S)$ have a positive semidefinite solution, then V and S are unique and $V\leq S,\forall S$. Moreover, if $L=L_V$, then $V=S$.

Proof: Fact (a) follows from:

$$\mathcal{L}(L,V)-\Psi_\nu(V)=\begin{bmatrix}(1-\nu)(L-L_V)^TP_V(L-L_V) & 0\\ 0 & 0\end{bmatrix}\geq 0,\forall L,V$$

where P_V is defined above.

In fact (b) the monotonicity of $\mathcal{L}$ follows from $\mathcal{L}(L,V_1)-\mathcal{L}(L,V_2)=\mathcal{L}(L,V_1-V_2)\big|_{U=0}\geq 0$. The monotonicity of Ψ_ν follows from fact (a) since $\Psi_\nu(V_1)=\mathcal{L}(L_{V_1},V_1)\geq\mathcal{L}(L_{V_1},V_2)\geq\Psi_\nu(V_1)$.

Fact (c) follows from:

$$\Psi_{\nu_1}(V)-\Psi_{\nu_2}(V)=(\nu_1-\nu_2)\begin{bmatrix}A^T(V_1B+V_{12})^TP_V^{-1}(V_1B+V_{12})A & A^T(V_1B+V_{12})\\ (V_1B+V_{12})^TA & P_V\end{bmatrix}\geq 0,\forall V$$

where the positive definiteness can be verified by taking the Shur complement and noting that $P_V>0$.

The proof for fact (d) is somewhat more technical and in the interest of space only a sketch is reported. The proof follows along the same lines of standard LQ-control [19], where it is shown that the null space of the matrices V and S corresponds to the unobservable subspace of $(A,W^{1/2})$, which, by hypothesis, is strictly stable. This property is sufficient to prove uniqueness of the solution. The inequality $V\leq S,\forall L$ follows directly from fact (a). The last statement $V=S$ if $L=L_V$ comes from fact (a) and the uniqueness of the solution for both operators $\mathcal{L}$ and Ψ_ν. ■

We are now ready to derive the optimal gain and stability conditions for the hold-input strategy:

Theorem 2: Consider the system defined by Equations (3) and the infinite horizon cost defined in Equation (4). Assume that the pair (A,B) is stabilizable and $(A,W^{1/2})$ is detectable. Then the closed loop system is mean square stabilizable if and only if there exists a positive semidefinite solution of the Riccati-like equation:

$$V^* \;= \Psi_\nu(V^*) \tag{24}$$

where $\Psi_\nu(V)$ is defined in Equation (23). The optimal cost is given by

$$J^*_\infty = \min_L J_\infty(L) = z_0^T V^* z_0$$

and the corresponding optimal gain is given by

$$L^* = L_{V^*} \tag{25}$$

where L_V is defined in Equation (22). The Riccati-like equation $V^* = \Psi_\nu(V^*)$ has a positive definite solution if and only if $\nu < \nu_c$, where ν_c is a critical packet loss probability, which depends on the pair (A, B). The fixed point V^* can be obtained as the limit of the sequence $V^*_{k+1} = \Psi_\nu(V^*_k), V^*_0 = 0$, i.e. $\lim_{k\to\infty} V^*_k = V^*$. If $u^a_{-1} = 0$, then the optimal cost reduces to $J^*_\infty = x_0^T V_1^* x_0$.

Proof: From Proposition 1 the mean square asymptotic stability of the closed loop system is equivalent to show that the corresponding infinite horizon cost is bounded. Also, from the definition of cost-to-go function we have that $J_\infty(L) = \lim_{N\to\infty} C_0^N(z_0) = \lim_{N\to\infty} \mathbb{E}[z_0^T V_0 z_0 | z_0] = \lim_{N\to\infty} z_0^T V_0 z_0$, where $V_0 = \mathcal{L}^N(L, V_N) = \mathcal{L}^{N+1}(L, 0)$. To simplify the notation we reverse time $V_{k+1} = \mathcal{L}(L, V_k)$ where $V_0 = 0$ so that $J_\infty(L) = \lim_{N\to\infty} z_0^T V_{N+1} z_0$. Let also consider the sequence $V^*_{k+1} = \Psi_\nu(V^*_k)$ where $V^*_0 = 0$. From monotonicity of the operators $\mathcal{L}$ and Ψ_ν given by fact (b) of Proposition 2, and from the fact that $V_1 = V_1^* \geq V_0 = V_0^* = 0$ it follows that V_k and V_k^* are monotonically increasing, i.e. $V_{k+1} \geq V_k$ and $V^*_{k+1} \geq V^*_k$ for all k. Since they are monotonically increasing, they either converge or are unbounded. If they converge, i.e. $\lim_{k\to} V_k = V$ and $\lim_{k\to} V^*_k = V^*$, then they also need to satisfy $V = \mathcal{L}(L, V)$ and $V^* = \Psi(V^*)$. Also, from fact (d) we have that $V \geq V^*, \forall L$. Therefore, if there exists a gain L such that V is bounded, then also V^* must be bounded, which means that if the system is mean square stabilizable then $V^* = \Psi_\nu(V^*)$ is bounded. Conversely, if V^* is bounded and we choose $L = L_{V^*}$ we must have $0 = V_0 \leq V^*$, therefore also $V_k = \mathcal{L}^k(L_{V^*}, 0) \leq \mathcal{L}^k(L_{V^*}, V^*) = V^*, \forall k$ from which it follows that $V \leq V^*$. Since $V^* \leq V$ by fact (d), we must have that $V = V^*$. This means that if $V^* = \Psi_\nu(V^*)$ is bounded, then the system is mean square asymptotically stable since by choosing the gain $L = L_{V^*}$ the corresponding cost is bounded. Implicitly, this sufficient condition provides also a constructive procedure to compute the optimal gain and the corresponding optimal cost by generating the sequence V^*_k from $V^*_0 = 0$.

The last part we need to prove is the existence of a critical probability ν_c. If A is strictly stable and we choose $L = 0$ in Equations (16)-(18), by direct inspection we see that $V = \mathcal{L}(0, V)$ exists if $\nu < 1$, therefore also V^* must exist since $V^* \leq V$. For $\nu = 1$, from $V^* = \Psi_1(V^*)$ follows that $V_1^* = W + A^T V_1^* A$, $V_{12}^* = A^T V_{12}^* + A^T V_1^* B$ and $V_2^* = V_2^* + BV_1^* B + V_{12}^* B + B^T {V_{12}^*}^T + U$. Clearly V_1^* and V_{12}^* exist since A is strictly stable, but V_2^* does not unless the terms BV_1^*B, V_{12}^*B, and U are identically null, therefore $\nu_c = 1$. If A is not strictly stable, then for $\nu = 1$ $V^* = \Psi_1(V^*)$ does not have a solution since we should have $V_1^* = W + A^T V_1^* A$. Conversely, for $\nu = 0$ we have $V_{12}^* = 0, V_2^* = 0$ and $V_1^* = W + A^T V_1^* A - A^T V_1^* B(B^T V_1^* B + U)^{-1} B^T V_1^* A$. The last equality is the standard Riccati equation that under the hypothesis (A, B) stabilizable and (A, W) detectable has a unique positive semidefinite solution, therefore for $\nu = 1$ the equation $V^* = \Psi_1(V^*)$ has a solution. Let us indicate V^*_ν the solution $V^*_\nu = \Psi_\nu(V^*_\nu)$ for a generic ν. If there exist a $\bar\nu$ such that $V^*_{\bar\nu}$ exists, then by fact (b) of Proposition 2 we have that for all $\nu \leq \bar\nu$ we must have $V^*_\nu \leq V^*_{\bar\nu}$, therefore a solution exists also for a smaller packet loss probability. Conversely, if there exists a $\bar\nu$ such that $V^*_{\bar\nu}$ does not exist, then also V^*_ν for $\nu \geq \bar\nu$ does not exist. Therefore, by continuity, there must exist a $\nu_c \in (0, 1)$ such that $V^* = \Psi_\nu(V^*)$ exists for $\nu > \nu_c$ and does not exists for $\nu < \nu_c$. ■

Note that the hypothesis $u^a_{-1} = 0$ is a natural choice which allows a fair comparison between the zero-input strategy and the hold-input strategy. Few remarks are in order. The first remark is that the previous theorem states that we can compute the optimal gain L^* and the corresponding optimal cost

J^*_∞ as the solution of a Riccati-like equation, which can be computed numerically through an iterative procedure. The second remark is that if the system A is unstable, then there is a critical loss probability ν_c above which the closed loop system cannot be stabilized by any linear feedback. In general, it is hard to find the value for ν_c in closed form. However, using a bisection method based on whether the solution $V^* = \Psi_\nu(V^*)$ exists, we can numerically compute it with any desired accuracy. The last remark is that the optimal gain L^* does not depend on the initial conditions (x_0, u^a_{-1}). Although this is clear for classic LQ control and for the zero-input strategy, it is not trivial from Equation (15) that the matrix V for hold-input strategy is convex in the gain L, since a multivariate quadratic function is not necessarily convex.

So far we have shown how to compute the optimal gain for both the zero-input strategy and for hold-input strategy. However, we have not yet determined whether one strategy is better than the other. In the next section, we will compare the performance of the two strategies for scalar systems, for which we can find closed form expressions for the gain L and the performance J_∞.

V. Hold-input vs zero-input: the scalar case

Without loss of generality, we assume that $B = 1$, $A = a$, $W = w$, and $x_0 = 1$. We start by assuming that $U = 0$, which corresponds to a scenario where we look for the fastest converging controller in mean square sense. In fact, for $\nu = 0, U = 0$ we obtain the usual dead-beat controller. If we substitute the systems parameters into Equations (8) and (9) for the zero-input strategy we get:

$$\begin{aligned} s^*_\infty &= w + a^2 s^*_\infty - (1-\nu)a^2 s^*_\infty = w + \nu a^2 s^*_\infty \\ &= \frac{w}{1-\nu a^2} \end{aligned} \tag{26}$$

$$\ell^*_z = -a \tag{27}$$

Note that if the open loop system is unstable, i.e. $|a| > 1$, then the optimal solution exists, i.e. $s^*_\infty \geq 0$, if and only if $\nu a^2 < 1$.

Similarly, if we substitute the system parameters into Equations (24) for the hold-input strategy we get:

$$\begin{aligned} v^*_{12} &= \nu a(v^*_1 + v^*_{12}) \Longrightarrow v^*_{12} = \frac{\nu a}{1-\nu a} v^*_1 \\ v^*_2 &= \nu(v^*_1 + 2v^*_{12} + v^*_2) \Longrightarrow v^*_2 = \frac{\nu(v^*_1 + 2v^*_{12})}{1-\nu} = \frac{\nu(1+\nu a)}{(1-\nu)(1-\nu a)} v^*_1 \\ v^*_1 &= w + a^2 v^*_\infty - (1-\nu)\frac{a^2(v^*_1 + v^*_{12})^2}{v^*_1 + v^*_2 + 2v^*_{12}} \end{aligned}$$

If we substitute the first two equations into the third one, we find the following expression for v^*_1 and the optimal gain ℓ^*_h given by Equation (25) can be readily computed as follows:

$$\begin{aligned} v^*_1 &= w + a^2 v^*_\infty - \frac{(1-\nu)^2}{1-\nu^2 a^2} a^2 v^*_\infty \\ &= \frac{w}{1 - \left(1 - \frac{(1-\nu)^2}{1-\nu^2 a^2}\right) a^2} \end{aligned} \tag{28}$$

$$\ell^*_h = -\frac{(1-\nu)a}{1+\nu a} \tag{29}$$

If the open loop system is unstable then the optimal solution exists if and only if the denominator $\frac{w}{1-\left(1-\frac{(1-\nu)^2}{1-\nu^2 a^2}\right)a^2} = \frac{w(1-\nu^2 a^2)}{(1-\nu a^2)^2}$ is positive, which leads to the constraint $\nu|a| < 1$. The constraint $\nu|a| < 1$ is less restrictive that $\nu a^2 < 1$, therefore it seems that the hold-input strategy can stabilize the system for a larger packet loss probability than the zero-input strategy. However, we need not to forget that a necessary and sufficient stability condition for the hold-input strategy is that $V^* \geq 0$, which is equivalent to the conditions

$$v^*_1 \geq 0 \text{ and } v^*_1 - v^*_{12}(v^*_2)^{-1} v^*_{12} \geq 0$$

The first inequality is obviously satisfied, while the second, after some simple algebraic manipulation is given by:

$$v_1^* - v_{12}^*(v_2^*)^{-1}v_{12}^* = v_1^* \frac{1-\nu a^2}{1-\nu^2 a^2}$$

which is positive if and only if $\nu a^2 < 1$, thus recovering the same stability condition of the zero-input strategy.

We now show that the zero-input strategy gives a better performance than the hold-input strategy. This is equivalent to show that:

$$0 \leq s^* \leq v_1^* \Longleftrightarrow \nu \leq 1 - \tfrac{(1-\nu)^2}{1-\nu^2 a^2} \Longleftrightarrow \tfrac{(1-\nu)(1-\nu^2 a^2)-(1-\nu)^2}{1-\nu^2 a^2} \geq 0 \Longleftrightarrow \tfrac{\nu(1-\nu)(1-\nu a^2)}{1-\nu^2 a^2} \geq 0 \Longleftrightarrow \nu \leq \tfrac{1}{a^2} \quad (30)$$

which is always true since the feasibility condition is $\nu a^2 < 1$. Figure 2 shows a graphical representation of Equations (26) and (28), where $A = 1.2$, $B = W = 1$, and $U = 0$. In Figure 3 it is shown a typical

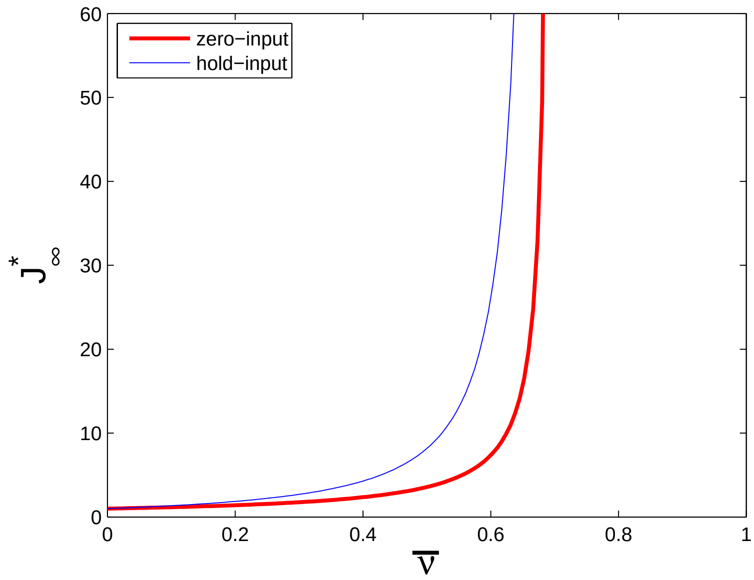

Fig. 2. Minimum cost J_∞ for $A = 1.2$, $B = W = x_0 = 1$, $U = 0$ under zero-input and hold-input control architectures. The critical loss probability for this systems is $\nu_c = 1/1.2^2 = 0.694$.

realization for an unstable system, $A = 1.2$, with packet loss probability $\nu = 0.5$, using optimal gain $\ell_z^* = -a = -1.2$ for the zero-input strategy and $\ell_h^* = -(1-\nu)a/(1+\nu a) = -0.375$ for the hold-input strategy. Note that the first control packet is lost and the state x starts to diverge, however as soon as a packet arrives the zero-hold strategy drives the system to zero, while the hold-input requires a longer time.

To validate the analytical equations derived in this paper, we computed the empirical total cost J_∞^{emp} by averaging 10000 run starting with the initial condition $x_0 = 1$ and $u_{-1} = 0$, for $A = 1.2$, $B = W = 1$, $U = 0$ and $\nu = 0.5$ for different values of the feedback gain ℓ. The analytical optimal gains ℓ_z^* and ℓ_h^*, and the corresponding minimum cost $J_{\infty,z}^* = s^*$ and $J_{\infty,h}^* = v_1^*$ given by Equations (26-(29), are computed and shown in Figure 4, which are consistent with the empirical curves.

So far we have considered only the case $U = 0$, i.e. the case when the input it is not penalized. Figure 5 shows the minimum cost obtained for the system where $A = 1.2$, $B = W = 1$, and $U = 10$. Very interestingly, there is range of values of the packet loss probability ν for which the hold-input strategy performs better than the zero-input strategy, while there is another range of values for which the converse occurs. This implies that, in general, it is not possible to state whether the hold-input strategy is always

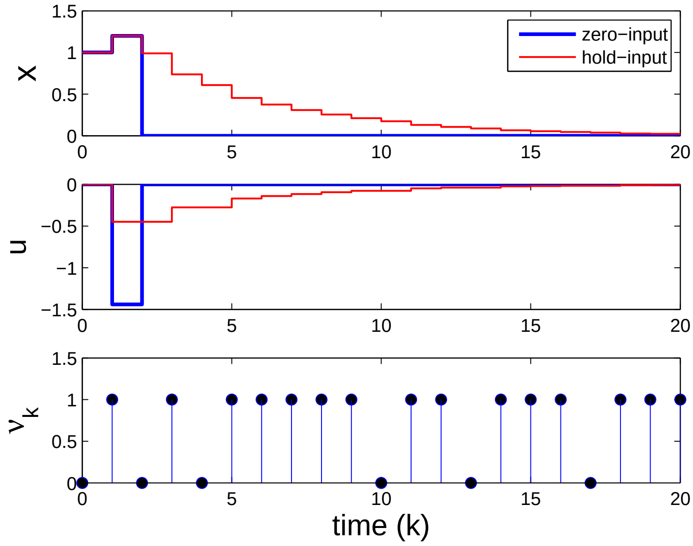

Fig. 3. A specific realization for $A = 1.2$, $x_0 = 1$, $\nu = 0.5$ under under optimal zero-input control, $\ell_z^* = -a = -1.2$ and optimal hold-input control, $\ell_h^* = -(1-\nu)a/(1+\nu a) = -0.375$.

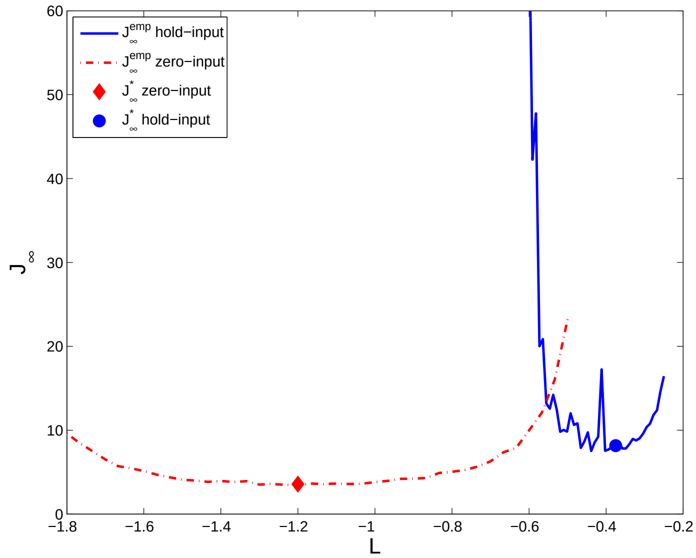

Fig. 4. Empirical total cost J_∞^{emp} for $A = 1.2$, $x_0 = 1$, $\nu = 0.5$ and obtained by averaging 10000 Monte Carlo runs under zero-input and hold-input control architectures. The analytical optimal gains ℓ^* and minimum total costs J_∞^* are also shown corresponding to the two strategies are shown.

better or worse than the hold-input strategy, even for simple scalar linear systems. We can summarize the previous results in the following proposition:

Proposition 3: Let us consider a scalar system where $A = a$, $B = 1$, $W = w = 1$ and $U = u \geq 0$, under the zero-hold control strategy and the hold-input control strategies. The fastest rate of convergence in terms of mean square state error $\mathbb{E}[x_k^2]$, corresponding to setting $u = 0$, is given by the zero-hold strategy, independency of packet loss probability ν and open loop dynamics a. If the system is unstable, i.e. $|a| \geq 1$, then both the hold-input control and the zero-input control can stabilize the system if and only if $\nu < \nu_c = \frac{1}{|a|^2}$. Finally, there exist regions of values for the parameters (u, ν) for which the hold-input strategy provides a smaller cost J_∞^* than the zero-hold strategy, and regions where the converse occurs.

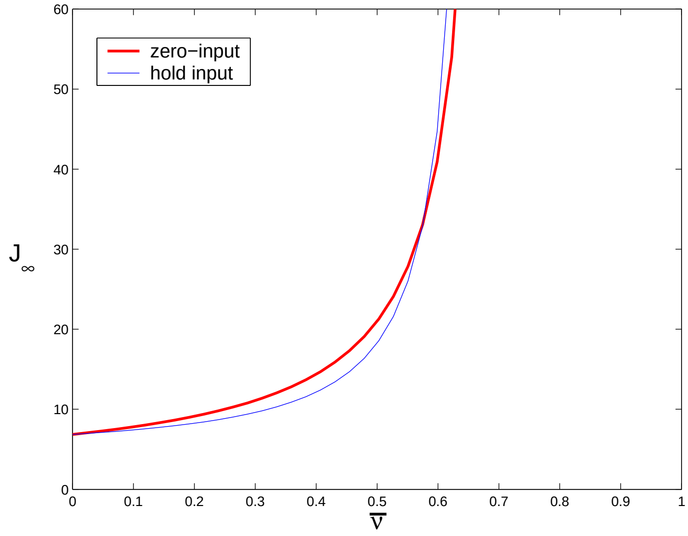

Fig. 5. Minimum cost J_∞ for $A = 1.2$, $B = W = x_0 = 1$, $U = 10$ under zero-input and hold-input control architectures.

VI. Conclusion

In this paper we studied LQ-like performance of the hold-input and zero-input strategies for control systems for which the control packets are subject to loss. These are the simplest and most commonly adopted strategies in the literature. We derived explicit expressions for computing the optimal static controller gains when control packets are lost according to a Bernoulli process. Interestingly, we showed that none of these two control schemes can be claimed to be superior to the other, even in simple scalar systems. However, the tools developed in this paper can be used to evaluate which architecture performs once the packet loss probability and the systems parameters are known.

We want to remark that although the zero-input strategy has been proposed in the literature mainly for mathematical reasons, in many situations it performs better than the hold-input strategy, thus encouraging further investigation in experimental settings and justifying its use in networked control systems.

References

[1] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE, special issue*, vol. 95, no. 1, pp. 138–162, January 2007.

[2] B. Sinopoli, C. Sharp, L. Schenato, S. Schaffert, and S. Sastry, "Distributed control applications within sensor networks," *Proceedings of the IEEE, Special Issue on Distributed Sensor Networks*, vol. 91, no. 8, pp. 1235–1246, August 2003.

[3] N. Nahi, "Optimal recursive estimation with uncertain observation," *IEEE Transaction on Information Theory*, vol. 15, no. 4, pp. 457–462, 1969.

[4] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," *Automatica*, vol. 34, no. 1, pp. 57–64, January 1998.

[5] O. Costa, "Stationary filter for linear minimum mean square error estimator of discrete-time markovian jump systems," *Transactions on Automatic Control*, vol. 48, no. 7, pp. 1351–1356, 2002.

[6] C. N. Hadjicostis and R. Touri, "Feedback control utilizing packet dropping network links," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 2, Las Vegas, NV, Dec. 2002, pp. 1205–1210.

[7] D. Giorgiev and D. Tilbury, "Packet-based control," in *Proc. of the 2004 Amer. Contr. Conf.*, June 2004, pp. 329–336.

[8] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, September 2004.

[9] L. Schenato, "Kalman filtering for networked control systems with random delay and packet loss," in *Conference of Mathematical Theroy of Networks and Systems (MTNS'06)*, Kyoto, Japan, July 2006.

[10] J. Nilsson, "Real-time control systems with delays," Ph.D. dissertation, Department of Automatic Control, Lund Institute of Technology, 1998.

[11] M. B. W. Zhang and S. Phillips, "Stability of networked control systems," *IEEE Control Systems Magazine*, vol. 21, no. 1, pp. 84–99, February 2001.

[12] Q. Ling and M. Lemmon, "Optimal dropout compensation in networked control systems," in *IEEE conference on decision and control*, Maui, HI, December 2003.

[13] N. Elia and J. Eisembeis, “Limitation of linear control over packet drop networks,” in *Proceedings of IEEE Conference on Decision and Control*, vol. 5, Bahamas, December 2004, pp. 5152–5157.
[14] O. C. Imer, S. Yuksel, and T. Basar, “Optimal control of dynamical systems over unreliable communication links,” *To appear in Automatica*, July 2006.
[15] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, “Foundations of control and estimation over lossy networks,” *Proceedings of the IEEE, special issue*, vol. 95, no. 1, pp. 163–187, January 2007.
[16] N. Elia, “Remote stabilization over fading channels,” *Systems and Control Letters*, vol. 54, pp. 237–249, 2005.
[17] V. Gupta, S. Adlakha, B. Sinopoli, and A. Goldsmith, “Towards receding horizon networked control,” *Submitted to Automatica*, 2007.
[18] M. Epstein, L. Shi, S. D. Cairano, and R. M. Murray, “Control over a network: Using actuation buffers and reducing transmission frequency,” in *2007 European Control Conference (ECC’07)*, Kos, Greece, July 2007.
[19] G. Chen, G. Chen, and S. Hsu, *Linear Stochastic Control Systems*. CRC Press, 1995.
[20] O. Costa, M. Fragoso, and R. Marques, *Discrete-Time Markov Jump Linear Systems (Probability and its Applications)*. Springer, 2004.